Ternary Fuzzy Semigroups

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Abstract

Ternary fuzzy semigroups are defined and some examples are given. Various properties of ternary fuzzy semigroups are also given. It is furthermore proved that the homomorphic image of a ternary fuzzy semigroup is a ternary fuzzy semigroup.

Keywords: Ternary Semigroup; Ternary Fuzzy Semigroup; Ternary Fuzzy Subsemigroup.

Introduction

Firstly ternary algebraic operations were introduced in the XIXth century by A. Cayley [11]. Zadeh’s classical paper [14] of 1965 introduced the concept of fuzzy set and fuzzy set operation. The concept of fuzzy subgroups was introduced by Rosenfeld [9] in 1971. Since then the considerable work has been done in different algebraic structures.

In this paper we have investigated some properties of ternary fuzzy semigroups. We show that the homomorphic image of a ternary fuzzy semigroup having the suprimum property is a ternary fuzzy semigroup. Through this note a set $T$ will denote a ternary semigroup unless otherwise stated and $I$ stand for a complete lattice with least element zero and the greatest element 1. The join and meet operations in $I$ are denoted by $\lor$ and $\land$, respectively.

Preliminaries

Definition 2.1 [7] A nonempty set $T$ is called a ternary semigroup if a ternary operation $[\cdot]$ on $T$ is defined and satisfies the following associative law.

$$[[x_1 x_2 x_3] x_4 x_5] = [x_1 [x_2 x_3 x_4] x_5] = [x_1 x_2 [x_3 x_4 x_5]]$$

For all $x_i \in T$, $1 \leq i \leq 5$. 
Example 2.2 \( T = \{-i, 0, i \mid i = \sqrt{-1} \} \) is a ternary semigroup under the multiplication over complex number, while \( T \) is not a binary semigroup under the multiplication over complex number.

Example 2.3 \( T = \{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \} \) is ternary semigroup under matrix multiplication.

Definition 2.4 A ternary subsemigroup is a subset \( H \) of a ternary semigroup \( T \) such that \( [H H H] \subseteq H \).

Ternary Fuzzy Semigroups

Definition 3.1 Let \( T \) be a ternary semigroup. Define a fuzzy subset \( A: T \to I \) by

\[
A(x * y * z) \geq A(x) \land A(y) \land A(z) \quad \forall x, y, z \in T.
\]

Where \( * \) infix ternary operation. Then \( A \) is called ternary fuzzy semigroup of \( T \).

Definition 3.2 A fuzzy subset \( A: T \to I \) is said to be a ternary fuzzy subsemigroup if it is ternary fuzzy semigroup.

Example 3.3 Let \( T = \langle N, \cdot \rangle \) be a ternary semigroup with usual multiplication operation \( \cdot \). Define \( A: T \to I \) by

\[
A(x) = \begin{cases} 
1 & \text{if } x \in (8) \\
\frac{1}{2} & \text{if } x \in (4) - (8) \\
\frac{1}{3} & \text{if } x \in (2) - (8) \\
0 & \text{otherwise}
\end{cases}
\]

Then \( A \) is ternary fuzzy semigroup of \( T \).

Example 3.4 Let \( T = \langle N, + \rangle \) be a ternary semigroup with usual addition operation \( + \). Define \( A: T \to I \) by

\[
A(x) = \begin{cases} 
1 & \text{if } x \in (6) \\
\alpha & \text{if } x \in (3) - (6) \\
\beta & \text{if } x \in (2) - (6) \\
0 & \text{otherwise}
\end{cases}
\]

With \( 1 > \alpha > \beta > 0 \). Then \( A \) is not a ternary fuzzy semigroup of \( T \).
**Proposition 3.5** The intersection of any set of ternary fuzzy semigroups of $T$ is a ternary fuzzy semigroup of $T$.

**Proposition 3.7** If $A$ is ternary fuzzy semigroup of $T$. Then $\alpha-$ cut of $A$ is ternary subsemigroup of $T$ for all $\alpha \in I$

**Proof:** Let $x, y, z \in A_\alpha$

\[\Rightarrow A(x) \geq \alpha, A(y) \geq \alpha, A(z) \geq \alpha\]

\[\Rightarrow A(x \ast y \ast z) \geq A(x) \land A(y) \land A(z)\]

\[\Rightarrow A(x \ast y \ast z) \geq \alpha \land \alpha \land \alpha = \alpha\]

\[\Rightarrow x \ast y \ast z \in A_\alpha\]

Hence $A_\alpha$ is a ternary subsemigroup of $S$.

Alternatively, let $\alpha \in I$ and suppose that $\alpha-$ cut of $A \neq \emptyset$ is ternary subsemigroup of $T$. We must show that $A$ satisfies

1. $A(x \ast y \ast z) \geq A(x) \land A(y) \land A(z)$

for all $x, y, z \in T$.

If condition 1) is false then there exists $x_0, y_0, z_0 \in T$ such that

$A(x \ast y \ast z) < A(x_0) \land A(y_0) \land A(z_0)$

for all $x, y, z \in T$.

Taking $\alpha_0 = \frac{1}{2}(A(x_0 \ast y_0 \ast z_0)+A(x_0) \land A(y_0) \land A(z_0))$ we have,

$A(x \ast y \ast z) < \alpha_0 < A(x_0) \land A(y_0) \land A(z_0)$

for all $x, y, z \in T$.

It follows that $x_0, y_0, z_0 \in \alpha-$ cut of $A$ and $x_0, y_0, z_0 \notin \alpha_0-$ cut of $A$ which is contradiction. Hence condition 1) is true. The proof of remaining cases will be similar to case 1).

**Proposition 3.8** If $A$ is a ternary fuzzy subsemigroup of $T$ and supp $(A) \neq \emptyset$, then supp$(A)$ is a ternary subsemigroup of $T$

**Proof:** Let $A: T \rightarrow I$ be a ternary fuzzy subsemigroup of $T$ and supp $(A) \neq \emptyset$.

Let $x, y, z \in \text{supp (A)}$

\[\Rightarrow A(x \ast y \ast z) \geq A(x) \land A(y) \land A(z) > 0\]

\[\Rightarrow A(x \ast y \ast z) > 0\]

\[\Rightarrow x \ast y \ast z \in \text{supp (A)}\]
Proposition 3.9 A subset $H \subseteq T$ is a ternary subsemigroup of $T$ if and only if the characteristic function $\chi_H$ is a ternary fuzzy semigroup of $T$.

Proposition 3.10 The ternary fuzzy subsemigroup generated by $\chi_T$ of a set $T$ is just characteristic function of the ternary semigroup generated by $T$.

Proof: If ternary fuzzy semigroup $A$ of $T$ is contained in $\chi_T$, we have $A = 1$ for $x, y, z \in T$. But as $A$ is ternary fuzzy semigroup, we have

$$A(x * y * z) \geq A(x) \land A(y) \land A(z)$$

For any composite of elements of $T$. i.e. $A = 1$. This will imply that $A \supseteq \chi_T$. Thus $\chi(T)$ (the characteristic function generated by $T$) is a subset of the intersection of all such $A$'s. While conversely $\chi(T)$ itself is such a $A$ (by proposition 3.7).

Proposition 3.11 If $A$ is a ternary fuzzy semigroup of $T$. Then

$$A(x) = \sup \{ \alpha \in I \mid x \in A^\alpha \} \text{ for all } x \in T$$

Proof: Let $\delta(x) = \sup \{ \alpha \in I \mid x \in A^\alpha \} \text{ for all } x \in T$ and let $\varepsilon > 0$ be given, Then $\delta - \varepsilon < \alpha$ for some $\alpha \in I$, such that $x \in A^\alpha$. It follows that $\delta \leq A(x)$ since $\varepsilon$ is arbitrary.

We prove that $A(x) < \delta$. Let $A(x) = \beta$. Then $x \in A^\beta$ and hence $\beta \in \{ \alpha \in I \mid x \in A^\alpha \}$

$$A(x) \leq \beta < \sup \{ \alpha \in I \mid x \in A^\alpha \} = \delta.$$  

Thus,

$$A(x) = \delta = \sup \{ \alpha \in I \mid x \in A^\alpha \}$$

Proposition 3.12 Let $\{ A_i \}_{i \in I}$ be a family of ternary fuzzy semigroups of $T$, then $\{ A_i \}_{i \in I}$ is a complete lattice.

Proof: To prove that $\langle \{ A_i \}_{i \in I}, \cup, \cap, 0, 1 \rangle$ is a lattice under the set inclusion. Let $x, y$ be arbitrary elements of $T$, then

$$( \bigcup A_i ) (x * y * z) = \bigvee_i ( A_i (x * y * z))$$

$$\geq \bigvee_i ( A_i (x) \land A_i(y) \land A_i(z))$$

$$= ( \bigvee_i A_i(x)) \land ( \bigvee_i A_i(y)) \land ( \bigvee_i A_i(z))$$

$$= ( \bigcup A_i ) (x) \land ( \bigcup A_i ) (y) \land ( \bigcup A_i ) (z)$$
And,
\[(\bigcap_{i} A_i)(x \star y \star z) = \bigwedge_{i} (A_i(x \star y \star z))\]
\[\geq \bigwedge_{i} (A_i(x) \wedge A_i(y) \wedge A_i(z))\]
\[= \bigwedge_{i} A_i(x) \wedge \bigwedge_{i} A_i(y) \wedge \bigwedge_{i} A_i(z)\]
\[= (\bigcap A_i)(x) \wedge (\bigcap A_i)(y) \wedge (\bigcap A_i)(z)\]

Also, \(0 \in T, 0(x \star y \star z) = 0 \forall x, y, z \in T\). \(1 \in T, 1(x \star y \star z) = 1 \forall x, y, z \in T\). i.e. least and greatest elements of \(T\) are the constant function \(0\) and \(1\) respectively. Therefore, \(\{A_i\}_{i \in I}\) is complete lattice.

**Homomorphism of Ternary Fuzzy Semigroups**

**Definition 4.1** If \(A\) is a fuzzy set in a ternary semigroup \(S\), and \(f\) is a function from \(S\) to \(T\), then the fuzzy set \(B\) in \(T\) (i.e. \(f(S)\) ) defined by the membership grade
\[B(T) = \bigvee_{x \in f^{-1}(y)} A(S) \forall y \in T\]
\[= 0 \text{ if } f^{-1}(y) = \emptyset\]

\(B\) is called the image of \(A\) under \(f\) and it is denoted by \(f(A)\).

**Definition 4.2** Let \(f\) be a function from a ternary semigroup \(S\) to \(T\) and \(B\) is fuzzy subset in \(T\). Then the fuzzy set \(A\) in \(S\) defined by the membership grade
\[A(x) = B(f(x)) = (B \circ f)(x) \text{ for all } x \in S\]
is called preimage (or inverse image )of \(B\) under \(f\). It is denoted by \(f^{-1}(B)\)

**Definition 4.3** Let \(A\) be a fuzzy subset of a ternary semigroup \(S\). Then, \(A\) is said to have a sup. property if, for any subset \(B \subseteq A\), there exists \(b_0 \in B\) such that,
\[A(b) = \bigvee_{b \in B} A(b) \text{ for all } x \in S\]
\[A(b) = \bigvee_{b \in B} A(b) \text{ for all } x \in S\]

Briefly, \(A\) is said to have sup property if every subset of \(A(x)\) has a maximal element.

**For example:** If \(A\) takes only finitely many values (in particular, if it is a characteristic function), it has a sup property.

**Proposition 4.4** A homomorphism image of a ternary fuzzy semigroup having the sup. Property is a ternary fuzzy semigroup.

**Proof:** Let \(f: S \rightarrow T\) be a homomorphism of a ternary fuzzy semigroup \(S\) onto a ternary
fuzzy semigroup $T$. Let $A : S \rightarrow I$ be a ternary fuzzy semigroup of $S$. Let $f(x), f(y) \in f(S)$.

Select $x_0 \in f^{-1}(f(x)), y_0 \in f^{-1}(f(y))$ and $z_0 \in f^{-1}(f(z))$ such that,

$$A(x_0) = \vee_{y \in f^{-1}(f(x))} A(y)$$

$$A(y_0) = \vee_{y \in f^{-1}(f(y))} A(y)$$

$$A(z_0) = \vee_{y \in f^{-1}(f(z))} A(y)$$

existence of $x_0, y_0, z_0$ is by sup. property.

respectively then,

$$f(S) (f(x) * f(y) * f(z)) = \vee_{z \in f^{-1}(f(x) * f(y) * f(z))} A(z)$$

$$\geq \vee_{f(z) = f(x) * f(y) * f(z)} A(z)$$

$$\geq \vee_{z = x_0 * y_0 * z_0} A(z)$$

$$\geq A(x_0 * y_0 * z_0)$$

$$\geq A(x_0) \land A(y_0) \land A(z_0)$$

$$= \vee_{y \in f^{-1}(f(x))} A(y) \land \vee_{y \in f^{-1}(f(y))} A(y) \land \vee_{y \in f^{-1}(f(z))} A(y)$$

$$\therefore f(S)(f(x) * f(y) * f(z)) = f(S)(f(x)) \land f(S)(f(y)) \land f(S)(f(z))$$

Hence $f(S)$ is a ternary fuzzy semigroup of $T$

**Proposition 4.5** A homomorphic, preimage of a ternary fuzzy semigroup is a ternary fuzzy semigroup.

**Proof**: Let $f : S \rightarrow T$ be an onto homomorphism. Let $B : T \rightarrow I$ be a ternary fuzzy semigroup of $T$ and define a fuzzy set $A : S \rightarrow I$ by the membership grade,

$$A(x) = B(f(x)) \quad \forall x \in S$$

Consider,

$$A(x * y * z) = f(S)(f(x) * f(y) * f(z)) \quad \forall x, y, z \in S$$

$$= f(S)(f(x) * f(y) * f(z))$$

$$\geq f(S)(f(x) \land f(S)(f(y) \land f(S)(f(z)))$$

$$= A(x) \land A(y) \land A(z)$$

Thus,

$$A(x * y * z) \geq A(x) \land A(y) \land A(z)$$

Therefore $A$ is a ternary fuzzy semigroup of $S$. 
Remark 4.6 If \( f: S \to T \) is not an epimorphism then image of a ternary fuzzy semigroup of a ternary semigroup of \( S \) need not be a ternary fuzzy semigroup of \( S \).

Proposition 4.7 If \( f: S \to T \) is an epimorphism of a ternary semigroup \( S \) onto \( T \) and if \( A \) and \( B \) are ternary fuzzy semigroups of \( S \), then

1. \( f(A \cap B) \subseteq f(A) \cap f(B) \)
2. \( f(A \cup B) = f(A) \cup f(B) \)

Proof: 1. If \( f(A) \) and \( f(B) \) are ternary fuzzy semigroups of \( T \). Clearly, \( f(A) \cap f(B) \) is a fuzzy semigroup of \( T \). Let \( y \in T \) be arbitrary element. Then

\[
[f(A) \cap f(B)](y) = f(A)(y) \land f(B)(y)
\]

\[
\therefore [f(A) \cap f(B)](y) = (\bigvee_{x \in f^{-1}(y)} A(x)) \land (\bigvee_{x \in f^{-1}(y)} B(x))
\]

\[
\geq \bigvee_{x \in f^{-1}(y)} (A(x) \land B(x))
\]

\[
= \bigvee_{x \in f^{-1}(y)} (A \cap B)(x)
\]

\[
= f(A \cap B)(x)
\]

Thus, \( f(A \cap B) \subseteq f(A) \cap f(B) \).

2. Let \( y \in T \) be arbitrary element. Then

\[
[f(A) \cup f(B)](y) = f(A)(y) \lor f(B)(y)
\]

\[
= \bigvee_{x \in f^{-1}(y)} (A(x) \lor B(x))
\]

\[
= \bigvee_{x \in f^{-1}(y)} (A \cup B)(x)
\]

\[
= f(A \cup B)
\]

Thus, \( f(A \cup B) = f(A) \cup f(B) \).

Remark 4.8 The above proposition is also true for more general case, if \( \{T_i\}_{i \in I} \) is a family of ternary fuzzy sets in \( S \) (a ternary semigroup) and if \( f:S \to T \) is a function, then

\[
f(\bigcap_{i \in I} T_i) \subseteq \bigcap_{i \in I} f(T_i)
\]

(Equality holds since \( I \) is infinitely distributive)

and,

\[
f(\bigcup_{i \in I} T_i) = \bigcup_{i \in I} f(T_i)
\]

Definition 4.9 If \( A \) and \( A' \) are ternary fuzzy semigroups of \( T \), then we define,
A \ast A' : T \to L by,
\[(A \ast A')(x) = \bigvee_{y,z \in T} \{A(y) \land A'(z)\}, y, z \in T\]

**Generalization:** Let $A_i$, $i \in \mathbb{N}$ are ternary fuzzy semigroups of $T_i$. Then the product of $T_i$ ($i \in \mathbb{N}$) is the function defined by the membership grade,
\[A_1 \ast A_2 \ast A_3 \ast A_4 \ast \ldots \ast A_n : T_1 \ast T_2 \ast T_3 \ast T_4 \ast \ldots \ast T_n \to \text{I}\]

**Lemma 4.10** Let $A$ be ternary fuzzy semigroup of a finite ternary semigroup $T$. For positive integer $k$ and $x, y, z$ in $T$ we have $T(x_k) \geq T(x)$

**Proof:** We use induction on $k$, for $k > 0$. Clearly result is true for $k = 0, 1$.

If $k = 2$, then
\[T(x^2) \geq T(x \ast x) \geq \land \{T(x), T(x)\} = T(x)\]

Make the hypothesis
\[T(x^s \ast x) \geq \land \{T(x^s), T(x)\} = T(x)\]

This completes the induction.

If $k < 0$. Then, $T(x^k) = T(x^{-1})^k$, hence for $k < 0$ the result is not true, since $T$ is a ternary semigroup.

**Proposition 4.11** If $S_1$ and $S_2$ be two ternary semigroups and let $A$ and $B$ are ternary fuzzy subsets of $S_1$. Suppose $f$ is a homomorphism of $S_1$ into $S_2$. Then,

\[f(A \ast B) = f(A) \ast f(B)\]

**Proof:** For any $y \in f(S_1)$, we have
\[(f(A) \ast f(B))(y) = \bigvee_{y_1, y_2 \in S_2} \left\{ \bigvee_{x_1 \in f^{-1}(y_1)} \right. \left. \left( A(x_1) \ast B(x_2) \right) \right\}\]

\[(x_1, x_2 \in S_1; y_1, y_2 \in S_2)\]

\[(f(A) \ast f(B))(y) = \bigvee_{y_1, y_2 \in S_2} \left\{ \bigvee_{x_1 \in f^{-1}(y_1)} \right. \left. \left( A(x_1) \ast B(x_2) \right) \right\}\]

\[(x_1, x_2 \in S_1; y_1, y_2 \in S_2)\]

\[x_2 \in f^{-1}(y_2)\]

\[(x_1, x_2 \in S_1; y_1, y_2 \in S_2)\]

\[= \bigvee_{x_1, x_2 \in f^{-1}(x)} \left\{ A(x_1) \ast B(x_2) \right\} x_1, x_2 \in S_1\]

\[= \bigvee_{x \in f^{-1}(y)} \left\{ x \ast \bigvee_{x_1, x_2 \in f^{-1}(x)} \left( A(x_1) \ast B(x_2) \right) \right\} x_1, x_2 \in S_1; x \in S_1\]
\[\begin{align*}
\{ (A \ast B)(x) \} & \subseteq S_1 \\
= f(A \ast B)(y) \\
\therefore f(A \ast B) = f(A) \ast f(B)
\end{align*}\]

**Proposition 4.12** Let \( \{A_i\}_{i \in I} \) and \( \{B_j\}_{j \in I} \) be the families of ternary fuzzy semigroups of \( S \) and \( T \) respectively. If \( f: S \rightarrow T \) is a function. Then, the following assertions hold.

1. \( f(\bigcap A_i) \subseteq \bigcap f(A_i) \)
2. \( f(\bigcup A_i) = \bigcup f(A_i) \)
3. \( f^{-1}(\bigcap B_j) = \bigcap f^{-1}(B_j) \)
4. \( f^{-1}(\bigcup B_j) = \bigcup f^{-1}(B_j) \)

**Proof:**

1. Let \( y \in T \)
\[f(\bigcap A_i)(y) = \bigvee_{x \in f^{-1}(y)} (\bigcap A_i(x))\]
\[= \bigvee_{x \in f^{-1}(y)} \bigwedge_{i \in I} A_i(x)\]
\[\leq \bigwedge_{i \in I} \bigvee_{x \in f^{-1}(y)} A_i(x) \quad (\because \sup(inf) < inf(sup))\]
\[= \bigwedge_{i \in I} f(A_i)(y)\]
\[= \bigcap f(A_i)(y)\]
\[\therefore f(\bigcap A_i)(y) \subseteq (\bigcap f(A_i))(y)\]
\[\therefore f(\bigcap A_i) \subseteq \bigcap f(A_i)\]

2. Let \( y \in T \)
\[f(\bigcup A_i)(y) = \bigvee_{x \in f^{-1}(y)} (\bigcup A_i(x))\]
\[= \bigvee_{x \in f^{-1}(y) \in I} A_i(x)\]
\[= \bigvee_{i \in I} \bigwedge_{x \in f^{-1}(y)} A_i(x)\]
\[\therefore f(\bigcup A_i) = \bigcup f(A_i)\]

3. Let \( x \in S \)
Proposition 4.13 Let $f$ be a mapping from a ternary semigroup $S$ to $T$ and let $\{A\}_{i \in I}$ and $\{B_i\}_{i \in I}$ be the families of sets in $S$ and $T$ respectively. Then the following assertions hold.

1. \[ f^{-1}(f(A)) \supseteq A, \quad \forall A \in S. \]

In particular, if $f$ is injection, then

2. \[ f^{-1}(f(A)) = A, \quad \forall A \in S \]

3. \[ ff^{-1}(B) \leq B, \quad \forall B \in T. \]

In particular, if $f$ is a surjection, then

4. \[ ff^{-1}(B) = B, \quad \forall B \in T \]

Proof: 

For any given $A \in S$, we have that

\[ f^{-1}(\bigcap_{j} B_j)(x) = (\bigcap_{j} f^{-1}(B_j))(x) \]

\[ \bigwedge_{j} f^{-1}(B_j)(x) = \bigcap_{j} f^{-1}(B_j)(x) \]

\[ f^{-1}(\bigcap_{j} B_j) = \bigcap_{j} f^{-1}(B_j) \]

4. Let $x \in S$

\[ f^{-1}(\bigcup_{j} B_j)(x) = (\bigcup_{j} B_j)(f(x)) \]

\[ \bigvee_{j} B_j(f(x)) = \bigvee_{j} f^{-1}(B_j)(x) = \bigvee_{j} f^{-1}(B_j)(x) \]

\[ f^{-1}(\bigcup_{j} B_j) = \bigcup_{j} f^{-1}(B_j) \]

\[ \vdots \]

\[ f^{-1}(f(A)) \supseteq A, \quad \forall A \in S. \]
\[ f^{-1}(f(A))(x) = f(A)(f(x)) \]
\[ = \bigvee_{z \in S} A(z), z = f^{-1}(f(x)) \]
\[ = \bigvee_{f(z) = f(x)} A(z), z \in S \]
\[ \geq A(z), \forall x \in S \]
\[ \therefore f^{-1}(f(A)) \geq A, \forall A. \]

In particular, when \( f \) is injection, we have
\[ f^{-1}((f(A))(x)) = \bigvee_{f(z) = f(x)} A(z), z \in S \]
\[ = A(x) \ \forall \ x \in S. \]

This shows that assertion (1) holds.

For any given \( b \in T \), we have that
\[ f^{-1}(f(B))(y) = \bigvee_{x \in S} f^{-1}(B)(x), x = f^{-1}(y) \]
\[ = \bigvee_{y = f(x)} f^{-1}(B)(x), x \in S \]
\[ = \bigvee_{x \in f^{-1}(y)} (B(f(x)), x \in S \]
\[ = \begin{cases} B(y) & \text{for } y \in T \\ 0 & \text{otherwise} \end{cases} \]
\[ \leq B(y) \ \forall \ y \in T \]
\[ \therefore f \left( f^{-1}(B) \right)(y) = B(y), \ \forall y \in T. \]

When \( f \) is a surjection. This shows that assertion 2) holds.

References


