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An Approximate Solution to Prandtl's Equations for a Laminar Flow over a Semi- Infinite Horizontal Flat Plate

[1*]J. Venetis

School of Applied Mathematics and Physical Sciences, NTUA, Section of Mechanics, Athens, Greece.

Abstract

In this paper, a further mathematical analysis of some previous results of the author's ongoing investigation regarding Prandtl's equations for boundary layer flows is accomplished. In particular, the objective of the present work is the performance of an approximate analytical solution to Prandtl's system of equations for a laminar, isothermal, incompressible and steady boundary layer viscous flow of a Newtonian fluid, over a semi - infinite horizontal flat plate. It should be emphasized beforehand, that the overall procedure leading to this solution, in opposition with the majority of phenomenological approaches in the current literature, does not introduce either dimensional arguments or functional notations with respect to any characteristic sizes of the laminar flow (e.g. Reynolds number).

In addition, by the use of some basic concepts of Differential Calculus, a qualitative investigation of velocity profiles within the boundary layer along the flat plate has been carried out. The novelty of the present work is that the proposed explicit formula for the governing velocity of the boundary layer flow is performed in terms of purely algebraic representations. Hence, it may be much more appropriate and useful for the computational procedures which are inserted into applied Fluid Dynamics techniques and/or other engineering practices.

Keywords: boundary layer flow, flat plate, Prandtl's equations

* *Corresponding author: School of Applied Mathematics and Physical Sciences, NTUA, Section of Mechanics, Athens, Greece.*

1. INTRODUCTION

The first exact analytical solution to Prandtl's equations for an incompressible, steady and laminar flows over a horizontal planar surface was obtained from Blasius by means of a combination of differential and dimensional analysis of the flow field [1]. The main assumption of this method is that the dimensionless parameter V_x / V_∞ depends on the horizontal transposition. Here, let us remark that V_∞ denotes free stream velocity. Also, the same fundamental assumption was adopted by Falkner – Skan approach of similar boundary layer flows [2,3]. In the meanwhile, in opposition with Blasius solution which is exact, Karman and Pohlhausen method towards an approximate solution to Prandtl's Equations, assumes an approximate shape of the velocity profile [4]. This method, which is also known as Momentum Integral Method, changes the two Prandtl's partial differential equations into a single differential equation which is applicable only when the pressure gradient term is zero. Besides, a remarkable progress was made, concerning the existence and the properties of analytical solutions to unsteady Prandtl's equations [5,6,7]. On the other hand, an alternative exact closed – from solution to these equations was proposed by Venetis et al. without dimensional arguments and functional notations with respect to the main characteristic sizes of the flow, (e.g. Re number) [8]. Further, in Ref. [9] the obtaining of an approximate explicit solution to Prandtl's equations for isothermal, incompressible and steady boundary layer flows of Newtonian fluids, in a coordinate free context was performed, whilst for a detailed study on the stability of Prandtl's boundary layers one may refer to a valuable work by Grenier et al. [10]. In Ref. [11] rigorous closed form solutions to unsteady free convection flow of a second srade fluid over an oscillating vertical plate were derived, whereas a thorough investigation on unsteady convection flows over a moving vertical flat plate was carried out in Ref. [12]. A similar remarkable study on unsteady natural convection flows past a vertical infinite flat plate was performed in Ref. [13], while in Ref. [14] an important computational treatment to Prandtl–Blasius laminar viscous flow over a semi-infinite flat plate can be found.

In Ref. [15], a theoretical investigation of a steady boundary layer flow near the inflection point of a smooth mathematical curve was carried out, whereas a prominent study on magneto-hydro-dynamic (MHD) convection flows over a stretched vertical flat plate was performed in Ref. [16]. In Ref. [17] a useful computational approach to 2D laminar forced convective viscous flows over a flat plate was carried out, whilst an analytical solution of forced-convective boundary-layer flow over a flat plate was proposed in Ref. [18]. In Ref. [19] an implicit finite difference solution of a boundary layer heat flow past a flat plate was derived, whilst for a detailed study on the role of pressure gradient on laminar boundary layer over a permeable surface with convective boundary conditions, one may refer to Ref. [20]. The intention of the present work is the obtaining of an approximate analytical solution to Prandtl's system of equations for laminar, isothermal, incompressible and steady boundary layer flows of Newtonian fluids over a horizontal semi - infinite flat plate. To this end, a closed – form solution to these fundamental equations was derived on the basis of the results appearing in Ref. [8]. Here, we have to emphasize that some simplifications were primarily taken into account in order to obtain this explicit expression. Apparently, these simplifications arise from Prandtl's fundamental assumptions for boundary layer flows.

2. TOWARDS AN APPROXIMATE SOLUTION TO PRANDTL'S EQUATIONS

Let us write out Prandtl's boundary layer equations, for a motionless Cartesian frame of reference which is positioned at the origin of a semi - infinite horizontal flat plate:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} = 0 \tag{1}$$

$$V_x \frac{\partial V_x}{\partial x} + V_y \frac{\partial V_x}{\partial y} = -\frac{1}{\rho} \cdot \frac{dP}{dx} + \frac{\mu}{\rho} \cdot \frac{\partial^2 V_x}{\partial y^2} \tag{2}$$

Evidently the term P denotes the algebraic summation of barometric and manometric pressure and indeed it is assumed to depend only on the variable x [1].

In addition, for isothermal incompressible boundary layer flows of a Newtonian fluid, the well known from the literature Prandtl's simplified assumptions hold [1]:

$$V_y << V_x \tag{3}$$

$$\frac{\partial V_x}{\partial x} << \frac{\partial V_x}{\partial y} \tag{4}$$

$$\frac{\partial^2 V_x}{\partial x^2} << \frac{\partial^2 V_x}{\partial y^2} \tag{5}$$

$$P = P(x) \tag{6}$$

Moreover, the pressure $P(x)$ can be estimated by the following relationship [1]

$$\frac{dP(x)}{dx} = \rho \cdot V_\infty \cdot \frac{dV_\infty}{dx} \tag{7}$$

where V_∞ denotes the free stream velocity, which is supposed to be obtained from a potential flow field.

In this context, eqn. (7) informs us that the gradient of pressure can be specified by means of a frictionless flow field.

This concept implies that all sequential velocity distributions are similar in shape to one another at different axial locations, having changes only as concerns the quantity scale across boundary layer thickness. In this framework, the geometry of flow field inside boundary layer can be studied without violating the nature of the problem, by means of sequential velocity profiles, which correspond injectively to the successive rates of the longitudinal displacement measured from the origin of the semi – infinite flat plate where the rectangular Cartesian frame of reference was positioned. In Ref. [8], it was proved that the governing velocity component along such a boundary layer, can be represented as

$$V_x(x,y) = \frac{h(x)}{B} \cdot \frac{y^2}{2} + \frac{\phi(x) \cdot y}{B} + \frac{f(x,y)}{B} + \frac{C \cdot y}{B} - \frac{f(x,0)}{B} \tag{8}$$

where the constant B is given as

$$B = \frac{\mu}{\rho} \tag{9}$$

The notation μ denotes the dynamic viscosity of the Newtonian fluid whereas the notation ρ denotes its density.

Moreover, according to non – slip boundary condition the following expression for the governing velocity component holds

$$V_x(x,0) = 0 \tag{10}$$

In Ref. [8] it was shown that the introduced expression for $V_x(x, y)$ satisfies eqn. (10).

In addition, in Ref. [8] it was also proved that the continuous function f appearing in eqn. (8) satisfies the following relationship

$$\lim_{y \to +\infty} \frac{f(x, y)}{y^2} = \text{a} \tag{11}$$

Moreover, the quantities h and ϕ are arbitrary continuous single valued functions and a is an arbitrary real constant.

Besides, let us remark that since $V_y(x, y)$ is not the governing velocity component, its variations do not affect the boundary layer flow in a significant manner.

Concurrently, according to Prandtl's simplified assumptions it implies that $V_y << V_x$.

In this framework, by considering the displacement x at the longitudinal direction of the semi – infinite horizontal flat plate as a parameter, eqn. (11) can be modified as:

$$\lim_{y \to +\infty} \frac{f(y)}{y^2} = \text{a} \tag{12a}$$

The above relationship can be equivalently written in the form of a functional equation

$$\lim_{y \to +\infty} \left| \left(\frac{f(y) - f(y-1)}{2y-1} - \text{a} \right) \right| = 0 \tag{12b}$$

Further, an application of L'Hôpital's rule to eqn. (12a) yields

$$\lim_{y \to +\infty} \frac{f'(y)}{y} = 2\text{a} \tag{12c}$$

The above relationship can be equivalently written in the form of a functional equation

$$\lim_{y \to +\infty} \left| f'(y) - f'(y-1) \right| = 2\text{a} \tag{12d}$$

Hence, in the same context, eqn. (8) can be simplified to yield

$$V_x(y) = C_1 \cdot \frac{y^2}{2} + C_2 y + \frac{f(y)}{B} + C_3 \tag{13}$$

Now, in order to accommodate our mathematical analysis, let us set

$$\frac{f(y)}{B} + C_3 = \sigma(y) \tag{14}$$

Evidently, the continuous function $\sigma(y)$ also satisfy both eqns. (12a,b)

Next, by differentiating eqn. (13) with respect to the independent variable y one finds

$$\frac{dV_x(y)}{dy} = C_1 y + C_2 + \frac{d\sigma(y)}{dy} \tag{15a}$$

Then, the following inequalities [21] signify whether the natural logarithm of $V_x(y)$ is a convex (or a concave) function respectively,

$$\left(\frac{dV_x(y)}{dy} \right)^2 \leq V_x(y) \cdot \frac{d^2 V_x(y)}{dy^2} \tag{15b}$$

or

$$\left(\frac{dV_x(y)}{dy} \right)^2 \geq V_x(y) \cdot \frac{d^2 V_x(y)}{dy^2} \tag{15c}$$

In the meanwhile, it is also known from Differential Calculus [21] that a single - valued continuous function $g:[0,+\infty) \to R$ such that $g(0) = 0$ is convex over the domain $(0,+\infty)$ if and only if the function $\frac{g(x)}{x}$ is strictly increasing over the domain $(0,+\infty)$.

Thus, we can write out:

$$\frac{V_x(y)}{y} = C_1 \cdot \frac{y}{2} + C_2 + \frac{\sigma(y)}{y} \tag{16}$$

and therefore

$$\frac{dV_x(y)}{dy} = \frac{C_1}{2B} + \frac{1}{B}\,\frac{\sigma'(y)\,y - \sigma(y)}{y^2} \tag{17}$$

Hence, in order to investigate in an easier manner where the governing velocity component $V_x(y)$ of the laminar boundary layer flow, is a convex (or a concave) function respectively, it is enough to solve with respect to y the following two inequalities:

$$\frac{C_1}{2B} + \frac{1}{B}\,\frac{\sigma'(y)\,y - \sigma(y)}{y^2} > 0 \tag{18a}$$

or

$$\frac{C_1}{2B} + \frac{1}{B}\,\frac{\sigma'(y)\,y - \sigma(y)}{y^2} < 0 \tag{18b}$$

In this framework, inequality (18a) yields

$$\sigma(y) - \sigma'(y)\,y < y^2\,\frac{C_1}{2} \Leftrightarrow$$

$$\frac{\sigma(y)}{y} - \sigma'(y) < y\,\frac{C_1}{2} \tag{19 a}$$

In addition, inequality (18b) yields

$$\sigma(y) - \sigma'(y)\,y > y^2\,\frac{C_1}{2} \Leftrightarrow$$

$$\frac{\sigma(y)}{y} - \sigma'(y) > y\,\frac{C_1}{2} \tag{19 b}$$

Here, we have to emphasize that all velocity distributions along the semi - infinite flat plate are similar in shape to one another at different axial locations, having changes only as concerns the quantity scale across boundary layer thickness.

Then, let us return to eqn. (13) and in order to facilitate our mathematical analysis, let us approach the single valued function continuous f by the following algebraic representation

$$f(y) = k_1 \ln(y+1) + k_2 + \mathrm{a}y^2 + by + d \tag{20}$$

where k_1, k_2, a, b and d are strictly positive real constants

Thus it implies that

$$\frac{f(y)}{y^2} = \mathrm{a} + \frac{b}{y} + \frac{d}{y^2} + \frac{k_1 \ln(y+1) + k_2}{y^2} \tag{21a}$$

Evidently eqn. (12a) is automatically satisfied. Moreover, the validity of the functional equation (12b) can be easily verified as follows

$$\lim_{y \to +\infty} \left| \left(\frac{\mathrm{a}y^2 + by + k_1 \ln\left(\frac{y+1}{y}\right) - \mathrm{a}(y-1)^2 - b(y-1)}{2y-1} - \mathrm{a} \right) \right| = 0 \Leftrightarrow$$

$$\lim_{y \to +\infty} \left| \left(\frac{-\mathrm{a} + 2\mathrm{a}y + b}{2y-1} - \mathrm{a} \right) \right| = 0 \Leftrightarrow$$

$$\lim_{y \to +\infty} \left| \left(\frac{\mathrm{a}(2y-1) + b}{2y-1} - \mathrm{a} \right) \right| = 0 \Leftrightarrow$$

$$\lim_{y \to +\infty} \left| \left(\mathrm{a} + \frac{b}{2y-1} - \mathrm{a} \right) \right| = 0 \Leftrightarrow$$

$$\lim_{y \to +\infty} \left| \left(\frac{b}{2y-1} \right) \right| = 0 \tag{21b}$$

Here, we took into account that the limit of the quantity $\ln\left(\frac{y+1}{y}\right)$ letting y tend to $+\infty$ vanishes.

In this context, eqn. (13) can be combined with (20) to yield:

$$V_x(y) = k_1 \ln(y+1) + k_2 + Ay^2 + Ky + \frac{\mathrm{a}y^2 + by + d}{B} + \frac{C \cdot y}{B} - \frac{d}{B} \Leftrightarrow$$

$$V_x(y) = k_1 \ln(y+1) + k_2 + \left(A + \frac{\mathrm{a}}{B} \right) y^2 + \left(K + \frac{b}{B} + C \right) y \tag{22a}$$

Concurrently, the first and second derivative of the above function with respect to y are estimated as

$$\frac{dV_x(y)}{dy} = \frac{k_1}{y+1} + 2\left(A + \frac{\text{a}}{B}\right)y + K + \frac{b}{B} + C \tag{22b}$$

$$\frac{d^2V_x(y)}{dy^2} = 2\left(A + \frac{\text{a}}{B}\right) - \frac{k_1}{(y+1)^2} \tag{22c}$$

Here one may observe that eqn. (22c) is more convenient to specify the inflection points of $V_x(y)$ when compared with inequalities (15a,b) and (19a,b) respectively.

Evidently, as we have already emphasized all velocity destributions within the laminar boundary layer along the semi - infinite flat plate expressed by eqn. (22) are similar in shape to one an other at different axial locations.

On the other hand, it is known from single valued Calculus [22] that given an nth degree polynomial Q with real constants and real roots as well, the following inequality holds

$$(n-1)\cdot\left(\frac{dQ(x)}{dx}\right)^2 \geq n\cdot Q(x)\cdot\left(\frac{d^2Q(x)}{dx^2}\right) \tag{23}$$

and therefore

$$Q(x) \leq \frac{n-1}{n}\cdot\left(\frac{dQ(x)}{dx}\right)^2\left(\frac{d^2Q(x)}{dx^2}\right)^{-1} \tag{24}$$

or equivalently

$$\max Q(x) = \frac{n-1}{n}\cdot\left(\frac{dQ(x)}{dx}\right)^2\left(\frac{d^2Q(x)}{dx^2}\right)^{-1} \tag{25}$$

Here, one may pinpoint that the summation $\left(A + \frac{\text{a}}{B}\right)y^2 + \left(K + \frac{b}{B} + C\right)y$ in the right hand side of eqn. (22a) constitutes a quadratic polynomial i.e. a polynomial of degree 2. Also, it is evident that this polynomial has two real roots.

Thus we can write out,

$$\frac{dQ(x)}{dx} = 2\left(A + \frac{\text{a}}{B}\right)\cdot y + \left(K + \frac{b}{B} + C\right) \tag{26}$$

and

$$\frac{d^2Q(x)}{dx^2} = 2\left(A + \frac{\text{a}}{B}\right) \tag{27}$$

In this context, eqn. (25) can be combined with (22a) to yield an upper bound of the governing velocity component $V_x(y)$:

$$V_x \le k_1 \ln(y+1) + k_2 + \frac{4\left(\left(A+\frac{\mathrm{a}}{B}\right)y + \left(K+\frac{b}{B}+C\right)\right)^2}{4\left(A+\frac{\mathrm{a}}{B}\right)} \Leftrightarrow$$

$$V_x \le k_1 \ln(y+1) + k_2 + \frac{\left(\left(A+\frac{\mathrm{a}}{B}\right)y + \left(K+\frac{b}{B}+C\right)\right)^2}{A+\frac{\mathrm{a}}{B}} \Leftrightarrow$$

$$V_x \le k_1 \ln(y+1) + k_2 + \left(A+\frac{\mathrm{a}}{B}\right)y^2 + 2y\left(K+\frac{b}{B}+C\right) + \frac{\left(K+\frac{b}{B}+C\right)^2}{A+\frac{\mathrm{a}}{B}} \tag{28}$$

In addition, since y is a positive real variable, it is known from Calculus [22] that the following inequality holds

$$\ln(y+1) \le \frac{y}{\sqrt{1+y}} \tag{29}$$

and therefore,

$$\max(k_1 \ln(y+1) + k_2) = \frac{k_1 y}{\sqrt{1+y}} + k_2 \tag{30}$$

Thus one infers,

$$\max V_x = \frac{k_1 y}{\sqrt{1+y}} + k_2 + \left(A+\frac{\mathrm{a}}{B}\right)y^2 + 2y\left(K+\frac{b}{B}+C\right) + \frac{\left(K+\frac{b}{B}+C\right)^2}{A+\frac{\mathrm{a}}{B}} \tag{31}$$

After all, eqn. (31) can be combined with eqn. (9) to yield

$$\max V_x = \frac{k_1 y}{\sqrt{1+y}} + k_2 + \left(A+\frac{\mathrm{a}\rho}{\mu}\right)y^2 + 2y\left(K+\frac{\mathrm{b}\rho}{\mu}+C\right) + \frac{\left(K+\frac{\mathrm{b}\rho}{\mu}+C\right)^2}{A+\frac{\mathrm{a}\rho}{\mu}} \tag{32}$$

3. DISCUSSION

The objective of the present investigation was the obtaining of an approximate explicit solution to Prandtl's equations for a general category of laminar isothermal, incompressible and steady boundary layer flows of Newtonian fluids over a planar horizontal semi- infinite flat plate. To this end, a closed – form solution to these significant system of equations was derived on the basis of the mathematical analysis performed in Ref. [8]. However, we have to emphasize that some simplifications were primarily taken into account in order to derive this analytical representation. These fundamental simplifications have been motivated by Prandtl's assumptions for laminar boundary layer flows. Therefore, to implement this closed – form expression to a realistic problem of a boundary layer flow one must always be ensured whether these simlifications hold. In this context, one has definitely to verify experimentally this concept before taking into consideration this relationship. Moreover, it should be clarified beforehand that the well known non – slip condition, which was taken into account to derive the final expression, cannot be explained and justified only by the existence of the cohesion. Thus, it is necessary to invoke the molecular, i.e. the discontinuous structure of the matter for the solid bounds as well as for the fluid. Indeed, the macroscopically appearing as smooth surface of a solid rigid body has cavitational irregularities in size order larger than that of the molecule. In this context, the molecules of the fluid which during their uncertain molecular motion strike against this surface loosing their macroscopic velocity, i.e. flow velocity. Now, according to the moleculokinetic interpretation of the cohesion, the molecules which escape from the irregularities of the solid wall and reenter in the flow, strike against the molecules that pass with the flow velocity and they accelerate with simultaneous deceleration of those which move beyond the wall. Hence, the velocity from zero on the solid wall increases away from it until settles the value that would be imposed by the solution of the corresponding potential flow. Moreover, a shortcoming of eqn. (12) is that although the limit of the quotient $\frac{f(y)}{y^2}$ letting y tend to infinity converges to a real number, the sign of this number remains unknown.

4. CONCLUSIONS

In this article, a further mathematical analysis of some previous results of the author's ongoing investigation regarding Prandtl's equations for a laminar boundary layer flow was performed. Specifically, this work aimed at presenting in a rigorous manner an approximate explicit solution to Prandtl's system of equations for laminar isothermal, incompressible and steady boundary layer flows of Newtonian fluids over a horizontal and semi - infinite flat plate. Nonetheless, we have to emphasize that some simplifications were primarily taken into account in order to obtain this analytical representation. These fundamental simplifications are motivated by Prandtl's assumptions for boundary layer flows. Hence, in order to adopt this closed – form expression for a boundary layer laminar flow, one should always substantiate whether these simplified assumptions hold. In this context, one has definitely to verify experimentally this concept before taking into consideration the proposed relationship.

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