Some Fixed Point Theorems for Occasionally Weakly Compatible Mappings in Fuzzy 2-Metric Spaces.

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Abstract

The Objective of this paper is to obtain some common fixed point theorems for occasionally weakly compatible mappings in fuzzy 2-metric spaces.

Keywords: Occasionally weakly compatible mappings, fuzzy 2- metric space.

Introduction

Fuzzy set was defined by Zadeh [30]. Kramosil and Michalek [16] introduced fuzzy metric space, George and Veeramani [8] modified the notion of fuzzy metric spaces with the help of continuous t-norms. Many researchers have obtained common fixed point theorems for mappings satisfying different types of commutativity conditions. Vasuki [27] proved fixed point theorems for R-weakly commutating mappings. Pant [20, 21, 22] introduced the new concept reciprocally continuous mappings and established some common fixed point theorems. Balasubramaniam et al.[6], have shown that Rhoades [24] open problem on the existence of contractive definition which generates a fixed point but does not force the mappings to be continuous at the fixed point, posses an affirmative answer. Pant and Jha [22] obtained some analogus results proved by Balasubramaniam et al. Recent literature on fixed point in fuzzy metric space can be viewed in [1, 2, 3, 4, 11, 18, 26].

The concept of 2-metric space was initiated by Gahler [28] whose abstract properties were suggested by the area function in Euclidian space. In a paper Sanjay kumar [29] discussed the concept of fuzzy 2-metric space akin to 2-metric spaces introduced by Gahler [28].

This paper presents some common fixed point theorems for more general commutative condition i.e. occasionally weakly compatible mappings in complete fuzzy 2-metric space.
Preliminary Notes

Definition 2.1 [30] A fuzzy set $A$ in $X$ is a function with domain $X$ and values in $[0,1]$.

Definition 2.2 [25] A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is a continuous t-norms if $*$ is satisfying conditions:
- $*$ is an commutative and associative;
- $*$ is continuous;
- $a * 1 = a$ for all $a \in [0,1]$;
- $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definitions 2.3 [8] A 3-tuple $(X, M, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t-norm and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$,
- $M(x, y, t) > 0$;
- $M(x, y, t) = 1$ if and only if $x = y$;
- $M(x, y, t) = M(y, x, t)$;
- $M(x, y, t) \cdot M(y, z, s) \leq M(x, z, t + s)$;
- $M(x, y, \cdot) : (0, \infty) \rightarrow (0,1]$ is continuous.

Then $M$ is called a fuzzy metric on $X$. Then $M(x, y, t)$ denotes the degree of nearness between $x$ and $y$ with respect to $t$.

Example 2.4 (Induced fuzzy metric [8]) Let $(X, d)$ be a metric space. Denote $a * b = ab$ for all $a, b \in [0,1]$ and let $M_d$ be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:
- $M_d = \frac{t}{t + d(x, y)}$.

Then $(X, M_d, *)$ is a fuzzy metric space.

Definition 2.5 A binary operation $*: [0,1] \times [0,1] \times [0,1] \rightarrow [0,1]$ is called a continuous t-norm if $([0,1], *)$ is an abelian topological monoid with unit $1$ such that $a * b * c \leq d * e * f$ whenever $a \leq d, b \leq e$ and $e \leq f$ for all $a, b, c, d, e, f \in [0,1]$.

In a paper Sanjay kumar [29 ] discussed the concept of fuzzy 2-metric space akin to 2-metric spaces introduced by Gahler [28]. We borrow some definitions, examples and remark given in [29].

Definition 2.6 [29] A triplet $(X, M, *)$ is a fuzzy 2-metric space if $X$ is an arbitrary
set, \(*\) is a continuous t-norm, and M is a fuzzy set in \(X^3 \times [0, \infty)\) satisfying the following conditions:

\[
M(x, y, a, 0) = 0
\]

\[
M(x, y, a, t) = 1 \text{ for all } t > 0 \text{ if and only if at least two of them are equal.}
\]

\[
M(x, y, a, t) = M(y, a, x, t) = M(a, y, x, t) \quad \text{(symmetric)}
\]

\[
M(x, y, a, r + s + t) \geq M(x, y, z, r) \ast M(x, z, a, s) \ast M(z, y, a, t) \text{ for all } x, y, z \in X
\]

and \(r, s, t > 0\).

\[
M(x, y, a, t) : [0, \infty) \to [0, 1] \text{ is left continuous for all } x, y, z, a \in X \text{ and } r, s, t > 0.
\]

\[
\lim_{n \to \infty} M(x, y, a, t) = 1 \text{ for all } x, y, a \in X, t > 0.
\]

**Example 2.7** [29] Let X be the set \(\{1, 2, 3, 4\}\) with 2-metric \(d\) defined by

\[
d(x, y, z) = \begin{cases} 0, & \text{if } x = y = z, \text{ and } \{x, y, z\} = \{1, 2, 3\} \\ 1, & \text{otherwise.} \end{cases}
\]

For each \(t \in [0, \infty)\), define \(a \ast b \ast c = abc\) and

\[
M(x, y, z, t) = \begin{cases} 0, & \text{if } t = 0. \\ \frac{t}{t + d(x, y, z)}, & \text{if } t > 0, \text{ where } x, y, z \in X. \end{cases}
\]

Then \((X, M, \ast)\) is a fuzzy 2-metric space.

**Definition 2.8** [29] (a) A sequence \(\{x_n\}\) in \((X, M, \ast)\) is convergent to \(x \in X\) if

\[
\lim_{n \to \infty} M(x_n, x, a, t) = 1 \text{ for each } t > 0.
\]

(b) A fuzzy 2-metric space, \((X, M, \ast)\) is called Cauchy if

\[
\lim_{n,m \to \infty} M(x_n, x_m, a, t) = 1
\]

for each \(t > 0\).

(c) A fuzzy 2-metric space in which every Cauchy sequence is convergent is said to be complete.

**Definition 2.9** A pair of self-mappings \((f, g)\) of a fuzzy 2-metric space \((X, M, \ast)\) is said to be weakly commuting if \(M(fx, gx, a, t) > M(fx, gx, a, t)\) for all \(x \in X\) and \(t > 0\).

R-weakly commuting if there exist some \(R > 0\) such that

\[
M(fx, gx, a, t) \geq M(fx, gx, a, \frac{t}{R}).
\]

**Definition 2.10** Two self mappings \(f\) and \(g\) of a fuzzy 2-metric space \((X, M, \ast)\) are
called compatible if $\lim_{n \to \infty} M(fg x_n, g f x_n, a, t) = 1$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x$ for some $x$ in $X$.

**Definition 2.11** Two self maps $f$ and $g$ of a fuzzy 2-metric space $(X, M, \ast)$ are called reciprocally continuous on $X$ if $\lim_{n \to \infty} f x_n = f x$ and $\lim_{n \to \infty} g x_n = g x$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g x_n = x$ for some $x$ in $X$.

**Lemma 2.12** Let $(X, M, \ast)$ be a fuzzy 2-metric space. If there exists $q \in (0,1)$ such that $M(x, y, a, t) \geq M(x, y, a, t)$ for all $x, y \in X$ and $t > 0$, then $x = y$.

**Definition 2.13** [1] Let $X$ be a set, $f, g$ selfmaps of $X$. A point $x$ in $X$ is called a coincidence point of $f$ and $g$ iff $f x = g x$. We shall call $w = f x = g x$ a point of coincidence of $f$ and $g$.

**Definition 2.14** [13] A pair of maps $S$ and $T$ is called weakly compatible pair if they commute at coincidence points.

**Definition 2.15** [1] Two self maps $f$ and $g$ of a set $X$ are occasionally weakly compatible (owc) iff there is a point $x$ in $X$ which is a coincidence point of $f$ and $g$ at which $f$ and $g$ commute.

A. Al-Thagafi and Naseer Shahzad [5] shown that occasionally weakly is weakly compatible but converse is not true.

**Definition 2.16** [5] Let $R$ be the usual metric space. Define $S, T: R \to R$ by $S x = 2x$ and $T x = x^2$ for all $x \in R$. Then $S x = T x$ for $x = 0, 2$ but $S 0 = T 0$ and $S 2 \neq T 2$. $S$ and $T$ are occasionally weakly compatible self maps but not weakly compatible.

**Lemma 2.17** [14] Let $X$ be a set, $f, g$ owc self maps of $X$. If $f$ and $g$ have a unique point of coincidence, $w = f x = g x$, then $w$ is the unique common fixed point of $f$ and $g$.

**Main Results**

**Theorem 3.1** Let $(X, M, \ast)$ be a complete fuzzy 2-metric space and Let $A, B, S$ and $T$ be self mappings of $X$. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exist $q \in (0,1)$ such that

$$M(A x, B y, a, q t) \geq \alpha_1 M(S x, T y, a, t) + \alpha_2 M(A x, T y, a, t) + \alpha_3 M(B y, S x, a, t) \quad (3.1)$$

For all $x, y \in X$, where $\alpha_1, \alpha_2, \alpha_3 > 0$, $\alpha_1 + \alpha_2 + \alpha_3 > 1$ then there exist a unique point $w \in X$ such that $A w = S w = w$ and a unique point $z \in X$ such that $B z = T z = z$. Moreover, $z = w$, so that there is a unique common fixed point of $A, B, S$ and $T$.

**Proof:** Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc, so there are points $x, y \in X$ such that
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Let \( Ax = Sx \) and \( By = Ty \). We claim that, \( Ax = By \). If not, by inequality (3.1)

\[
M(Ax, By, a, qt) \geq \alpha M(Sx, Ty, a, t) + \alpha_2 M(Ax, Ty, a, t) + \alpha_3 M(By, Sx, a, t)
\]

\[
= \alpha_1 M(Ax, By, a, t) + \alpha_2 M(Ax, By, a, t) + \alpha_3 M(By, Ax, a, t)
\]

\[
= (\alpha_1 + \alpha_2 + \alpha_3) M(Ax, By, a, t)
\]

a contradiction, since \((\alpha_1 + \alpha_2 + \alpha_3) > 1\). And by Lemma (2.12) \(Ax=By\), i.e. \(Ax = Sx = By = Ty\). Suppose that there is another point \( z \) such that \( Az = Sz \) then by (3.1) we have \( Az = Sz = By = Ty \). So, \( Ax = Az \) and \( w = Ax = Sx \) is the unique point of coincidence of \( A \) and \( S \). By Lemma (2.17) \( w \) is the only common fixed point of \( A \) and \( S \), i.e. \( w = Aw = Sw \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \).

Assume that \( w \neq z \). We have

\[
M(w, z, a, qt) = M(Aw, Bz, a, qt)
\]

\[
\geq \alpha_1 M(Sw, Tz, a, t) + \alpha_2 M(Aw, Tz, a, t) + \alpha_3 M(Bz, Sw, a, t)
\]

\[
= \alpha M(w, z, a, t) + \alpha_2 M(w, z, a, t) + \alpha_3 M(z, w, a, t)
\]

\[
= (\alpha_1 + \alpha_2 + \alpha_3) M(w, z, a, t)
\]

a contradiction, since \((\alpha_1 + \alpha_2 + \alpha_3) > 1\). And by Lemma (2.12) \(z=w\). Also by Lemma (2.17), \( z \) is the common fixed point of \( A, B, S \) and \( T \). The uniqueness of the fixed point holds from (3.1).

**Theorem 3.2** Let \((X, M, \ast)\) be a complete fuzzy 2-metric space and Let \( A, B, S \) and \( T \) be self mappings of \( X \). Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc. If there exist \( q \in (0,1) \) such that

\[
M(Ax, By, a, qt) \geq \alpha \min\{M(Sx, Ty, a, t), M(Sx, Ax, a, t)\}
\]

\[
+ \beta \min\{M(By, Ty, a, t), M(Ax, Ty, a, t)\}
\]

\[
+ \gamma M(By, Sx, a, t)
\]

(3.2)

For all \( x, y \in X \), where, \( \alpha, \beta, \gamma > 0, (\alpha + \beta + \gamma) > 1 \) then there exist a unique point \( w \in X \) such that \( Aw = Sw = w \) and a unique point \( z \in X \) such that \( Bz = Tz = z \). Moreover, \( z = w \), so that there is a unique common fixed point of \( A, B, S \) and \( T \).

**Proof:** Let the pairs \( \{A, S\} \) and \( \{B, T\} \) be owc, so there are points \( x, y \in X \) such that \( Ax = Sx \) and \( By = Ty \). We claim that, \( Ax = By \). If not, by inequality (3.2)

\[
M(Ax, By, a, qt) \geq \alpha \min\{M(Sx, Ty, a, t), M(Sx, Ax, a, t)\}
\]

\[
+ \beta \min\{M(By, Ty, a, t), M(Ax, Ty, a, t)\}
\]

\[
+ \gamma M(By, Sx, a, t)
\]
\[
\begin{align*}
&= \alpha \min \{M(Ax, By, a, t), M(Ax, Ax, a, t)\} \\
&+ \beta \min \{M(By, By, a, t), M(Ax, By, a, t)\} \\
&+ \gamma M(By, Ax, a, t) \\
&= \alpha \min \{M(Ax, By, a, t), 1\} + \beta \min \{1, M(Ax, By, a, t)\} \\
&+ \gamma M(By, Ax, a, t) \\
&= \alpha M(Ax, By, a, t) + \beta M(Ax, By, a, t) + \gamma M(Ax, By, a, t) \\
&= (\alpha + \beta + \gamma)M(Ax, By, a, t)
\end{align*}
\]

a contradiction, since \((\alpha + \beta + \gamma) > 1\). And by Lemma (2.12) \(Ax = By\), i.e. \(Ax = Sx = By = Ty\). Suppose that there is another point \(z\) such that \(Az = Sz\) then by (3.2) we have \(Az = Sz = By = Ty\). So, \(Ax = Az\) and \(w = Ax = Sx\) is the unique point of coincidence of \(A\) and \(S\). By Lemma (2.17) \(w\) is the only common fixed point of \(A\) and \(S\), i.e. \(w = Aw = Sw\). Similarly there is a unique point \(z \in X\) such that \(z = Bz = Tz\).

Assume that \(w \neq z\). We have
\[
M(w, z, a, qt) = M(Aw, Bz, a, qt)
\]
\[
\geq \alpha \min \{M(Sw, Tz, a, t), M(Sw, Aw, a, t)\} \\
+ \beta \min \{M(Bz, Tz, a, t), M(Aw, Tz, a, t)\} \\
+ \gamma M(Bz, Sw, a, t) \\
= \alpha \min \{M(w, z, a, t), 1\} + \beta \min \{1, M(w, z, a, t)\} \\
+ \gamma M(z, w, a, t) \\
= (\alpha + \beta + \gamma)M(w, z, a, t)
\]

a contradiction, since \((\alpha + \beta + \gamma) > 1\). And by Lemma (2.12) \(z = w\). Also by Lemma (2.17) \(z = w\) is the common fixed point of \(A\), \(B\), \(S\) and \(T\). Therefore the uniqueness of the fixed point holds from (3.2).

**Theorem 3.3** Let \((X, M, *)\) be a complete fuzzy 2-metric space and Let \(A\), \(B\), \(S\) and \(T\) be self mappings of \(X\). Let the pairs \(\{A, S\}\) and \(\{B, T\}\) be owc. If there exist \(q \in (0, 1)\) such that
\[
M(Ax, By, a, qt) \geq \min \{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(Ty, Ax, a, t), M(By, Ty, a, t)\}, M(By, Sx, a, t)\}
\]
\[(3.3)\]

For all \(x, y \in X\), and \(t > 0\) then there exist a unique point \(w \in X\) such that \(Aw = Sw = w\) and a unique point \(z \in X\) such that \(Bz = Tz = z\). Moreover, \(z = w\), so that there is a unique common fixed point of \(A\), \(B\), \(S\) and \(T\).
**Proof:** Let the pairs \{A, S\} and \{B, T\} be owc, so there are points \(x, y \in X\) such that \(Ax = Sx\) and \(By = Ty\). We claim that, \(Ax = By\). If not, by inequality (3.3)

\[
M(Ax, By, a, qt) \geq \min\{M(Sx, Ty, a, t), M(Ax, Sx, a, t), M(Ty, Ax, a, t), M(By, Ty, a, t)\},
\]

\[
M(By, Sx, a, t);\]

\[
= \min\{M(Ax, By, a, t), M(Ax, Ax, a, t), M(By, By, a, t)\}
\]

\[
, M(By, Ax, a, t)\}
\]

\[
= \min\{M(Ax, By, a, t), 1, M(By, Ax, a, t)\,1, M(By, Ax, a, t)\}
\]

\[
= M(Ax, By, a, t)\]

Therefore \(Ax = By\), i.e. \(Ax = Sx = By = Ty\). Suppose that there is another point \(z\) such that \(Az = Sz\) then by (3.3) we have \(Az = Sz = By = Ty\). So, \(Ax = Az\) and \(w = Ax = Sx\) is the only common point of coincidence of \(A\) and \(S\). By Lemma (2.17) \(w\) is the unique common fixed point of \(A\) and \(S\), i.e. \(w = Aw = Sw\). Similarly there is a unique point \(z \in X\) such that \(z = Bz = Tz\).

Assume that \(w \neq z\). We have

\[
M(w, z, a, qt) = M(Aw, Bz, a, qt)
\]

\[
\geq \min\{M(Sz, Tz, a, t), M(Aw, Sw, a, t), M(Tz, Aw, a, t), M(Bz, Tz, a, t)\}
\]

\[
M(Bz, Sw, a, t))\}
\]

\[
= \min\{M(w, z, a, t), M(w, w, a, t), M(z, z, a, t), M(z, w, a, t)\}
\]

\[
= M(w, z, a, t)\]

Therefore we have \(z = w\) by Lemma (2.17) and \(z\) is the common fixed point of \(A, B, S\) and \(T\). The uniqueness of fixed point holds from (3.3).

**Theorem 3.4** Let \((X, M, *)\) be a complete fuzzy 2-metric space and \(A, B, S\) and \(T\) be self mappings of \(X\). Let the pairs \{A, S\} and \{B, T\} be owc. If there exist \(q \in (0, 1)\) such that

\[
M(Ax, By, a, qt) \geq \phi\{M(Sx, Ty, a, t), M(Sx, Ax, a, t), M(By, Ty, a, t), M(Ax, Ty, a, t)\}
\]

\[
, M(By, Sx, a, t)\};\]

(3.4)

For all \(x, y \in X\), and \(\phi: [0, 1]^3 \rightarrow [0, 1]\) such that \(\phi(t, t, t, t) > t\) for all \(0 < t < 1\) then there exist a unique common fixed point of \(A, B, S\) and \(T\).

**Proof:** Let the pairs \{A, S\} and \{B, T\} be owc, there are points \(x, y \in X\) such that \(Ax = Sx\) and \(By = Ty\). We claim that, \(Ax = By\). If not, by inequality (3.4)
a contradiction, therefore by Lemma (2.12) $A x = B y$, i.e. $A x = S x = B y = T y$. Suppose that there is another point $z$ such that $A z = S z$ then by (3.4) we have $A z = S z = B y = T y$. So, $A x = A z$ and $w = A x = S x$ is the unique point of coincidence of $A$ and $S$. By Lemma (2.17) $w$ is the only common fixed point of $A$ and $S$, i.e. $w = A w = S w$. Similarly there is a unique point $z \in X$ such that $z = B z = T z$.

Thus $z$ is a common fixed point of $A$, $B$, $S$ and $T$. The uniqueness of the fixed point holds from (3.4).

**Theorem 3.5** Let $(X, M, \ast)$ be a complete fuzzy 2-metric space and Let $A$, $B$, $S$ and $T$ be self mappings of $X$. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exist $q \in (0, 1)$ such that

$$M(A x, B y, a, q t) \geq \phi(M(S x, T y, a, t), M(A x, A x, a, t), M(B y, B y, a, t))$$

(3.5)

For all $x, y \in X$, and $\phi : [0, 1] \rightarrow [0, 1]$ such that $\phi(t, t, t) > t$ for all $0 < t < 1$ then there exist a unique common fixed point of $A$, $B$, $S$ and $T$.

**Proof:** The proof follows from Theorem 3.4.

**Theorem 3.6** Let $(X, M, \ast)$ be a complete fuzzy 2-metric space and Let $A$, $B$, $S$ and $T$ be self mappings of $X$. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exist $q \in (0, 1)$ such that

$$M(A x, B y, a, q t) \geq \phi(M(S x, T y, a, t), M(S x, A x, a, t), M(B y, T y, a, t))$$

$$M(A x, T y, a, t), M(B y, S x, a, t))$$

(3.6)

For all $x, y \in X$, and $\phi : [0, 1] \rightarrow [0, 1]$ such that $\phi(t) > t$ for all $0 < t < 1$ then there exist a unique common fixed point of $A$, $B$, $S$ and $T$.

**Proof:** The Proof follows from Theorem 3.4.

**Theorem 3.7** Let $(X, M, \ast)$ be a complete fuzzy 2-metric space and Let $A$, $B$, $S$ and $T$ be self mappings of $X$. Let the pairs $\{A, S\}$ and $\{B, T\}$ be owc. If there exist a point $q \in (0, 1)$ for all $x, y \in X$ and $t > 0$ such that
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\[ M(Ax, By, a, qt) \geq M(Sx, Ty, a, t) \] \quad (3.7)

then there exist a unique common fixed point of A, B, S and T.

**Proof:** Let the pairs \{A, S\} and \{B, T\} be owc, there are points \( x, y \in X \) such that \( Ax = Sx \) and \( By = Ty \). We claim that \( Ax = By \). If not, by inequality (3.7)

\[ M(Ax, By, a, qt) \geq M(Sx, Ty, a, t) \]

= \( M(Ax, By, a, t) \)

a contradiction, therefore by Lemma (2.12) \( Ax = By \), i.e. \( Ax = Sx = By = Ty \). Suppose that there is another point \( z \) such that \( Az = Sz \) then by (3.7) we have \( Az = Sz = By = Ty \). So, \( Ax = Az \) and \( w = Ax = Sx \) is the unique point of coincidence of A and S. By Lemma (2.17) \( w \) is the only common fixed point of A and S, i.e. \( w = Aw = Sw \). Similarly there is a unique point \( z \in X \) such that \( z = Bz = Tz \).

Assume that \( w \neq z \). We have

\[ M(w, z, a, qt) = M(Aw, Bz, a, t) \]

\[ \geq M(Sw, Tz, a, t) \]

= \( M(w, z, a, t) \)

Therefore we have \( z = w \) by Lemma 2.17 and \( z \) is the common fixed point of A, B, S and T. The uniqueness of the fixed point holds from (3.7).

**Theorem 3.8** Let \((X, M, *)\) be a complete fuzzy 2-metric space and Let A and S be self mappings of X. Let A and S are owc. If there exist a point \( q \in (0, 1) \) for all \( x, y \in X \) and \( t > 0 \) such that

\[ M(Sx, Sy, a, qt) \geq \alpha_1 M(Ax, Ay, a, t) + \alpha_2 M(Sx, Ay, a, t) + \alpha_3 M(Sy, Ax, a, t) \] \quad (3.8)

for all \( x, y \in X \), where \( \alpha_1, \alpha_2, \alpha_3 > 0, (\alpha_1 + \alpha_2 + \alpha_3) > 1 \), then there exist a unique common fixed point of A and S.

**Proof:** Let the pair \{A, S\} be owc, so there is a point \( x \in X \) such that \( Ax = Sx \). We claim that \( Sx = Sy \). By inequality (3.8)

\[ M(Sx, Sy, a, qt) \geq \alpha_1 M(Ax, Ay, a, t) + \alpha_2 M(Sx, Ay, a, t) + \alpha_3 M(Sy, Ax, a, t) \]

\[ = \alpha_1 M(Sx, Sy, a, t) + \alpha_2 M(Sx, Sy, a, t) + \alpha_3 M(Sy, Sx, a, t) \]

\[ = (\alpha_1 + \alpha_2 + \alpha_3) M(Sx, Sy, a, t) \]

a contraction, since \( (\alpha_1 + \alpha_2 + \alpha_3) > 1 \). Therefore \( Sx = Sy \). Therefore \( Ax = Ay \) and
Ax is unique. From Lemma 2.17, A and S have a unique fixed point.

**Theorem 3.9** Let \((X, M, \ast)\) be a complete fuzzy 2-metric space and Let A and S be self mappings of X. Let A and S are owc. If there exist a point \(q \in (0,1)\) for all \(x, y \in X\) and \(t > 0\) such that

\[
M(Sx, Sy, a, qt) \geq \alpha M(Ax, Ay, a, t) + \beta \min\{M(Ax, Ay, a, t), M(Sx, Ax, a, t), M(Sy, Ay, a, t)\} \tag{3.9}
\]

for all \(x, y \in X\), where \(\alpha, \beta > 0\), \((\alpha + \beta) > 1\). Then A and S have a unique common fixed point.

**Proof:** Let the pair \(\{A, S\}\) be owc, so there is a point \(x \in X\) such that \(Ax = Sx\). We claim that \(Sx = Sy\). By inequality (3.9)

\[
M(Sx, Sy, a, qt) \geq \alpha M(Ax, Ay, a, t) + \beta \min\{M(Ax, Ay, a, t), M(Sx, Ax, a, t), M(Sy, Ay, a, t)\}
\]

\[
= \alpha M(Sx, Sy, a, t) + \beta \min\{M(Sx, Sy, a, t), M(Sx, Sx, a, t), M(Sy, Sy, a, t)\}
\]

\[
= \alpha M(Sx, Sy, a, t) + \beta \min\{M(Sx, Sy, a, t), 1, 1\}
\]

\[
= (\alpha + \beta)M(Sx, Sy, a, t)
\]

a contraction, since \((\alpha + \beta) > 1\). Therefore \(Sx = Sy\). Therefore \(Ax = Ay\) and Ax is unique. From Lemma 2.17, A and S have a unique fixed point.

**Conclusion**

In this paper we prove some fixed point theorems for a pair of occasionally weakly compatible mappings in fuzzy 2-metric space.

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**References**


Some Fixed Point Theorems