International Journal of Difference Equations
ISSN 0973-6069, Volume 6, Number 1, pp. 39–51 (2011)
http://campus.mst.edu/ijde

Overview of Weyl–Titchmarsh Theory for Second Order Sturm–Liouville Equations on Time Scales

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Abstract

In this paper we present an overview of the basic Weyl–Titchmarsh theory for second order Sturm–Liouville equations on time scales. We construct the $m(\lambda)$-function, the Weyl solution, and the Weyl disk. We justify the terminology "disk" by its geometric properties, show explicitly the coordinates of the center of the disk, and calculate its radius. We show that the dichotomy regarding the square-integrable solutions known in the continuous time and discrete theory works in the same way for general time scales.

AMS Subject Classifications: *Primary* 34B20. *Secondary* 34N05, 34B27, 34B24.
Keywords: Time scale; Weyl–Titchmarsh theory; Second order Sturm–Liouville dynamic equation; $m(\lambda)$-function; Weyl disk; Weyl circle; Limit point case; Limit circle case.

1 Introduction

The purpose of this paper is to provide an overview of the Weyl–Titchmarsh theory for the second order Sturm–Liouville dynamic equation

$$-\big(p(t)\,x^{\Delta}\big)^{\Delta}+q(t)\,x^{\sigma}=\lambda\,w(t)\,x^{\sigma},\quad t\in[a,\infty)_{\mathbb{T}},\tag{1.1}$$

where $\mathbb{T}$ is a time scale which is unbounded above, i.e., $\mathbb{T}=[a,\infty)_{\mathbb{T}}$ with $a:=\min\mathbb{T}$. The coefficients $p(\cdot),q(\cdot),w(\cdot)$ are real piecewise rd-continuous functions on $[a,\infty)_{\mathbb{T}}$ such that

$$\inf_{t\in[a,b]_{\mathbb{T}}}|p(t)|>0\quad\text{for all } b\in(a,\infty)_{\mathbb{T}}\quad\text{and}\quad w(t)>0\quad\text{for all } t\in[a,\infty)_{\mathbb{T}},\tag{1.2}$$

Received February 23, 2011; Accepted May 10, 2011
Communicated by Martin Bohner

and $\lambda \in \mathbb{C}$ is a spectral parameter. The function $p(\cdot)$ is allowed to change its sign. For the details of the calculus on time scales see Section 2.

It is well known that the second order Sturm–Liouville differential equations

$$-(p(t)\,x')' + q(t)\,x = \lambda\,w(t)\,x, \quad t \in [a,\infty), \tag{1.3}$$

can be divided into two cases depending on the count of its square-integrable solutions. Namely, in the *limit point case* there is exactly one (up to a multiplicative constant) square-integrable solution, and in the *limit circle case* there are two linearly independent square-integrable solutions. This dichotomy was initially investigated (by using a geometrical approach) by Weyl in his paper [48] from 1910. One of the most important contributions in extending this theory was made by Titchmarsh in the series of papers from 1939–1945 (especially in papers [42–44] from 1941), which were summarized in his book [45]. He re-proved Weyl's results by using an alternative method and established many properties of the fundamental function appearing in this theory, the so-called $m(\lambda)$-function. Hence in honour of the pioneers of this theory, it is called the *Weyl–Titchmarsh theory*.

When studying differential equation (1.3), a crucial role is played by the solutions $\theta(\cdot), \phi(\cdot) : [a,\infty) \to \mathbb{C}$ satisfying the following boundary conditions

$$\theta(a) = \sin\varphi, \quad p(a)\,\theta'(a) = \cos\varphi, \qquad \phi(a) = -\cos\varphi, \quad p(a)\,\phi'(a) = \sin\varphi, \tag{1.4}$$

where $\varphi \in [0,\pi)$. Weyl considered in his paper [48] equation (1.3) with continuous $p(\cdot)$ and $q(\cdot)$, $p(\cdot) > 0$, $w(\cdot) \equiv 1$ on $[a,\infty)$, and boundary conditions (1.4) in which $\varphi = \frac{\pi}{2}$. On the other hand, Titchmarsh studied in his book [45] equation (1.3) with $p(\cdot) \equiv 1$, continuous $q(\cdot)$, and $w(\cdot) \equiv 1$ on $[a,\infty)$, but with general boundary conditions in the form (1.4) having $\varphi \in [0,\pi)$.

Their results were extended in many ways. First of all, there were weakened conditions put on the coefficients of the differential equation. For an overview of the progress in this way we refer to [14, Section 1]. In addition, Sims discussed in [38] equation (1.3) with $p(\cdot) \equiv 1$, $w(\cdot) \equiv 1$, and he allowed $q(\cdot)$ to be a complex function. This change gives (surprisingly) a new limit point behavior which does not occur when $q(\cdot)$ is real-valued, namely, there are two linearly independent square-integrable solutions while the equation is in the limit point case. Therefore, paper [38] started the development of the so-called *Titchmarsh–Sims–Weyl theory*.

The investigation of the Weyl–Titchmarsh theory for the second order difference equations

$$b_{k-1}y_{k-1} + a_k y_k + b_k y_{k+1} = \lambda y_k, \quad k \in \mathbb{N},$$

where $a_k \in \mathbb{R}$ and $b_k > 0$ for all $k \in \mathbb{N} \cup \{0\}$, was initiated by Hellinger and Nevanlinna in their independent papers from 1922, see [16, 32]. In the 20th century, the discrete theory was not studied so intensively like the continuous theory, but some results concerning our treatment can be found e.g. in [1, 5, 6, 8, 39]. In particular, in [5, 6] the second order difference equation

$$-\Delta\,(p_k\,\Delta x_k) + q_k x_{k+1} = \lambda\, w_k x_{k+1}, \quad k \in \mathbb{N}$$

was investigated, where p_k, q_k, w_k are real and satisfy $p_k \neq 0$ and $w_k > 0$. A comprehensive summary of the history of the Weyl–Titchmarsh theory for the second order differential and difference equations can be found in the expository paper [14] by Everitt.

It is known that every Sturm–Liouville differential equation can be written in the form of the linear Hamiltonian differential system, see [2] for the even order equations and [46] for arbitrary order equations. As a consequence, since middle seventies of the 20th century a constant attention is paid for the study of the Weyl–Titchmarsh theory for continuous time linear Hamiltonian systems, see [2, 7, 9, 10, 15, 18–25, 27–31, 33–35]. The discrete analogy of these continuous time results, i.e., the Weyl–Titchmarsh theory for discrete linear Hamiltonian systems and more generally for discrete symplectic systems, was given in [11, 36, 40] and [4, 12].

Most recent extensions of the Weyl–Titchmarsh theory are represented by dynamic equations on time scales. Following the results in [48], the elements of the Weyl–Titchmarsh theory for equation (1.1) with $p(\cdot) = w(\cdot) \equiv 1$ and the time scale analogue of boundary conditions (1.4) with $\varphi = \frac{\pi}{2}$, see also (3.2), is considered in [49]. Similarly, when $p(\cdot)$ is piecewise continuously nabla-differentiable and $w(\cdot) \equiv 1$, the results in [48] are generalized in [26] to equation (1.1). A classification of the limit point and limit circle cases for the second order dynamic equations with mixed time scale delta and nabla derivatives and nonzero continuous coefficients is given in [47]. The Weyl–Titchmarsh theory for linear Hamiltonian dynamic systems was studied in [41]. For the purpose of this paper we consider the system

$$\left.\begin{aligned} x^{\Delta}(t) &= A(t)\,x^{\sigma}(t) + B(t)\,u(t), \\ u^{\Delta}(t) &= [C(t) - \lambda\,W(t)]\;x^{\sigma}(t) - A^*(t)\,u(t), \end{aligned}\right\} \quad t \in [a,\infty)_{\mathbb{T}}, \tag{1.5}$$

which has the spectral parameter in the second equation. The coefficients $A(\cdot)$, $B(\cdot)$, $C(\cdot)$, and $W(\cdot)$ are rd-continuous complex $n \times n$ matrix-valued functions such that $B(\cdot)$, $C(\cdot)$, $W(\cdot)$ are Hermitian matrices, $W(\cdot)$ is nonnegative definite, and $I - \mu(t)\,A(t)$ is invertible. Here A^* denotes the conjugate transpose of the indicated matrix. Finally, it is well known that linear Hamiltonian dynamic system (1.5) is a special case of the time scale symplectic system

$$\left.\begin{aligned} x^{\Delta}(t) &= \mathcal{A}(t)\,x(t) + \mathcal{B}(t)\,u(t), \\ u^{\Delta}(t) &= \mathcal{C}(t)\,x(t) + \mathcal{D}(t)\,u(t) - \lambda\,\mathcal{W}(t)\,x^{\sigma}(t), \end{aligned}\right\} \quad t \in [a,\infty)_{\mathbb{T}}, \tag{$\mathcal{S}$}$$

where $\mathcal{A}(\cdot)$, $\mathcal{B}(\cdot)$, $\mathcal{C}(\cdot)$, $\mathcal{D}(\cdot)$, $\mathcal{W}(\cdot)$ are piecewise rd-continuous complex $n \times n$ matrix-valued functions on $[a,\infty)_{\mathbb{T}}$, $\mathcal{W}(\cdot)$ is Hermitian and nonnegative definite on $[a,\infty)_{\mathbb{T}}$,

$$\mathcal{S}^*(t)\,\mathcal{J} + \mathcal{J}\mathcal{S}(t) + \mu(t)\,\mathcal{S}^*(t)\,\mathcal{J}\mathcal{S}(t) = 0, \;\; \mathcal{S}(t) = \begin{pmatrix} \mathcal{A}(t) & \mathcal{B}(t) \\ \mathcal{C}(t) & \mathcal{D}(t) \end{pmatrix}, \;\; \mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

System ($\mathcal{S}$) generalizes and unifies the continuous, discrete, and dynamic linear Hamiltonian systems, and consequently continuous, discrete, and dynamic Sturm–Liouville equations.

In this paper we present an overview of the Weyl–Titchmarsh theory for equation (1.1) in which the function $p(\cdot)$ is merely piecewise rd-continuous (without any requirement on the existence of its delta-derivative or nabla-derivative) and $w(\cdot)$ is piecewise rd-continuous, satisfying (1.2). In particular, an important role in this theory is played by the $m(\lambda)$-*function*, whose natural properties

$$m(\bar{\lambda}) = \overline{m(\lambda)} \quad \text{and} \quad (\operatorname{Im}\lambda)\,(\operatorname{Im} m(\lambda)) > 0 \quad \text{for } \lambda \notin \mathbb{R}$$

remain true on general time scales, where $\operatorname{Im}\nu$ denotes the imaginary part of the complex number ν. We construct the so-called *Weyl solution* and *Weyl disk*. We justify the terminology "disk" by its geometric properties, show explicitly the coordinates of the center of the disk, and calculate its radius. We show that the dichotomy mentioned above works in the same way (especially, that the Weyl solution is square-integrable) and present a necessary and sufficient criterion for the limit point case. Finally, we consider a nonhomogeneous problem associated with equation (1.1). We define the *Green function* and use it for expressing a solution of the nonhomogeneous problem. These results complete the study in [49], in which the square-integrable Weyl solution and the center and radius of the Weyl disk were obtained for the special case of (1.1) as we discuss above.

Our method is to transform equation (1.1) into a 2×2 symplectic dynamic system ($\mathcal{S}$), which allows us to apply the results from [37]. Also, it needs to be mentioned that when the coefficients of equation (1.1) satisfy the assumption that

$$p(\cdot), q(\cdot), w(\cdot) \text{ are rd-continuous and (1.2) holds,} \tag{1.6}$$

then some of our results, but not all of them, follow from [41], where equation (1.1) is considered as the special case of linear Hamiltonian system (1.5). In spite of that the results of this paper are derived as a special case of the time scale symplectic system ($\mathcal{S}$) in [37], their particular forms for the second order dynamic equation (1.1) are now published for the first time. Therefore, this paper provides a straightforward unification and extension of the Weyl–Titchmarsh theory for the second order Sturm–Liouville differential and difference equations.

Concerning the structure of this paper, we recall in the next section some important notation from the theory of time scales. The main results of this paper are then displayed in Section 3, where at the end we quote the corresponding sources.

2 Sturm–Liouville Equations on Time Scales

For the basic notation and terminology of dynamic on time scales we refer to [3]. The definitions of piecewise rd-continuous functions ($\mathrm{C}_{\mathrm{prd}}$) and piecewise rd-continuously delta-differentiable functions ($\mathrm{C}^1_{\mathrm{prd}}$) can be found in [17, 37].

For a given $\lambda \in \mathbb{C}$, a function $x(\cdot,\lambda) : [a,\infty)_{\mathbb{T}} \to \mathbb{C}$ is said to be a solution of equation (1.1) if $x(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}}$, $\big(p\, x^{\Delta}\big)(\cdot,\lambda) \in \mathrm{C}^1_{\mathrm{prd}}$, and equation (1.1) is satisfied for all $t \in [a,\infty)_{\mathbb{T}}$. The existence and uniqueness of solutions of (1.1) together with the initial conditions $x(t_0) = x_0$ and $x^{\Delta}(t_0) = x_1$, where $x_0, x_1 \in \mathbb{C}$ are given constants, follows through the next lemma by the corresponding result for the linear systems, see [13, Corollary 7.12].

Lemma 2.1. *Equation* (1.1) *is equivalent with the time scale symplectic system* ($\mathcal{S}$) *of the form*

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\Delta} = \begin{pmatrix} 0 & 1/p(t) \\ q(t) & \mu(t)\, q(t)/p(t) \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} - \begin{pmatrix} 0 \\ \lambda\, w(t)\, x^{\sigma} \end{pmatrix},$$

where $u(t) := p(t)\, x^{\Delta}(t)$*, or equivalently with the Hamiltonian system* (1.5) *of the form*

$$\begin{pmatrix} x \\ u \end{pmatrix}^{\Delta} = \begin{pmatrix} 0 & 1/p(t) \\ q(t) - \lambda\, w(t) & 0 \end{pmatrix} \begin{pmatrix} x^{\sigma} \\ u \end{pmatrix}.$$

Proof. The statements follow by straightforward calculations. □

For any two functions $x(\cdot), y(\cdot) \in \mathrm{C}^1_{\mathrm{prd}}$ on $[a,\infty)_{\mathbb{T}}$ we define their *Wronskian* by

$$W[x(t), y(t)] = x(t)\, y^{\Delta}(t) - x^{\Delta}(t)\, y(t) \quad \text{for all } t \in [a,\infty)_{\mathbb{T}}, \tag{2.1}$$

which will be used in the formulation of our results similarly to [45].

3 Main Results

Throughout this paper we assume that $\beta_1, \beta_2 \in \mathbb{C}$ are given numbers such that

$$\beta_1\bar{\beta}_1 + \beta_2\bar{\beta}_2 = 1, \quad \bar{\beta}_1\beta_2 - \beta_1\bar{\beta}_2 = 0. \tag{3.1}$$

Following (1.4), we denote by $\theta(\cdot,\lambda,\varphi)$ and $\phi(\cdot,\lambda,\varphi)$ the solutions of (1.1) satisfying the initial conditions

$$\left.\begin{aligned} \theta(a,\lambda,\varphi) &= \sin\varphi, & p(a)\,\theta^{\Delta}(a,\lambda,\varphi) &= \cos\varphi, \\ \phi(a,\lambda,\varphi) &= -\cos\varphi, & p(a)\,\phi^{\Delta}(a,\lambda,\varphi) &= \sin\varphi, \end{aligned}\right\} \tag{3.2}$$

where $\lambda \in \mathbb{C}$ and $\varphi \in [0,\pi)$. Our first result is associated with the regular spectral problem.

Theorem 3.1. *Consider the boundary value problem*

$$(1.1) \text{ with } \quad x(a)\, \sin\varphi + p(a)\, x^{\Delta}(a)\, \cos\varphi = 0, \quad \beta_1\, x(b) + \beta_2\, p(b)\, x^{\Delta}(b) = 0, \tag{3.3}$$

where $b \in [a,\infty)_{\mathbb{T}}$ *is fixed. Then* *(i) a number* $\lambda \in \mathbb{C}$ *is an eigenvalue of* (3.3) *if and only if*

$$\beta_1 \, \phi(b, \lambda, \varphi) + \beta_2 \, p(b) \, \phi^{\Delta}(b, \lambda, \varphi) = 0,$$

(ii) the eigenvalues of (3.3) *are real and the eigenfunctions corresponding to different eigenvalues are orthogonal with respect to the semi-inner product*

$$\langle x(\cdot), y(\cdot) \rangle_{w,b} := \int_a^b w(t) \, \bar{x}^{\sigma}(t) \, y^{\sigma}(t) \, \Delta t. \tag{3.4}$$

The following definition corresponds in the continuous case to [45, identity (2.1.5)].

Definition 3.2 ($m(\lambda)$-function)**.** For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $b \in [a, \infty)_{\mathbb{T}}$ we define the $m(\lambda)$*-function*

$$m(b, \lambda, \varphi) := -\frac{\beta_1 \, \theta(b, \lambda, \varphi) + \beta_2 \, p(b) \, \theta^{\Delta}(b, \lambda, \varphi)}{\beta_1 \, \phi(b, \lambda, \varphi) + \beta_2 \, p(b) \, \phi^{\Delta}(b, \lambda, \varphi)}.$$

In the next theorem we present the fundamental properties of the $m(\lambda)$-function.

Theorem 3.3. *The* $m(\lambda)$*-function is an entire function in* λ *and it satisfies*

$$\overline{m}(b, \lambda, \varphi) = m(b, \bar{\lambda}, \varphi) \quad \textit{for every } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

Now, we define the so-called *Weyl solution*, compare with [45, pg. 25].

Definition 3.4 (Weyl solution)**.** For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and $m \in \mathbb{C}$ we define the *Weyl solution*

$$\mathcal{X}(\cdot, \lambda, \varphi, m) := \theta(\cdot, \lambda, \varphi) + m \, \phi(\cdot, \lambda, \varphi). \tag{3.5}$$

Now, we describe how the $m(\lambda)$-function depends on the value of φ used in the initial conditions (3.2).

Theorem 3.5. *For any* $\varphi, \psi \in [0, \pi)$ *we have*

$$m(b, \lambda, \varphi) = \frac{\sin(\varphi - \psi) + \cos(\varphi - \psi) \, m(b, \lambda, \psi)}{\cos(\varphi - \psi) + \sin(\varphi - \psi) \, m(b, \lambda, \psi)}.$$

Now, we define the corresponding Weyl disk. The justification of this terminology follows from Theorem 3.8 bellow.

Definition 3.6 (Weyl disk)**.** By using the function

$$\mathcal{E}(m) := i \, \operatorname{sgn}(\operatorname{Im} \lambda) \, p(b) \, W \left[\mathcal{X}(b, \bar{\lambda}, \varphi, m), \mathcal{X}(b, \lambda, \varphi, m) \right]$$

with fixed $b \in [a, \infty)_{\mathbb{T}}$ we construct the Weyl disk to be the set

$$D(b, \lambda) := \{ m \in \mathbb{C} : \; \mathcal{E}(m) \le 0 \}.$$

A characterization of elements of the Weyl disk is formulated in the following theorem. In this result the numbers $\beta_1, \beta_2 \in \mathbb{C}$ satisfy only the first identity in (3.1), while the second identity from (3.1) is replaced by an inequality.

Theorem 3.7. *The number $m \in \mathbb{C}$ belongs to the Weyl disk $D(b,\lambda)$ if and only if there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that* (3.1)*(i) holds, $\bar{\beta}_1\beta_2 - \beta_1\bar{\beta}_2 \geq 0$, and*

$$\beta_1\, \mathcal{X}(b,\lambda,\varphi,m) + \beta_2\, p(b)\, \mathcal{X}^{\Delta}(b,\lambda,\varphi,m) = 0.$$

Moreover, in this case we have $m = m(b,\lambda,\varphi)$.

The following properties represent the time scales analogies of the geometric characterization of the Weyl disk obtained for the continuous time case in [45, pg. 24].

Theorem 3.8. *For $b \in [a,\infty)_{\mathbb{T}}$, the Weyl disk $D(b,\lambda)$ has the form*

$$D(b,\lambda) = \{c(b,\lambda) + r(b,\lambda)\, v,\ v \in \mathbb{C} \text{ with } \|v\| \leq 1\},$$

where the center $c(b,\lambda) \in \mathbb{C}$ is the point

$$c(b,\lambda) = -W[\phi(b,\bar{\lambda}), \theta(b,\lambda)] \,/\, W[\phi(b,\bar{\lambda}), \phi(b,\lambda)],$$

and the radius $r(b,\lambda) \in \mathbb{R}$ is given by

$$1/r(b,\lambda) = \big|p(b)\, W[\phi(b,\bar{\lambda}), \phi(b,\lambda)]\big|.$$

From now on, we consider the corresponding singular spectral problem. For this case we use the semi-inner product which is the limit of $\langle\cdot,\cdot\rangle_{w,b}$ as $b \to \infty$, i.e.,

$$\langle x(\cdot), y(\cdot)\rangle_w := \int_a^\infty w(t)\, \bar{x}^\sigma(t)\, y^\sigma(t)\, \Delta t.$$

First of all we use the nesting property of the Weyl disks and define the limiting Weyl disk.

Theorem 3.9. *The Weyl disks $D(b,\lambda)$ are closed, convex, and nested with respect to $b \to \infty$. Hence, the so-called limiting Weyl disk*

$$D_+(\lambda) := \bigcap_{b \in [a,\infty)_{\mathbb{T}}} D(b,\lambda)$$

is closed, convex, and nonempty. Moreover, the limits $\lim_{b\to\infty} c(b,\lambda)$ and $\lim_{b\to\infty} r(b,\lambda)$ exist and define the center $c_+(\lambda)$ and the radius $r_+(\lambda)$ of the limiting Weyl disk $D_+(\lambda)$, i.e.,

$$c_+(\lambda) = \lim_{b\to\infty} c(b,\lambda), \quad r_+(\lambda) = \lim_{b\to\infty} r(b,\lambda) \geq 0.$$

Next, we consider the linear space of complex square-integrable $\mathrm{C}^1_{\mathrm{prd}}$ functions on the interval $[a,\infty)_{\mathbb{T}}$, i.e.,

$$L^2_w := \left\{ x \in \mathrm{C}^1_{\mathrm{prd}},\ \|\,x(\cdot)\|^2_w := \int_a^\infty w(t)\,|x^\sigma(t)|^2\,\Delta t < \infty \right\}.$$

The following theorem says that the Weyl solution is square-integrable, which corresponds to [45, inequality (2.1.9)] in the continuous time case.

Theorem 3.10. *For any $m \in D_+(\lambda)$ we have*

$$\|\,\mathcal{X}(\cdot,\lambda,\varphi,m)\|^2_w \leq (\operatorname{Im} m)\,/\,\operatorname{Im}\lambda.$$

From Theorem 3.10 it follows that there is always at least one square-integrable solution. Hence, it is natural to classify the second order Sturm–Liouville equations on time scales of the form (1.1) depending on their number of linearly independent square-integrable solutions.

Definition 3.11. Equation (1.1) is said to be in the *limit point case* if, for every $\lambda \in \mathbb{C}\setminus\mathbb{R}$, there is exactly one (up to a multiplicative constant) square-integrable solution on $[a,\infty)_{\mathbb{T}}$, while it is said to be in the *limit circle case* if there are two linearly independent square-integrable solutions on $[a,\infty)_{\mathbb{T}}$.

The limit point case is characterized in the next theorem.

Theorem 3.12. *The following statements are equivalent.*

(i) Equation (1.1) *is in the limit point case.*

(ii) For every $\lambda \in \mathbb{C}\setminus\mathbb{R}$ we have $r_+(\lambda) = 0$, and consequently $D_+(\lambda) = \{c_+(\lambda)\}$.

(iii) For every $\lambda,\nu \in \mathbb{C}\setminus\mathbb{R}$ and every square-integrable solutions $x_1(\cdot,\lambda)$ and $x_2(\cdot,\nu)$ of (1.1) *with the spectral parameter equal to λ and ν, respectively, we have*

$$\lim_{t\to\infty} p(t)\,W[x_1(t,\bar\lambda), x_2(t,\nu)] = 0.$$

Now, for $f(\cdot) \in L^2_w$ we consider the nonhomogeneous equation

$$\big(p(t)\,x^\Delta\big)^\Delta + q(t)x^\sigma = \lambda w(t)\,x^\sigma + w(t)\,f^\sigma(t), \quad t \in [a,\infty)_{\mathbb{T}}. \tag{3.6}$$

The form of the solution of the nonhomogeneous problem given in the following theorem corresponds to the continuous time case in [45, identity (2.6.1)]. For $m \in D_+(\lambda)$ we define the *Green function*

$$G(t,s,\lambda) := \begin{cases} [\theta(t,\lambda) + m\,\phi(t,\lambda)]\,\phi(s,\lambda), & s \in [a,t]_{\mathbb{T}}, \\ \phi(t,\lambda)\,[\theta(s,\lambda) + m\,\phi(s,\lambda)]\,, & s \in [t,\infty)_{\mathbb{T}}. \end{cases}$$

Theorem 3.13. *The function*

$$\hat{x}(t,\lambda) = -\int_a^\infty G(t,\sigma(s),\lambda)\, w(s) f^\sigma(s)\, \Delta s, \quad t \in [a,\infty)_{\mathbb{T}},$$

solves equation (3.6) *and satisfies the boundary conditions*

$$\hat{x}(a,\lambda)\, \sin\varphi + p(a)\, \hat{x}^\Delta(a,\lambda)\, \cos\varphi = 0,$$
$$\lim_{t\to\infty} p(t)\, W\, [\mathcal{X}(t,\bar{\nu},\varphi,m), \hat{x}(t,\lambda)] = 0$$

for every $\nu \in \mathbb{C} \setminus \mathbb{R}$. *Moreover, we have the estimate*

$$\|\hat{x}(\cdot,\lambda)\|_w \leq \|f(\cdot)\|_w \,/\, |\mathrm{Im}\,\lambda|.$$

Remark 3.14. The statements of the results in this paper follow from the corresponding results in [37], and some of them under (1.6) in [41]. More precisely, Theorems 3.1, 3.7–3.10 and Definition 3.6 have their counterparts both in [41] and [37]. The remaining results in this paper follow from [37] alone.

Acknowledgements

The research was supported by the Czech Science Foundation under grant 201/09/J009, by the research project MSM 0021622409 of the Ministry of Education, Youth, and Sports of the Czech Republic, and by the grant MUNI/A/0964/2009 of Masaryk University.

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