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Lyapunov-Type Inequalities for Discrete Riemann–Liouville Fractional Boundary Value Problems

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Abstract

In this article, we obtain a few Lyapunov-type inequalities for two-point discrete fractional boundary value problems involving Riemann–Liouville type backward differences. To illustrate the applicability of established results, we deduce criteria for the nonexistence of nontrivial solutions and estimate lower bounds for eigenvalues of the corresponding eigenvalue problems. We also apply these inequalities to deduce criteria for the nonexistence of real zeros of certain discrete Mittag–Leffler functions.

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1 Introduction

In 1907, Lyapunov [25] provided a necessary condition, known as the Lyapunov inequality, for the existence of a nontrivial solution of Hill's equation associated with Dirichlet boundary conditions. Due to its importance, this inequality has been generalized in many forms. For a detailed introduction on the Lyapunov inequality and its applications, we refer [5, 29, 30, 34, 36, 37] and the references therein.

In this line, Cheng [6] developed a discrete analogue of the classical Lyapunov inequality:

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Theorem 1.1 (See [6]). *If the boundary value problem*

$$\begin{cases} \big(\Delta^2 u\big)(t-1) + q(t)y(t) = 0, & t \in \mathbb{N}_a^b, \\ u(a-1) = 0, \quad u(b+1) = 0, \end{cases}$$

has a nontrivial solution, where q is a nonnegative function defined on $\mathbb{N}_a^b$, then

$$\sum_{s=a}^{b} q(s) \geq \begin{cases} \dfrac{4(b-a+2)}{(b-a+1)(b-a+3)}, & \text{if } (b-a) \text{ is odd}, \\ \dfrac{4}{(b-a+2)}, & \text{if } (b-a) \text{ is even}. \end{cases}$$

Following Cheng's work, several authors have established Lyapunov-type inequalities for various classes of discrete boundary value problems [7, 10, 18, 19, 26, 35, 38]. On the other hand, Ferreira [14] generalized Theorem 1.1 for α^{th}-order $(1 < \alpha \leq 2)$ Riemann–Liouville type forward differences.

Theorem 1.2 (See [14]). *If the fractional boundary value problem*

$$\begin{cases} \big(\Delta^\alpha u\big)(t) + q(t+\alpha-1)u(t+\alpha-1) = 0, & t \in \mathbb{N}_0^{b+1}, \quad b \geq 2, \\ y(\alpha-2) = 0, \quad y(\alpha+b+1) = 0, \end{cases}$$

has a nontrivial solution, where q is a nonnegative function defined on $\{(\alpha-1), (\alpha-1)+(\alpha-1), (\alpha-1)+2(\alpha-1), \ldots, \alpha+b\}$, then

$$\sum_{s=0}^{b+1} q(s+\alpha-1) > \begin{cases} 4\Gamma(\alpha)\dfrac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b}{2}+2)}{(b+2\alpha)(b+2)\Gamma^2(\frac{b}{2}+\alpha)\Gamma(b+3)}, & \text{if } b \text{ is even}, \\ \Gamma(\alpha)\dfrac{\Gamma(b+\alpha+2)\Gamma^2(\frac{b+3}{2})}{\Gamma(b+3)\Gamma^2(\frac{b+1}{2}+\alpha)}, & \text{if } b \text{ is odd}. \end{cases}$$

Following Ferreira's work, authors of [9, 16] have established Lyapunov-type inequalities for various classes of delta fractional boundary value problems. Motivated by these developments, in this article, we derive Lyapunov-type inequalities for two-point nabla fractional boundary value problems of Riemann–Liouville type.

2 Preliminaries

Denote the set of all real numbers by $\mathbb{R}$. Define

$$\mathbb{N}_a := \{a, a+1, a+2, \ldots\} \text{ and } \mathbb{N}_a^b := \{a, a+1, a+2, \ldots, b\}$$

for any $a, b \in \mathbb{R}$ such that $b - a \in \mathbb{N}_1$. Assume that empty sums and products are taken to be 0 and 1, respectively.

Definition 2.1 (See [3])**.** The backward jump operator $\rho : \mathbb{N}_a \to \mathbb{N}_a$ is defined by

$$\rho(t) = \max\{a, (t-1)\}, \quad t \in \mathbb{N}_a.$$

Definition 2.2 (See [24, 31])**.** The Euler gamma function is defined by

$$\Gamma(z) := \int_0^\infty t^{z-1} e^{-t} dt, \quad \Re(z) > 0.$$

Using the reduction formula

$$\Gamma(z+1) = z\Gamma(z), \quad \Re(z) > 0,$$

the Euler gamma function can be extended to the half-plane $\Re(z) \leq 0$ except for $z \neq 0, -1, -2, \ldots$

Definition 2.3 (See [15])**.** For $t \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$ and $r \in \mathbb{R}$ such that $(t+r) \in \mathbb{R} \setminus \{\ldots, -2, -1, 0\}$, the generalized rising function is defined by

$$t^{\overline{r}} := \frac{\Gamma(t+r)}{\Gamma(t)}, \quad 0^{\overline{r}} := 0.$$

Definition 2.4 (See [3])**.** Let $u : \mathbb{N}_a \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The first order backward (nabla) difference of u is defined by

$$\big(\nabla u\big)(t) := u(t) - u(t-1), \quad t \in \mathbb{N}_{a+1},$$

and the N^{th}-order nabla difference of u is defined recursively by

$$\big(\nabla^N u\big)(t) := \Big(\nabla\big(\nabla^{N-1} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Definition 2.5 (See [15])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $N \in \mathbb{N}_1$. The N^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-N} u\big)(t) := \frac{1}{(N-1)!} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{N-1}} u(s), \quad t \in \mathbb{N}_{a+1}.$$

We define

$$\big(\nabla_a^{-0} u\big)(t) = u(t), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.6 (See [15])**.** Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$ and $\nu > 0$. The ν^{th}-order nabla sum of u based at a is given by

$$\big(\nabla_a^{-\nu} u\big)(t) := \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\nu-1}} u(s), \quad t \in \mathbb{N}_{a+1}.$$

Definition 2.7 (See [15]). Let $u : \mathbb{N}_{a+1} \to \mathbb{R}$, $\nu > 0$ and choose $N \in \mathbb{N}_1$ such that $N-1 < \nu \leq N$. The Riemann–Liouville type ν^{th}-order nabla difference of u is given by

$$\big(\nabla_a^{\nu} u\big)(t) := \Big(\nabla^N \big(\nabla_a^{-(N-\nu)} u\big)\Big)(t), \quad t \in \mathbb{N}_{a+N}.$$

Theorem 2.8 (See [15]). *Let ν, $\mu > 0$ and $u : \mathbb{N}_{a+1} \to \mathbb{R}$. Then,*

$$\Big(\nabla_a^{\nu}\big(\nabla_a^{-\mu} u\big)\Big)(t) = \big(\nabla_a^{\nu-\mu} u\big)(t), \quad t \in \mathbb{N}_{a+1}.$$

Theorem 2.9 (See [21]). *We observe the following properties of gamma and generalized rising functions.*

1. $\Gamma(t) > 0$ *for all* $t > 0$.
2. $t^{\overline{\alpha}}(t+\alpha)^{\overline{\beta}} = t^{\overline{\alpha+\beta}}$.
3. *If* $t \leq r$, *then* $t^{\overline{\alpha}} \leq r^{\overline{\alpha}}$.
4. *If* $\alpha < t \leq r$, *then* $r^{\overline{-\alpha}} \leq t^{\overline{-\alpha}}$.

Theorem 2.10 (See [1]). *Let $\nu \in \mathbb{R}^+$ and $\mu \in \mathbb{R}$ such that μ, $\mu+\nu$ and $\mu-\nu$ are nonnegative integers. Then,*

$$\nabla_a^{-\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\nu+1)}(t-a)^{\overline{\mu+\nu}}, \quad t \in \mathbb{N}_a,$$
$$\nabla_a^{\nu}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\nu+1)}(t-a)^{\overline{\mu-\nu}}, \quad t \in \mathbb{N}_a.$$

Definition 2.11 (See [15]). For $|p| < 1$, $\alpha > 0$ and $\beta \in \mathbb{R}$, we define the nabla Mittag–Leffler function by

$$E_{p,\alpha,\beta}(t,a) = \sum_{k=0}^{\infty} p^k \frac{(t-a)^{\overline{\alpha k+\beta}}}{\Gamma(\alpha k+\beta+1)}, \quad t \in \mathbb{N}_a.$$

Clearly, we have that $E_{0,\alpha,0}(t,a) = 1$ and $E_{p,\alpha,0}(a,a) = 1$.

Theorem 2.12 (See [15]). *Assume $|p| < 1$, $\alpha > 0$ and $\beta \in \mathbb{R}$. Then,*

$$\nabla_{\rho(a)}^{\nu} E_{p,\alpha,\beta}(t,\rho(a)) = E_{p,\alpha,\beta-\nu}(t,\rho(a)), \quad t \in \mathbb{N}_a.$$

Theorem 2.13 (See [15]). *Assume $\nu > 0$ and $N-1 < \nu \leq N$. Then, a general solution of*

$$\big(\nabla_a^{\nu} u\big)(t) = 0, \quad t \in \mathbb{N}_{a+N},$$

is given by

$$u(t) = C_1(t-a)^{\overline{\nu-1}} + C_2(t-a)^{\overline{\nu-2}} + \ldots + C_N(t-a)^{\overline{\nu-N}}, \quad t \in \mathbb{N}_a,$$

where C_1, C_2, $\cdots$, $C_N \in \mathbb{R}$.

Theorem 2.14 (See [15]). *Assume $\nu > 0$, $N-1 < \nu \leq N$ and $|c| < 1$. Then, a general solution of*

$$\big(\nabla^{\nu}_{\rho(a)}u\big)(t) + cu(t) = 0, \quad t \in \mathbb{N}_{a+N},$$

is given by

$$u(t) = C_1 E_{-c,\nu,\nu-1}(t,\rho(a)) + C_2 E_{-c,\nu,\nu-2}(t,\rho(a)) + \ldots + C_N E_{-c,\nu,\nu-N}(t,\rho(a)),$$

for $t \in \mathbb{N}_a$, where C_1, C_2, $\cdots$, $C_N \in \mathbb{R}$.

3 Left-Focal Type Boundary Value Problem

In this section, we derive a few important properties of the Green's function associated with a left-focal type discrete fractional boundary value problem and obtain the corresponding Lyapunov-type inequality.

Theorem 3.1. *Let $1 < \alpha < 2$ and $h : \mathbb{N}^b_{a+2} \to \mathbb{R}$. The discrete fractional boundary value problem*

$$\begin{cases} \big(\nabla^{\alpha}_{a}u\big)(t) + h(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ \big(\nabla^{\alpha-1}_{a}u\big)(a+1) = 0,\ u(b) = 0, \end{cases} \tag{3.1}$$

has the unique solution

$$u(t) = \sum_{s=a+2}^{b} G_l(t,s)h(s), \quad t \in \mathbb{N}^b_{a+1}, \tag{3.2}$$

where

$$G_l(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \dfrac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}}, & t \in \mathbb{N}^{\rho(s)}_{a+1}, \\ \dfrac{1}{\Gamma(\alpha)} \Big[\dfrac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}} - (t-s+1)^{\overline{\alpha-1}}\Big], & t \in \mathbb{N}^b_s. \end{cases} \tag{3.3}$$

Proof. Applying $\nabla^{-\alpha}_{a}$ on both sides of (3.1) and using Theorem 2.13, we have

$$u(t) = -\big(\nabla^{-\alpha}_{a}h\big)(t) + C_1(t-a)^{\overline{\alpha-1}} + C_2(t-a)^{\overline{\alpha-2}}, \quad t \in \mathbb{N}_{a+1}, \tag{3.4}$$

for some C_1, $C_2 \in \mathbb{R}$. Applying $\nabla^{\alpha-1}_{a}$ on both sides of (3.4) and using Theorems 2.8 and 2.10, we have

$$\big(\nabla^{\alpha-1}_{a}u\big)(t) = -\big(\nabla^{-1}_{a}h\big)(t) + C_1\Gamma(\alpha), \quad t \in \mathbb{N}_{a+1}. \tag{3.5}$$

Using $\left(\nabla_a^{\alpha-1}u\right)(a+1)=0$ in (3.5), we get

$$C_1=\frac{h(a+1)}{\Gamma(\alpha)}.$$

Using $u(b)=0$ in (3.4), we get

$$C_2=\frac{1}{(b-a)^{\overline{\alpha-2}}\Gamma(\alpha)}\sum_{s=a+2}^{b}(b-s+1)^{\overline{\alpha-1}}h(s).$$

Substituting the values of C_1 and C_2 in (3.4), we have

$$\begin{aligned}
u(t) &= -\frac{1}{\Gamma(\alpha)}\sum_{s=a+1}^{t}(t-s+1)^{\overline{\alpha-1}}h(s)+\frac{h(a+1)}{\Gamma(\alpha)}(t-a)^{\overline{\alpha-1}}\\
&\quad+\frac{(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}\Gamma(\alpha)}\sum_{s=a+2}^{b}(b-s+1)^{\overline{\alpha-1}}h(s)\\
&= -\frac{1}{\Gamma(\alpha)}\sum_{s=a+2}^{t}(t-s+1)^{\overline{\alpha-1}}h(s)+\frac{(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}\Gamma(\alpha)}\sum_{s=a+2}^{b}(b-s+1)^{\overline{\alpha-1}}h(s)\\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=a+2}^{t}\Big[\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}}-(t-s+1)^{\overline{\alpha-1}}\Big]h(s)\\
&\quad+\frac{1}{\Gamma(\alpha)}\sum_{s=t+1}^{b}\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}}h(s)\\
&= \sum_{s=a+2}^{b}G_l(t,s)h(s).
\end{aligned}$$

This concludes the proof. □

First, we show that this Green's function is nonnegative and obtain an upper bound for the Green's function and its integral.

Theorem 3.2. *The Green's function $G_l(t,s)$ satisfies $G_l(t,s)\ge 0$ for $(t,s)\in\mathbb{N}_{a+1}^{b}\times\mathbb{N}_{a+2}^{b}$.*

Proof. Clearly,

$$G_l(b,s)=0,\quad s\in\mathbb{N}_{a+2}^{b}.$$

Assume $(t,s)\in\mathbb{N}_{a+1}^{b-1}\times\mathbb{N}_{a+2}^{b}$. For $t\in\mathbb{N}_{a+1}^{\rho(s)}$, consider

$$G_l(t,s)=\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-2}}}=\frac{\Gamma(b-s+\alpha)\Gamma(t-a+\alpha-2)\Gamma(b-a)}{\Gamma(\alpha)\Gamma(b-s+1)\Gamma(b-a+\alpha-2)\Gamma(t-a)}.$$

It follows from (1) of Theorem 2.9 that $G_l(t,s) > 0$. Now, suppose $t \in \mathbb{N}_s^{b-1}$. Since $t < b$ and $(2-\alpha) < (t-a) < (b-a)$, by Theorem 2.9, we have

$$(t-s+1)^{\overline{\alpha-1}} < (b-s+1)^{\overline{\alpha-1}} \text{ and } (b-a)^{\overline{\alpha-2}} < (t-a)^{\overline{\alpha-2}},$$

implying that

$$\begin{aligned} G_l(t,s) &= \frac{1}{\Gamma(\alpha)}\Big[\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}} - (t-s+1)^{\overline{\alpha-1}}\Big] \\ &> \frac{1}{\Gamma(\alpha)}\Big[(b-s+1)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big] > 0. \end{aligned}$$

This concludes the proof. $\square$

Theorem 3.3. *The maximum of the Green's function $G_l(t,s)$ defined in* (3.3) *is given by*

$$\max_{(t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b} G_l(t,s) = \frac{(b-a-1)}{(\alpha-1)}.$$

Proof. Fix $s \in \mathbb{N}_{a+2}^b$. Let $t \in \mathbb{N}_{a+2}^{\rho(s)}$. Consider

$$\begin{aligned} \nabla_t\big[G_l(t,s)\big] &= \frac{1}{\Gamma(\alpha)}\frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-2}}}\nabla_t\big[(t-a)^{\overline{\alpha-2}}\big] \\ &= -\frac{(2-\alpha)}{\Gamma(\alpha)}\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-3}}}{(b-a)^{\overline{\alpha-2}}} \\ &= -\frac{(2-\alpha)}{\Gamma(\alpha)}\frac{\Gamma(b-s+\alpha)\Gamma(t-a+\alpha-3)\Gamma(b-a)}{\Gamma(b-s+1)\Gamma(b-a+\alpha-2)\Gamma(t-a)}. \end{aligned}$$

It follows from (1) of Theorem 2.9 that $\nabla_t\big[G_l(t,s)\big] < 0$ implying that $G_l(t,s)$ is a decreasing function of t. Now, suppose $t \in \mathbb{N}_s^b$. Consider

$$\begin{aligned} \nabla_t\big[G_l(t,s)\big] &= \frac{1}{\Gamma(\alpha)}\Big[\frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-2}}}\nabla_t\big[(t-a)^{\overline{\alpha-2}}\big] - \nabla_t\big[(t-s+1)^{\overline{\alpha-1}}\big]\Big] \\ &= -\frac{(2-\alpha)}{\Gamma(\alpha)}\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-3}}}{(b-a)^{\overline{\alpha-2}}} - \frac{(\alpha-1)}{\Gamma(\alpha)}(t-s+1)^{\overline{\alpha-2}} \\ &= -\frac{(2-\alpha)}{\Gamma(\alpha)}\frac{\Gamma(b-s+\alpha)\Gamma(t-a+\alpha-3)\Gamma(b-a)}{\Gamma(b-s+1)\Gamma(b-a+\alpha-2)\Gamma(t-a)} \\ &\quad - \frac{(\alpha-1)}{\Gamma(\alpha)}\frac{\Gamma(t-s+\alpha-1)}{\Gamma(t-s+1)}. \end{aligned}$$

From (1) of Theorem 2.9, we have

$$\nabla_t\big[G_l(t,s)\big] < 0,$$

implying that $G_l(t,s)$ is a decreasing function of t. Now, we examine the Green's function to determine whether the maximum for a fixed s will occur at $(a+1,s)$, $(a+2,s)$ or (s,s). We have

$$G_l(a+1,s) = \frac{(b-s+1)^{\overline{\alpha-1}}}{(\alpha-1)(b-a)^{\overline{\alpha-2}}},$$

$$G_l(a+2,s) = \frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-2}}},$$

and

$$G_l(s,s) = \frac{(b-s+1)^{\overline{\alpha-1}}(s-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)(b-a)^{\overline{\alpha-2}}} - 1.$$

Clearly, $G_l(a+2,s) < G_l(a+1,s)$. Since $(2-\alpha) < 2 \leq (s-a)$, by Theorem 2.9, we have

$$(s-a)^{\overline{\alpha-2}} \leq 2^{\overline{\alpha-2}},$$

implying that

$$\begin{aligned} G_l(a+1,s) - G_l(s,s) &= \frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-2}}}\Big[\frac{1}{(\alpha-1)} - \frac{(s-a)^{\overline{\alpha-2}}}{\Gamma(\alpha)}\Big] \\ &\geq \frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-2}}}\Big[\frac{1}{(\alpha-1)} - 1\Big]. \end{aligned} \tag{3.6}$$

It follows from Theorem 2.9 that the term

$$\frac{(b-s+1)^{\overline{\alpha-1}}}{(b-a)^{\overline{\alpha-2}}} = \frac{\Gamma(b-s+\alpha)\Gamma(b-a)}{\Gamma(b-a+\alpha-2)\Gamma(b-s+1)}$$

in the expression (3.6) is positive. Since

$$\frac{1}{(\alpha-1)} - 1 > 0,$$

we obtain $G_l(s,s) < G_l(a+1,s)$. Thus,

$$\max_{(t,s)\in\mathbb{N}_{a+1}^b\times\mathbb{N}_{a+2}^b} G_l(t,s) = \max_{s\in\mathbb{N}_{a+2}^b} G_l(a+1,s) = \frac{(b-a-1)^{\overline{\alpha-1}}}{(\alpha-1)(b-a)^{\overline{\alpha-2}}} = \frac{(b-a-1)}{(\alpha-1)}.$$

This concludes the proof. □

Theorem 3.4. *The following inequality holds for the Green's function $G_l(t,s)$ from* (3.3).

$$\sum_{s=a+2}^{b} G_l(t,s) \leq \frac{(b-a-1)(b-a+\alpha-2)}{\alpha(\alpha-1)}, \quad t\in\mathbb{N}_{a+1}^b.$$

Proof. Consider

$$\sum_{s=a+2}^{b} G_l(t,s) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Big[\frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}} - (t-s+1)^{\overline{\alpha-1}} \Big]$$
$$+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b} \frac{(b-s+1)^{\overline{\alpha-1}}(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}}$$
$$= \frac{(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}} \sum_{s=a+2}^{b} \frac{(b-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - \sum_{s=a+2}^{t} \frac{(t-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}$$
$$= \frac{(t-a)^{\overline{\alpha-2}}}{(b-a)^{\overline{\alpha-2}}} \frac{(b-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)} - \frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.$$

We now find the maximum of this expression with respect to $t \in \mathbb{N}_{a+1}^{b}$. From Theorem 2.9, we have

$$\frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(\alpha+1)\Gamma(t-a-1)} \geq 0, \quad t \in \mathbb{N}_{a+1}^{b}.$$

Since $(2-\alpha) < 1 \leq (t-a)$, by Theorem 2.9, we have

$$(t-a)^{\overline{\alpha-2}} \leq 1^{\overline{\alpha-2}},$$

implying that

$$\sum_{s=a+2}^{b} G_l(t,s) \leq \frac{(b-a-1)^{\overline{\alpha}}}{\alpha(\alpha-1)(b-a)^{\overline{\alpha-2}}} = \frac{(b-a-1)(b-a+\alpha-2)}{\alpha(\alpha-1)}.$$

This concludes the proof. □

We are now able to formulate a Lyapunov-type inequality for the left-focal type discrete boundary value problem.

Theorem 3.5. *If the following discrete fractional boundary value problem*

$$\begin{cases} \big(\nabla_a^{\alpha} u\big)(t) + q(t)y(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ \big(\nabla_a^{\alpha-1} u\big)(a+1) = 0, \ u(b) = 0, \end{cases} \tag{3.7}$$

has a nontrivial solution, then

$$\sum_{s=a+2}^{b} |q(s)| \geq \frac{(\alpha-1)}{(b-a-1)}. \tag{3.8}$$

Proof. Let $\mathfrak{B}$ be the Banach space of functions $u : \mathbb{N}^b_{a+1} \to \mathbb{R}$ endowed with norm

$$\|u\| = \max_{t \in \mathbb{N}^b_{a+1}} |u(t)|.$$

It follows from Theorem 3.1 that a solution to (3.7) satisfies the equation

$$u(t) = \sum_{s=a+2}^{b} G(t,s)q(s)u(s).$$

Hence,

$$\begin{aligned}
\|u\| = \max_{t \in \mathbb{N}^b_{a+1}} \Big| \sum_{s=a+2}^{b} G(t,s)q(s)u(s) \Big| &\le \max_{t \in \mathbb{N}^b_{a+1}} \Big[\sum_{s=a+2}^{b} G(t,s)|q(s)||u(s)| \Big] \\
&\le \|u\| \Big[\max_{t \in \mathbb{N}^b_{a+1}} \sum_{s=a+2}^{b} G(t,s)|q(s)| \Big] \\
&\le \|u\| \Big[\max_{(t,s) \in \mathbb{N}^b_{a+1} \times \mathbb{N}^b_{a+2}} G(t,s) \Big] \sum_{s=a+2}^{b} |q(s)|,
\end{aligned}$$

or, equivalently,

$$1 \le \Big[\max_{t \in \mathbb{N}^b_{a+1},\, s \in \mathbb{N}^b_{a+2}} G(t,s) \Big] \sum_{s=a+2}^{b} |q(s)|.$$

An application of Theorem 3.3 yields the result. $\square$

Now, we discuss three applications of Theorem 3.5. First, we obtain a criterion for the nonexistence of nontrivial solutions of (3.7).

Theorem 3.6. *Assume that* $1 < \alpha < 2$ *and*

$$\sum_{s=a+2}^{b} |q(s)| < \frac{(\alpha - 1)}{(b - a - 1)}. \tag{3.9}$$

Then, the discrete fractional boundary value problem (3.7) *has no nontrivial solution on* $\mathbb{N}^b_{a+1}$.

Next, we estimate a lower bound for eigenvalues of the eigenvalue problem corresponding to (3.7).

Theorem 3.7. *Assume that* $1 < \alpha < 2$ *and* u *is a nontrivial solution of the eigenvalue problem*

$$\begin{cases} \big(\nabla^{\alpha}_{a} u\big)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}^b_{a+2}, \\ \big(\nabla^{\alpha-1}_{a} u\big)(a+1) = 0, \ u(b) = 0, \end{cases} \tag{3.10}$$

where $u(t) \neq 0$ *for each* $t \in \mathbb{N}_{a+2}^{b-1}$*. Then,*

$$|\lambda| \geq \frac{(\alpha - 1)}{(b - a - 1)^2}. \tag{3.11}$$

Finally, we deduce a criterion for the nonexistence of real zeros of certain nabla Mittag-Leffler functions.

Theorem 3.8. *Let* $1 < \alpha < 2$*. Then, the function* $\lambda E_{-\lambda,\alpha,\alpha-1}(t, 0) + E_{-\lambda,\alpha,\alpha-2}(t, 0)$ *has no real zeros for*

$$|\lambda| < \frac{(\alpha - 1)}{(n - 1)^2}.$$

Proof. Let $a = 0$, $b = n \in \mathbb{N}_2$ and consider the eigenvalue problem

$$\begin{cases} \big(\nabla_0^{\alpha} u\big)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_2^n, \\ \big(\nabla_0^{\alpha-1} u\big)(1) = 0, \ u(n) = 0. \end{cases} \tag{3.12}$$

By Theorem 2.14, a general solution of (3.12) is given by

$$u(t) = C_1 E_{-\lambda,\alpha,\alpha-1}(t, 0) + C_2 E_{-\lambda,\alpha,\alpha-2}(t, 0), \quad t \in \mathbb{N}_1, \tag{3.13}$$

where $C_1, C_2 \in \mathbb{R}$. Applying $\nabla_0^{\alpha-1}$ on both sides of (3.13), we get

$$\big(\nabla_0^{\alpha-1} u\big)(t) = C_1 E_{-\lambda,\alpha,0}(t, 0) - \lambda C_2 E_{-\lambda,\alpha,\alpha-1}(t, 0), \quad n \in \mathbb{N}_1. \tag{3.14}$$

Using $\big(\nabla_0^{\alpha-1} u\big)(1) = 0$, we get $C_1 = \lambda C_2$. Using $u(n) = 0$, we have that the eigenvalues $\lambda \in \mathbb{R}$ of (3.12) are the solutions of

$$\lambda E_{-\lambda,\alpha,\alpha-1}(n, 0) + E_{-\lambda,\alpha,\alpha-2}(n, 0) = 0, \tag{3.15}$$

and the corresponding eigenfunctions are given by

$$u(t) = \lambda E_{-\lambda,\alpha,\alpha-1}(t, 0) + E_{-\lambda,\alpha,\alpha-2}(t, 0), \quad t \in \mathbb{N}_1. \tag{3.16}$$

By Theorem 3.5, if a real eigenvalue λ of (3.12) exists, i.e., λ is a zero of (3.12), then $|\lambda| \geq \dfrac{(\alpha - 1)}{(n - 1)^2}$. This concludes the proof. □

4 Right-Focal Type Boundary Value Problem

In this section, we derive a few important properties of the Green's function associated with a right-focal type discrete fractional boundary value problem and obtain the corresponding Lyapunov-type inequality.

Theorem 4.1. *Let $1 < \alpha < 2$ and $h : \mathbb{N}_{a+2}^{b} \to \mathbb{R}$. The discrete fractional boundary value problem*

$$\begin{cases} \big(\nabla_a^{\alpha} u\big)(t) + h(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ u(a+1) = 0, \ \big(\nabla_a^{\alpha-1} u\big)(b) = 0, \end{cases} \tag{4.1}$$

has the unique solution

$$u(t) = \sum_{s=a+2}^{b} G_r(t,s)h(s), \quad t \in \mathbb{N}_{a+1}^{b}, \tag{4.2}$$

where

$$G_r(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)}(t-a-1)^{\overline{\alpha-1}}, & t \in \mathbb{N}_{a+1}^{\rho(s)}, \\ \dfrac{1}{\Gamma(\alpha)}\big[(t-a-1)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\big], & t \in \mathbb{N}_{s}^{b}. \end{cases} \tag{4.3}$$

Proof. Using $\big(\nabla_a^{\alpha-1} u\big)(b) = 0$ in (3.5), we get

$$C_1 = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{b} h(s).$$

Using $u(a+1) = 0$ in (3.4), we get

$$C_2 = -\frac{1}{\Gamma(\alpha-1)} \sum_{s=a+2}^{b} h(s).$$

Substituting the values of C_1 and C_2 in (3.4), we have

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-s+1)^{\overline{\alpha-1}} h(s) + \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} \sum_{s=a+1}^{b} h(s) \\ &- \frac{(t-a)^{\overline{\alpha-2}}}{\Gamma(\alpha-1)} \sum_{s=a+2}^{b} h(s) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Big[(t-a)^{\overline{\alpha-1}} - (\alpha-1)(t-a)^{\overline{\alpha-2}} - (t-s+1)^{\overline{\alpha-1}}\Big] h(s) \\ &+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b} \Big[(t-a)^{\overline{\alpha-1}} - (\alpha-1)(t-a)^{\overline{\alpha-2}}\Big] h(s) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Big[(t-a-1)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big] h(s) \end{aligned}$$

$$+\frac{1}{\Gamma(\alpha)}\sum_{s=t+1}^{b}(t-a-1)^{\overline{\alpha-1}}h(s)=\sum_{s=a+2}^{b}G_r(t,s)h(s).$$

This concludes the proof. □

First, we show that this Green's function is nonnegative and obtain an upper bound for the Green's function and its integral.

Theorem 4.2. *The Green's function $G_r(t,s)$ satisfies $G_r(t,s)\geq 0$ for $(t,s)\in\mathbb{N}_{a+1}^{b}\times\mathbb{N}_{a+2}^{b}$.*

Proof. Clearly,

$$G_r(a+1,s)=0,\quad s\in\mathbb{N}_{a+2}^{b}.$$

Assume $(t,s)\in\mathbb{N}_{a+2}^{b}\times\mathbb{N}_{a+2}^{b}$. For $t\in\mathbb{N}_{a+2}^{\rho(s)}$, consider

$$G_r(t,s)=\frac{(t-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}=\frac{\Gamma(t-a+\alpha-2)}{\Gamma(\alpha)\Gamma(t-a-1)}.$$

It follows from (1) of Theorem 2.9 that $G_r(t,s)>0$. Suppose $t\in\mathbb{N}_{s}^{b}$. Since $a+2\leq s$, from Theorem 2.9, we have

$$(t-s+1)^{\overline{\alpha-1}}\leq(t-a-1)^{\overline{\alpha-1}},$$

implying that

$$G_r(t,s)=\frac{1}{\Gamma(\alpha)}\Big[(t-a-1)^{\overline{\alpha-1}}-(t-s+1)^{\overline{\alpha-1}}\Big]\geq 0.$$

This concludes the proof. □

Theorem 4.3. *The maximum of the Green's function $G_r(t,s)$ defined in* (4.3) *is given by*

$$\max_{(t,s)\in\mathbb{N}_{a+1}^{b}\times\mathbb{N}_{a+2}^{b}}G_r(t,s)=\frac{(b-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Proof. Clearly,

$$G_r(a+1,s)=0,\quad s\in\mathbb{N}_{a+2}^{b}.$$

Fix $t\in\mathbb{N}_{a+2}^{b}$. For $s\in\mathbb{N}_{t+1}^{b}$, we have

$$\nabla_s\big[G_r(t,s)\big]=0,$$

implying that $G_r(t,s)$ is a constant function of s. Now, suppose $s\in\mathbb{N}_{a+2}^{t}$. Consider

$$\nabla_s\big[G_r(t,s)\big]=\frac{(\alpha-1)}{\Gamma(\alpha)}(t-s+2)^{\overline{\alpha-2}}=\frac{\Gamma(t-s+\alpha)}{\Gamma(\alpha-1)\Gamma(t-s+2)}.$$

It follows from (1) of Theorem 2.9 that

$$\nabla_s\big[G_r(t,s)\big] > 0,$$

implying that $G_r(t,s)$ is an increasing function of s. So, we examine the Green's function to determine whether the maximum for a fixed t will occur at (t,t) or $(t,t+1)$. We have

$$G_r(t,t+1) = \frac{(t-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)},$$

and

$$G_r(t,t) = \frac{(t-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} - 1.$$

Clearly,

$$G_r(t,t) < G_r(t,t+1), \quad t \in \mathbb{N}_{a+2}^{b},$$

and

$$\max_{t\in\mathbb{N}_{a+1}^{b}} G_r(t,t+1) = \frac{(b-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

Thus,

$$\max_{(t,s)\in\mathbb{N}_{a+1}^{b}\times\mathbb{N}_{a+2}^{b}} G_r(t,s) = \frac{(b-a-1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}.$$

This concludes the proof. □

Theorem 4.4. *The following inequality holds for the Green's function $G_r(t,s)$ from* (4.3).

$$\sum_{s=a+2}^{b} G_r(t,s) \le \frac{(b-a-1)^{\overline{\alpha-1}}(b-a-1)}{\Gamma(\alpha)}, \quad t \in \mathbb{N}_{a+1}^{b}.$$

Proof. Consider

$$\begin{aligned}
\sum_{s=a+2}^{b} G_r(t,s) &= \sum_{s=a+2}^{t} G_r(t,s) + \sum_{s=t+1}^{b} G_r(t,s) \\
&= \frac{1}{\Gamma(\alpha)} \sum_{s=a+2}^{t} \Big[(t-a-1)^{\overline{\alpha-1}} - (t-s+1)^{\overline{\alpha-1}}\Big] \\
&+ \frac{1}{\Gamma(\alpha)} \sum_{s=t+1}^{b} (t-a-1)^{\overline{\alpha-1}} \\
&= \frac{(t-a-1)^{\overline{\alpha-1}}(t-a-1)}{\Gamma(\alpha)} - \sum_{s=a+2}^{t} \frac{(t-s+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}
\end{aligned}$$

$$= \frac{(t-a-1)^{\overline{\alpha-1}}(t-a-1)}{\Gamma(\alpha)} - \frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)}.$$

We now find the maximum of this expression with respect to $t \in \mathbb{N}_{a+1}^{b}$. From Theorem 2.9, we have

$$\frac{(t-a-1)^{\overline{\alpha}}}{\Gamma(\alpha+1)} = \frac{\Gamma(t-a+\alpha-1)}{\Gamma(\alpha+1)\Gamma(t-a-1)} \geq 0,$$

and

$$(t-a-1)^{\overline{\alpha-1}} \leq (b-a-1)^{\overline{\alpha-1}},$$

implying that

$$\sum_{s=a+2}^{b} G_r(t,s) \leq \frac{(b-a-1)^{\overline{\alpha-1}}(b-a-1)}{\Gamma(\alpha)}.$$

This concludes the proof. □

We are now able to formulate a Lyapunov-type inequality for the right focal boundary value problem.

Theorem 4.5. *If the following discrete fractional boundary value problem*

$$\begin{cases} \left(\nabla_a^{\alpha} u\right)(t) + q(t)y(t) = 0, \quad t \in \mathbb{N}_{a+2}^{b}, \\ u(a+1) = 0, \ \left(\nabla_a^{\alpha-1} u\right)(b) = 0, \end{cases} \tag{4.4}$$

has a nontrivial solution, then

$$\sum_{s=a+2}^{b} |q(s)| \geq \frac{\Gamma(\alpha)}{(b-a-1)^{\overline{\alpha-1}}}. \tag{4.5}$$

Now, we discuss three applications of Theorem 4.5. First, we obtain a criterion for the nonexistence of nontrivial solutions of (4.4).

Theorem 4.6. *Assume that* $1 < \alpha < 2$ *and*

$$\sum_{s=a+2}^{b} |q(s)| < \frac{\Gamma(\alpha)}{(b-a-1)^{\overline{\alpha-1}}}. \tag{4.6}$$

Then, the discrete fractional boundary value problem (4.4) *has no nontrivial solution on* $\mathbb{N}_{a+1}^{b}$.

Next, we estimate a lower bound for eigenvalues of the eigenvalue problem corresponding to (4.4).

Theorem 4.7. *Assume that* $1 < \alpha < 2$ *and* u *is a nontrivial solution of the eigenvalue problem*

$$\begin{cases} \big(\nabla_a^\alpha u\big)(t) + \lambda y(t) = 0, \quad t \in \mathbb{N}_{a+2}^b, \\ u(a+1) = 0, \ \big(\nabla_a^{\alpha-1} u\big)(b) = 0, \end{cases} \tag{4.7}$$

where $u(t) \neq 0$ *for each* $t \in \mathbb{N}_{a+2}^{b-1}$. *Then,*

$$|\lambda| \geq \frac{\Gamma(\alpha)}{(b-a-1)(b-a-1)^{\overline{\alpha-1}}}. \tag{4.8}$$

Finally, we deduce a criterion for the nonexistence of real zeros of certain nabla Mittag-Leffler functions.

Theorem 4.8. *Let* $1 < \alpha < 2$. *Then, the function* $E_{-\lambda,\alpha,0}(t,0) + \lambda E_{-\lambda,\alpha,\alpha-1}(t,0)$ *has no real zeros for*

$$|\lambda| < \frac{\Gamma(\alpha)}{(n-1)(n-1)^{\overline{\alpha-1}}}.$$

Proof. Let $a = 0$, $b = n \in \mathbb{N}_2$ and consider the eigenvalue problem

$$\begin{cases} \big(\nabla_0^\alpha u\big)(t) + \lambda u(t) = 0, \quad t \in \mathbb{N}_2^n, \\ u(1) = 0, \ \big(\nabla_0^{\alpha-1} u\big)(n) = 0. \end{cases} \tag{4.9}$$

Using $u(1) = 0$ in (3.13), we get $C_1 = -C_2$. Using $\big(\nabla_0^{\alpha-1} u\big)(n) = 0$ in (3.14), we have that the eigenvalues $\lambda \in \mathbb{R}$ of (4.9) are the solutions of

$$E_{-\lambda,\alpha,0}(n,0) + \lambda E_{-\lambda,\alpha,\alpha-1}(n,0) = 0, \tag{4.10}$$

and the corresponding eigenfunctions are given by

$$u(t) = E_{-\lambda,\alpha,\alpha-1}(t,0) - E_{-\lambda,\alpha,\alpha-2}(t,0), \quad t \in \mathbb{N}_1. \tag{4.11}$$

By Theorem 4.5, if a real eigenvalue λ of (4.9) exists, i.e. λ is a zero of (4.9), then $|\lambda| \geq \dfrac{\Gamma(\alpha)}{(n-1)(n-1)^{\overline{\alpha-1}}}$. This concludes the proof. □

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