International Journal of Difference Equations
ISSN 0973-6069, Volume 6, Number 1, pp. 17–37 (2011)
http://campus.mst.edu/ijde

A Comparison Result for the Fractional Difference Operator

Christopher S. Goodrich
University of Nebraska-Lincoln
Department of Mathematics
Lincoln, NE 68588
s-cgoodri4@math.unl.edu

Abstract

In this paper, we deduce the Green's function for a ν-th order, $1 < \nu \leq 2$, discrete fractional boundary value problem with boundary conditions of the type $\alpha y(\nu-2) - \beta\Delta y(\nu-2) = 0$, $\gamma y(\nu+b) + \delta\Delta y(\nu+b) = 0$, for α, β, γ, $\delta \geq 0$ and $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$. This extends and generalizes the results of some recent papers. We then show that this Green's function satisfies a positivity property. From this we deduce a relatively general comparison result for boundary value problems of this sort. In particular, this shows that the fractional difference operator satisfies the same sort of comparison principle that is well-known in the integer-order case.

AMS Subject Classifications: Primary: 26A33, 39A12, 39A70.
Keywords: Discrete fractional calculus, Green's function, boundary value problem, positivity, comparison principle.

1 Introduction

The continuous fractional calculus has a long and important history. Recognized by applied scientists for its efficacy in modeling certain physical problems, it has recently attracted increasing and concerted attention from mathematicians, who have recognized that the fractional calculus possesses an interesting and relatively undeveloped mathematical theory. The papers [1, 8, 9, 13, 20–24] are a good survey of some of the recent research on the continuous fractional calculus, and this research spans initial value problems, boundary value problems, and certain applications, such as epidemic modeling

Received May 28, 2010; Accepted January 13, 2011
Communicated by Christopher C. Tisdell

and chemical kinetics, of the continuous fractional calculus. In particular, the monograph by I. Podlubny [22] is highly recommended for its concise survey of both the theory and application of the continuous fractional calculus.

Very recently there has been progress in developing the theory of the discrete fractional calculus. In several recent papers both by Atici and Eloe [2–4, 6] and by the present author [12, 14–18] some basic results for discrete fractional IVPs and BVPs have been deduced. Moreover, in a recent paper by Atici and Şengül [7], it was shown that fractional difference equations can be used to provide models for tumor growth by way of a fractional, discrete Gompertz equation. Thus, the theory of discrete fractional equations seems to hold promise for interesting biological and physical applications. We should also mention that there has recently been some initial progress made in extending the discrete fractional calculus to the somewhat more general time scale $h\mathbb{Z}$. The excellent article by Bastos, et al. [10] provides a detailed account of these developments. In summary, the broad area of fractional calculus on discrete time scales seems to be gaining interest among mathematicians.

More specifically, discrete two-point FBVPs have recently been considered in several papers. Atici and Eloe [6] considered the conjugate discrete fractional boundary value problem

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) = 0 = y(\nu+b) \end{cases},$$

where $1 < \nu \leq 2$. On the other hand, the present author considered in [14] the right-focal discrete fractional boundary value problem

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)) \\ y(\nu-2) = 0 = \Delta y(\nu+b) \end{cases},$$

where, again, $1 < \nu \leq 2$. In each of those papers existence theorems were proved for the associated boundary value problem. Moreover, important conditions on the Green's functions were also deduced. In addition, there has recently been some work on fractional nabla difference equations. In particular, a paper by Atici and Eloe [5] develops some of the basic theory.

Underlying each of the previous works is a common concern. Indeed, one of the interesting aspects of the fractional calculus (both discrete and continuous) is whether certain classical results hold, and, if so, whether their statement in the fractional case is the same as or different than their statement in the integer-order case. In some cases, well-known and even crucially important properties in the integer-order case (e.g., the Harnack-like inequality that certain Green's functions satisfy) fails in certain fractional problems (cf., [9] versus [13]). On the other hand, even if a given property remains true, it may have to be formulated differently, and this formulation may yield certain insight into the fractional problem that would be difficult to attain without the given property.

With these thoughts and motivations in mind, we are concerned in this work with establishing whether or not the discrete fractional difference operator satisfies a sort of comparison principle. The comparison principle we prove here is well-known in the integer-order case, but, so far as the author knows, has not been established in the fractional case.

To provide an appropriate framework in which to deduce this comparison result, we consider in this paper the very general discrete fractional boundary problem

$$\begin{cases} -\Delta^{\nu} y(t) = f(t+\nu-1, y(t+\nu-1)) \\ \alpha y(\nu-2) - \beta \Delta y(\nu-2) = 0 \\ \gamma y(\nu+b) + \delta \Delta y(\nu+b) = 0 \end{cases}, \tag{1.1}$$

where $t \in [0,b]_{\mathbb{N}_0}$, $\nu \in (1,2]$, and $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$ with α, β, γ, $\delta \geq 0$. This is a generalization of the problems considered in both [6] and [14], and certain of our results subsume those as special cases. In fact, the results of this work are, in part, a natural extension of certain of the earlier papers [6, 14].

In particular, the results in this work provide the following generalizations and contributions.

1. We show that the Green's function associated the very general problem (1.1) is nonnegative. As mentioned above, this generalizes some of the results in both [6] and [14]. More generally, it provides an essential ingredient for a program to study conditions under which problem (1.1) will have at least one positive solution. While we do not carry out that program in this work, the positivity of the Green's function provides an initial step in that direction. Of course, in the integer-order case such a program is well known. In addition, as pointed out in Section 4, it is rather technical to prove this result. The techniques used in the proof of Theorem 4.1 may be of interest to others working with discrete fractional problems.

2. We deduce a comparison-type theorem for the operator Δ^{ν}, with $1 < \nu \leq 2$. This is an obvious generalization of the well-known result in case $\nu = 2$.

3. We give, in Section 5 of this work, some consequences of the comparison principle. In particular, we explain how it implies a sort of concavity-type interpretation for the fractional difference. This seems interesting due to the fact that virtually all of the natural geometric meaning of the integer-order difference is lost when one passes to the fractional-order difference. (The same is essentially true in the continuous case.) While it is difficult to say at this juncture whether or not this particular interpretation will be important in future work on fractional discrete problems, it may be, and, in any case, it has not been reported in the literature, so far as we know.

In terms of the organization of this work, (1) above is developed in Sections 3 and 4, whereas (2) and (3) are developed in Section 5. Finally, we conclude Section 5 and this work with an example to illustrate our results.

2 Preliminaries

We first wish to collect some basic lemmas that will be important to us in the sequel. These and other related results and their proofs can be found, for example, in [2–4, 6]. We begin with some basic properties regarding the discrete fractional derivative. These results will play a decisive role in our proofs later in this paper.

Definition 2.1. We define

$$t^{\underline{\nu}} := \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)},$$

for any t and ν for which the right-hand side is defined. We also appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2.2. The ν-th fractional sum of a function f defined on the set $\mathbb{N}_a := \{a, a+1, \dots\}$, for $\nu > 0$, is defined to be

$$\Delta^{-\nu} f(t) = \Delta^{-\nu} f(t; a) := \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s),$$

where $t \in \{a+\nu, a+\nu+1, \dots\} =: \mathbb{N}_{a+\nu}$. We also define the ν-th fractional difference, where $\nu > 0$ and $0 \le N-1 < \nu \le N$ with $N \in \mathbb{N}$, to be $\Delta^{\nu} f(t) := \Delta^{N} \Delta^{-(N-\nu)} f(t)$, where $t \in \mathbb{N}_{a+N-\nu}$.

Next we require some operational properties of the fractional sum operator.

Lemma 2.3. *Let t and ν be any numbers for which $t^{\underline{\nu}}$ and $t^{\underline{\nu-1}}$ are defined. Then $\Delta t^{\underline{\nu}} = \nu t^{\underline{\nu-1}}$.*

Lemma 2.4. *Let $0 \le N-1 < \nu \le N$. Then $\Delta^{-\nu}\Delta^{\nu} y(t) = y(t) + C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} + \cdots + C_N t^{\underline{\nu-N}}$, for some $C_i \in \mathbb{R}$, with $1 \le i \le N$.*

Finally, we require the following lemma. While mathematically straightforward, this lemma is of crucial importance in the sequel. Indeed, its utility in proving certain theorems in the discrete fractional calculus is considerable. Its proof may be found in a paper by the author (see [14, Lemma 4.3]).

Lemma 2.5. *Fix $k \in \mathbb{N}$ and let $\{m_j\}_{j=1}^{k}$, $\{n_j\}_{j=1}^{k} \subseteq (0, \infty)$ such that*

$$\max_{1 \le j \le k} m_j \le \min_{1 \le j \le k} n_j$$

and that for at least one j_0, $1 \le j_0 \le k$, we have that $m_{j_0} < n_{j_0}$. Then for fixed $\alpha_0 \in (0,1)$, it follows that

$$\left(\frac{n_1}{n_1+\alpha_0} \cdot \dots \cdot \frac{n_k}{n_k+\alpha_0}\right)\left(\frac{m_1+\alpha_0}{m_1} \cdot \dots \cdot \frac{m_k+\alpha_0}{m_k}\right) > 1.$$

3 Derivation of Green's Function

In this section we wish to derive the Green's function for the problem

$$\begin{cases} -\Delta^{\nu} y(t) = h(t+\nu-1) \\ \alpha y(\nu-2) - \beta \Delta y(\nu-2) = 0 \\ \gamma y(\nu+b) + \delta \Delta y(\nu+b) = 0 \end{cases}, \tag{3.1}$$

where $h : \mathbb{N}_{\nu-2} \to \mathbb{R}$ is given, $t \in [0,b]_{\mathbb{N}_0}$, and α, β, γ, $\delta \geq 0$ with $\alpha\gamma + \alpha\delta + \beta\gamma \neq 0$. Before stating our fundamental theorem, let us introduce the following notation, which will be important in the sequel.

$$T_1 := \left\{(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0} \ : \ 0 \leq s < t-\nu+1 \leq b+2\right\}$$

$$T_2 := \left\{(t,s) \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \times [0, b+1]_{\mathbb{N}_0} \ : \ 0 \leq t-\nu+1 \leq s \leq b+2\right\}$$

Additionally, for the sake of space in the sequel, let us put $\phi_1(\alpha,\beta) := \alpha - \beta(\nu-2)$ and $\phi_2(\beta,\gamma,\delta,\nu,b) := \gamma\beta(\nu-1)(\nu+b)^{\underline{\nu-2}} + \delta\beta(\nu-1)(\nu-2)(\nu+b)^{\underline{\nu-3}}$, which will occur frequently in our subsequent arguments. We now state a fundamental theorem. Although the proof of this theorem is straightforward, we include it in any case since the formula for the Green's function is somewhat complicated in this context.

Theorem 3.1. *The unique solution to problem* (3.1) *is given by*

$$\sum_{s=0}^{b} G(t,s) h(s+\nu-1),$$

with

$$G(t,s) := \begin{cases} G_1(t,s), & (t,s) \in T_1 \\ G_2(t,s), & (t,s) \in T_2 \end{cases},$$

where

$$\begin{aligned} G_1(t,s) :=& \frac{\phi_1(\alpha,\beta) t^{\underline{\nu-1}} + \beta(\nu-1) t^{\underline{\nu-2}}}{\Gamma(\nu)} \\ &\times \frac{[\gamma(\nu+b-s-1)^{\underline{\nu-1}} + \delta(\nu-1)(\nu+b-s-1)^{\underline{\nu-2}}]}{\phi_1(\alpha,\beta)\left[\gamma(b+\nu)^{\underline{\nu-1}} + \delta(\nu-1)(b+\nu)^{\underline{\nu-2}}\right] + \phi_2(\beta,\gamma,\delta,\nu,b)} \\ &- \frac{1}{\Gamma(\nu)}(t-s-1)^{\underline{\nu-1}} \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} G_2(t,s) :=& \frac{\phi_1(\alpha,\beta) t^{\underline{\nu-1}} + \beta(\nu-1) t^{\underline{\nu-2}}}{\Gamma(\nu)} \\ &\times \frac{[\gamma(\nu+b-s-1)^{\underline{\nu-1}} + \delta(\nu-1)(\nu+b-s-1)^{\underline{\nu-2}}]}{\phi_1(\alpha,\beta)\left[\gamma(b+\nu)^{\underline{\nu-1}} + \delta(\nu-1)(b+\nu)^{\underline{\nu-2}}\right] + \phi_2(\beta,\gamma,\delta,\nu,b)}. \end{aligned} \tag{3.3}$$

Proof. Observe that the general solution to problem (3.1) is $y(t) = C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}} - \Delta^{-\nu} h(t+\nu-1)$. Note, furthermore, that

$$\begin{aligned} \alpha y(\nu-2) &= \alpha \left[C_1(\nu-2)^{\underline{\nu-1}} + C_2(\nu-2)^{\underline{\nu-2}} - \Delta^{-\nu} h(t+\nu-1)\big|_{t=\nu-2} \right] \\ &= \alpha C_2 \Gamma(\nu-1) \end{aligned} \tag{3.4}$$

and that

$$\beta \Delta y(\nu-2) = \beta \left[C_1(\nu-1)\Gamma(\nu-1) + C_2(\nu-2)\Gamma(\nu-1) \right], \tag{3.5}$$

where we have tacitly used the fact that $\Delta\Delta^{-\nu} y(t) = \Delta^{1-\nu} y(t)$. So, it follows by applying the left-hand boundary condition together with the results of (3.4) and (3.5) above that

$$\begin{aligned} 0 &= \alpha y(\nu-2) - \beta \Delta y(\nu-2) \\ &= \alpha C_2 \Gamma(\nu-1) - \beta \left[C_1(\nu-1)\Gamma(\nu-1) + C_2(\nu-2)\Gamma(\nu-1) \right], \end{aligned} \tag{3.6}$$

whence (3.6) implies that

$$C_2 = \frac{\beta(\nu-1)}{\alpha - \beta(\nu-2)} C_1. \tag{3.7}$$

We next calculate both that

$$\gamma y(\nu+b) = \gamma \left[C_1(\nu+b)^{\underline{\nu-1}} + C_2(\nu+b)^{\underline{\nu-2}} - \left[\Delta^{-\nu} h(t-\nu+1) \right]_{t=\nu+b} \right] \tag{3.8}$$

and that

$$\begin{aligned} &\delta \Delta y(\nu+b) \\ &= \delta \left[C_1(\nu-1)(\nu+b)^{\underline{\nu-2}} + C_2(\nu-2)(\nu+b)^{\underline{\nu-3}} - \left[\Delta^{1-\nu} h(t+\nu-1) \right]_{t=\nu+b} \right]. \end{aligned} \tag{3.9}$$

So, putting (3.8) and (3.9) into the right-hand boundary condition, we get that

$$\begin{aligned} 0 &= \gamma y(\nu+b) + \delta \Delta y(\nu+b) \\ &= \gamma \left[C_1(\nu+b)^{\underline{\nu-1}} + C_2(\nu+b)^{\underline{\nu-2}} - \left[\Delta^{-\nu} h(t-\nu+1) \right]_{t=\nu+b} \right] \\ &\quad + \delta \left[C_1(\nu-1)(\nu+b)^{\underline{\nu-2}} + C_2(\nu-2)(\nu+b)^{\underline{\nu-3}} - \left[\Delta^{1-\nu} h(t+\nu-1) \right]_{t=\nu+b} \right] \\ &= \gamma \left[C_1(\nu+b)^{\underline{\nu-1}} + C_2(\nu+b)^{\underline{\nu-2}} - \frac{1}{\Gamma(\nu)} \sum_{s=0}^{b+1} (\nu+b-s-1)^{\underline{\nu-1}} h(s+\nu-1) \right] \\ &\quad + \delta \left[C_1(\nu-1)(\nu+b)^{\underline{\nu-2}} + C_2(\nu-2)(\nu+b)^{\underline{\nu-3}} \right] \\ &\quad - \delta \cdot \frac{1}{\Gamma(\nu-1)} \sum_{s=0}^{b+1} (\nu+b-s-1)^{\underline{\nu-2}} h(s+\nu-1), \end{aligned} \tag{3.10}$$

whence by putting (3.7) into (3.10) we get that

$$\begin{aligned}
&\gamma\left[C_1(\nu+b)^{\underline{\nu-1}}+\frac{\beta(\nu-1)}{\alpha-\beta(\nu-2)}C_1(\nu+b)^{\underline{\nu-2}}\right]\\
&-\gamma\frac{1}{\Gamma(\nu)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-1}}h(s+\nu-1)\\
&+\delta\left[C_1(\nu-1)(\nu+b)^{\underline{\nu-2}}+\frac{\beta(\nu-1)}{\alpha-\beta(\nu-2)}C_1(\nu-2)(\nu+b)^{\underline{\nu-3}}\right]\\
&-\delta\cdot\frac{\nu-1}{\Gamma(\nu)}\sum_{s=0}^{b+1}(\nu+b-s-1)^{\underline{\nu-2}}h(s+\nu-1)=0.
\end{aligned}\tag{3.11}$$

Now, a routine calculation shows that

$$\begin{aligned}
C_1=&\frac{1}{\Gamma(\nu)}\times\frac{\phi_1(\alpha,\beta)}{\phi_1(\alpha,\beta)\left[\gamma(\nu+b)^{\underline{\nu-1}}+\delta(\nu-1)(\nu+b)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)}\\
&\times\sum_{s=0}^{b+1}\left[\gamma(\nu+b-s-1)^{\underline{\nu-1}}+\delta(\nu-1)(\nu+b-s-1)^{\underline{\nu-2}}\right]h(s+\nu-1),
\end{aligned}\tag{3.12}$$

from which it follows at once that

$$\begin{aligned}
C_2=&\frac{\beta(\nu-1)}{\Gamma(\nu)}\times\frac{1}{\phi_1(\alpha,\beta)\left[\gamma(\nu+b)^{\underline{\nu-1}}+\delta(\nu-1)(\nu+b)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)}\\
&\times\sum_{s=0}^{b+1}\left[\gamma(\nu+b-s-1)^{\underline{\nu-1}}+\delta(\nu-1)(\nu+b-s-1)^{\underline{\nu-2}}\right]h(s+\nu-1).
\end{aligned}\tag{3.13}$$

Finally, as the general solution to (3.1) is given by $y(t)=C_1t^{\underline{\nu-1}}+C_2t^{\underline{\nu-2}}-\Delta^{-\nu}h(t+\nu-1)$, it is clear that the general solution (3.1) is given by $\sum_{s=0}^{b+1}G(t,s)h(s+\nu-1)$, where G is defined as in the statement of the theorem. And this completes the proof. □

Remark 3.2. As is easily checked, in case $\alpha=\gamma=1$ and $\beta=\delta=0$, Theorem 3.1 yields the Green's function for the conjugate problem (see [6]). On the other hand, in case $\alpha=\delta=1$ and $\beta=\gamma=0$, Theorem 3.1 yields the Green's function for the right-focal problem (see [14]). Thus, Theorem 3.1 is a generalization of the related theorems proved in both [6] and [14].

Remark 3.3. Note that in case $s\neq b+1$, we may suitably rewrite the Green's function by using the fact that

$$\frac{(\nu+b-s-1)^{\underline{\nu-1}}}{b-s+1}=(\nu+b-s-1)^{\underline{\nu-2}},\tag{3.14}$$

for $s \in [0,b]_{\mathbb{N}_0}$. In particular, doing so, we find that

$$\begin{aligned} G_1(t,s) &:= \frac{\phi(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}}{\Gamma(\nu)(b-s+1)} \\ &\quad\times \frac{[\gamma(b-s+1)+\delta(\nu-1)]\,(\nu+b-s-1)^{\underline{\nu-1}}}{\phi_1(\alpha,\beta)\,[\gamma(b+\nu)^{\underline{\nu-1}}+\delta(\nu-1)(b+\nu)^{\underline{\nu-2}}]+\phi_2(\beta,\gamma,\delta,\nu,b)} \\ &\quad- \frac{1}{\Gamma(\nu)}(t-s-1)^{\underline{\nu-1}}, \end{aligned} \tag{3.15}$$

for $(t,s) \in T_1 \cap [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}$, and

$$\begin{aligned} G_2(t,s) &:= \frac{\phi_1(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}}{\Gamma(\nu)(b-s+1)} \\ &\quad\times \frac{[\gamma(b-s+1)+\delta(\nu-1)]\,(\nu+b-s-1)^{\underline{\nu-1}}}{\phi_1(\alpha,\beta)\,[\gamma(b+\nu)^{\underline{\nu-1}}+\delta(\nu-1)(b+\nu)^{\underline{\nu-2}}]+\phi_2(\beta,\gamma,\delta,\nu,b)}, \end{aligned} \tag{3.16}$$

for $(t,s) \in T_2 \cap [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b]_{\mathbb{N}_0}$. Moreover, since $G(t,b+1)$ is clearly positive for each admissible t, in the sequel we shall use the alternative forms of $G(t,s)$ as given by (3.15)–(3.16) above.

4 Positivity of Green's Function

In this section we wish to prove that the Green's function derived in Section 3 is positive on its domain. This important result will allow us in Section 5 to deduce a relatively general comparison theorem. Let us note that in the sequel, the functions ϕ_1 and ϕ_2 retain their meaning from Section 3.

Moreover, as was intimated in Section 1, whereas in both the continuous fractional case and both the continuous and discrete integer-order settings, it is often not difficult at all to verify that a Green's function is nonnegative, in the discrete fractional case the analysis can be quite subtle. In particular, as will be seen in the sequel, it is somewhat taxing to demonstrate that our Green's function, $G(t,s)$, is nonnegative.

Theorem 4.1. *Let $G(t,s)$ be as given in the statement of Theorem 3.1. Then $G(t,s) \geq 0$ for all $(t,s) \in [\nu-2,\nu+b]_{\mathbb{N}_{\nu-2}} \times [0,b+1]_{\mathbb{N}_0}$.*

Proof. As explained in Remark 3.3, in the proof of this theorem we shall use the alternative forms of the Green's function, $G(t,s)$, as provided in (3.15)–(3.16). With this in mind, let us consider $G_2(t,s)$ first. Note that the factor

$$\frac{\phi_1(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}}{(b-s+1)\Gamma(\nu)}$$

is clearly positive since $\phi_1(\alpha,\beta)\geq 0$, seeing as $-(\nu-2)\geq 0$. So, we need only worry about the factor

$$\frac{\left[\gamma(b-s+1)+\delta(\nu-1)\right](\nu+b-s-1)^{\underline{\nu-1}}}{\phi_1(\alpha,\beta)\left[\gamma(b+\nu)^{\underline{\nu-1}}+\delta(\nu-1)(b+\nu)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)},$$

which occurs in $G_2(t,s)$. As the numerator of this factor is clearly nonnegative, it suffices to show that the denominator of this factor is likewise nonnegative.

To this end, observe that it suffices to show that

$$\phi_1(\alpha,\beta)\left[\gamma(b+\nu)^{\underline{\nu-1}}+\delta(\nu-1)(b+\nu)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)>0 \tag{4.1}$$

in case $\alpha=\gamma=0$. Indeed, as α, γ are increased in (4.1), it is clear that the left-hand side of (4.1) will increase. Thus, if the inequality holds in case $\alpha=\gamma=0$, then it holds in general. Putting $\alpha=\gamma=0$ in the left-hand side of (4.1), we find that

$$\begin{aligned}&\left\{\phi_1(\alpha,\beta)\left[\gamma(b+\nu)^{\underline{\nu-1}}+\delta(\nu-1)(b+\nu)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)\right\}_{\alpha=\gamma=0}\\&=-\beta\delta(\nu-1)(\nu-2)(\nu+b)^{\underline{\nu-2}}+\delta\beta(\nu-1)(\nu-2)(\nu+b)^{\underline{\nu-3}}\\&=\beta\delta(\nu-1)(\nu-2)\left[(\nu+b)^{\underline{\nu-3}}-(\nu+b)^{\underline{\nu-2}}\right].\end{aligned} \tag{4.2}$$

Note that in (4.2) we find that $\beta\delta(\nu-1)(\nu-2)\leq 0$. So, if (4.2) is positive, it must be the case that $(\nu+b)^{\underline{\nu-3}}-(\nu+b)^{\underline{\nu-2}}\leq 0$. But note that

$$\begin{aligned}(\nu+b)^{\underline{\nu-3}}-(\nu+b)^{\underline{\nu-2}}&=\frac{\Gamma(\nu+b+1)}{\Gamma(b+4)}-\frac{\Gamma(\nu+b+1)}{\Gamma(b+3)}\\&=\frac{(-b-2)\Gamma(\nu+b+1)}{\Gamma(b+4)}\leq 0,\end{aligned} \tag{4.3}$$

from which it follows that (4.1) is true. Thus, $G_2(t,s)>0$ on its domain, as claimed.

We next argue that $G_1(t,s)>0$ on its domain. Observe from the statement of Theorem 3.1 that $G_1(t,s)>0$ if and only if

$$\begin{aligned}&\frac{\phi_1(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}}{\Gamma(\nu)(b-s+1)}\\&\times\frac{\left[\gamma(b-s+1)+\delta(\nu-1)\right](\nu+b-s-1)^{\underline{\nu-1}}}{\phi_1(\alpha,\beta)\left[\gamma(b+\nu)^{\underline{\nu-1}}+\delta(\nu-1)(b+\nu)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)}\\&-\frac{1}{\Gamma(\nu)}(t-s-1)^{\underline{\nu-1}}\\&>0.\end{aligned} \tag{4.4}$$

But this is equivalent to requiring that

$$\begin{aligned}&\left[\phi_1(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}\right]\left[\gamma(b-s+1)+\delta(\nu-1)\right]\\&\times(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\\&-\left[\phi_1(\alpha,\beta)\left[\gamma(\nu+b)^{\underline{\nu-1}}+\delta(\nu-1)(\nu+b)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)\right]\\&\times\Gamma(t-s)(b-s+1)>0.\end{aligned} \tag{4.5}$$

Now, let us put $A(\alpha)$ equal to the left-hand side of (4.5) above. We claim that $A'(\alpha) \geq 0$ for all admissible α, β, γ, δ, ν, b, s, and t. To prove this claim, observe first that

$$\begin{aligned} A'(\alpha) &= t^{\underline{\nu-1}}[\gamma(b-s+1)+\delta(\nu-1)](\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1) \\ &\quad - \left[\gamma(\nu+b)^{\underline{\nu-1}}\Gamma(t-s)+\delta(\nu-1)(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)\right](b-s+1) \\ &= \gamma\left[t^{\underline{\nu-1}}(b-s+1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\right] \\ &\quad + \gamma\left[-(\nu+b)^{\underline{\nu-1}}\Gamma(t-s)(b-s+1)\right] \\ &\quad + \delta\left[t^{\underline{\nu-1}}(\nu-1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\right] \\ &\quad + \delta\left[-(\nu-1)(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1)\right]. \end{aligned} \tag{4.6}$$

We claim now both that

$$t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)-(\nu+b)^{\underline{\nu-1}}\Gamma(t-s) > 0 \tag{4.7}$$

and that

$$t^{\underline{\nu-1}}(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)-(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1) > 0, \tag{4.8}$$

which will suffice to prove the positivity of $A'(\alpha)$ since $\gamma, \delta \geq 0$.

To show that (4.7) holds, observe that (4.7) is true if and only if

$$\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+2)}{\Gamma(t-\nu+2)\Gamma(b-s+1)\Gamma(t-s)\Gamma(\nu+b+1)} > 1. \tag{4.9}$$

To show that (4.9) holds, let s_0 be an arbitrary but fixed element of $[0,b]_{\mathbb{N}_0}$ and notice that t must have the form $t = s_0+k+\nu$, for some $k \in \mathbb{N}_0$ satisfying $0 \leq k \leq b-s_0$. But it then follows that

$$\begin{aligned} &\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+2)}{\Gamma(t-\nu+2)\Gamma(b-s+1)\Gamma(t-s)\Gamma(\nu+b+1)} \\ &= \frac{\Gamma\left(s_0+k+\nu+1\right)\Gamma\left(\nu+b-s_0\right)\Gamma(k+1)\Gamma(b+2)}{\Gamma\left(s_0+k+2\right)\Gamma\left(b-s_0+1\right)\Gamma(k+\nu)\Gamma(\nu+b+1)} \\ &= \frac{\left[(s_0+k+\nu)\cdots(k+\nu)\right]\left[(b+1)\cdots(b-s_0+1)\right]}{\left[(k+s_0+1)\cdots(k+1)\right]\left[(\nu+b)\cdots(\nu+b-s_0)\right]} \\ &= \left[\frac{(s_0+k+\nu)\cdots(k+\nu)}{(k+s_0+1)\cdots(k+1)}\right]\cdot\left[\frac{(b+1)\cdots(b-s_0+1)}{(\nu+b)\cdots(\nu+b-s_0)}\right]. \end{aligned} \tag{4.10}$$

Now, observe that we may apply Lemma 2.5 to

$$\left[\frac{(s_0+k+\nu)\cdots(k+\nu)}{(k+s_0+1)\cdots(k+1)}\right]\cdot\left[\frac{(b+1)\cdots(b-s_0+1)}{(\nu+b)\cdots(\nu+b-s_0)}\right]. \tag{4.11}$$

Indeed, each of the fractions in the square brackets in (4.11) above contains (s_0+1)-factors. Moreover, it is clear that we may choose α_0, m_j, and n_j, for $1 \le j \le s_0+1$, in such a way that the hypotheses of Lemma 2.5 hold. Thus, we conclude at once that

$$\left[\frac{(s_0+k+\nu)\cdots(k+\nu)}{(k+s_0+1)\cdots(k+1)}\right]\cdot\left[\frac{(b+1)\cdots(b-s_0+1)}{(\nu+b)\cdots(\nu+b-s_0)}\right] > 1,$$

whence inequality (4.7) holds, too, as desired.

Similarly, to show that (4.8) holds, observe that proving (4.8) is equivalent to proving that

$$\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+3)}{\Gamma(t-\nu+2)\Gamma(b-s+1)\Gamma(\nu+b+1)\Gamma(t-s)(b-s+1)} > 1. \tag{4.12}$$

To show that (4.12) is true, we can, as before, let s_0 be a fixed but arbitrary element of $[0,b]_{\mathbb{N}_0}$. Then it follows, just as before, that $t = s_0+k+\nu$, for some $k \in \mathbb{N}_0$ satisfying $0 \le k \le b-s_0$. But then we observe that

$$\begin{aligned}
&\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+3)}{\Gamma(t-\nu+2)\Gamma(b-s+1)\Gamma(\nu+b+1)\Gamma(t-s)(b-s+1)}\\
&=\frac{\Gamma(s_0+k+\nu+1)\,\Gamma(\nu+b-s_0)\,\Gamma(k+1)\Gamma(b+3)}{\Gamma(s_0+k+2)\,\Gamma(b-s_0+1)\,\Gamma(\nu+b+1)\Gamma(k+\nu)}\cdot\frac{1}{b-s_0+1}\\
&=\frac{k!(b+2)!\Gamma(s_0+k+\nu+1)\,\Gamma(\nu+b-s_0)}{(s_0+k+1)!\,(b-s_0)!\Gamma(\nu+b+1)\Gamma(k+\nu)}\cdot\frac{1}{b-s_0+1}\\
&=\frac{b+2}{b-s_0+1}\cdot\left[\frac{(k+s_0+\nu)\cdots(k+\nu)}{(k+s_0+1)\cdots(k+1)}\right]\cdot\left[\frac{(b+1)\cdots(b-s_0+1)}{(\nu+b)\cdots(\nu+b-s_0)}\right].
\end{aligned} \tag{4.13}$$

Since by Lemma 2.5 it is clear that the last line of (4.13) above is certainly greater than unity, we find that (4.13) holds, whence (4.8) holds, too. In summary, then, we have shown that $A'(\alpha) > 0$ for all admissible α, β, γ, δ, ν, b, s, and t. In particular, this means that $G_1(t,s)$ is an increasing function of α.

We show next that $G_1(t,s)$ is also an increasing function in γ. To this end, let us put

$$\begin{aligned}
M(\gamma) :=& \left[\phi_1(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}\right]\\
&\times[\gamma(b-s+1)+\delta(\nu-1)](\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\\
&-\left[\phi_1(\alpha,\beta)\left[\gamma(\nu+b)^{\underline{\nu-1}}+\delta(\nu-1)(\nu+b)^{\underline{\nu-2}}\right]+\phi_2(\beta,\gamma,\delta,\nu,b)\right]\\
&\times\Gamma(t-s)(b-s+1).
\end{aligned} \tag{4.14}$$

We claim that $G'(\gamma) \ge 0$ for all admissible α, β, γ, δ, ν, b, s, and t. To prove this claim, observe first that

$$\begin{aligned}
&M'(\gamma)\\
&=\left[\phi_1(\alpha,\beta)t^{\underline{\nu-1}}+\beta(\nu-1)t^{\underline{\nu-2}}\right](b-s+1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\\
&-\left[\phi_1(\alpha,\beta)(\nu+b)^{\underline{\nu-1}}+\beta(\nu-1)(\nu+b)^{\underline{\nu-2}}\right]\Gamma(t-s)(b-s+1).
\end{aligned} \tag{4.15}$$

Now, observe from (4.15) above that $G'(\gamma)$ can be rewritten in the form

$$\begin{aligned} M'(\gamma) &= \phi_1(\alpha,\beta)\left[t^{\underline{\nu-1}}(b-s+1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\right] \\ &\quad + \phi_1(\alpha,\beta)\left[-(\nu+b)^{\underline{\nu-1}}\Gamma(t-s)(b-s+1)\right] \\ &\quad + \beta(\nu-1)\left[t^{\underline{\nu-2}}(b-s+1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\right] \\ &\quad + \beta(\nu-1)\left[-(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1)\right]. \end{aligned} \tag{4.16}$$

So, to prove that $M'(\gamma)\geq 0$, it suffices to prove both that

$$t^{\underline{\nu-1}}(b-s+1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)-(\nu+b)^{\underline{\nu-1}}\Gamma(t-s)(b-s+1)>0 \tag{4.17}$$

and that

$$t^{\underline{\nu-2}}(b-s+1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)-(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1)>0. \tag{4.18}$$

To show that (4.17) holds, observe that it suffices to show that

$$\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+2)}{\Gamma(t-\nu+2)\Gamma(b-s+1)\Gamma(\nu+b+1)\Gamma(t-s)}\geq 1. \tag{4.19}$$

But this follows at once from the proof of (4.9) above. Hence, (4.19) and thus (4.17) hold, as desired.

On the other hand, to show that (4.18) holds, we note that its truth is implied by the inequality

$$\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+3)}{\Gamma(t-\nu+3)\Gamma(b-s+1)\Gamma(\nu+b+1)\Gamma(t-s)}\geq 1. \tag{4.20}$$

To prove this inequality, let s_0 be an arbitrary but fixed element of $[0,b]_{\mathbb{N}_0}$, from which it follows, as before, that t has the form $t=s_0+k+\nu$, for some $k\in\mathbb{N}_0$ satisfying $0\leq k\leq b-s_0$. Then we find that

$$\begin{aligned} &\frac{\Gamma(t+1)\Gamma(\nu+b-s)\Gamma(t-s-\nu+1)\Gamma(b+3)}{\Gamma(t-\nu+3)\Gamma(b-s+1)\Gamma(\nu+b+1)\Gamma(t-s)} \\ &= \frac{\Gamma\left(s_0+k+\nu+1\right)\Gamma\left(\nu+b-s_0\right)\Gamma(k+1)(b+2)!}{\Gamma\left(s_0+k+3\right)\Gamma\left(b-s_0+1\right)\Gamma(\nu+b+1)\Gamma(k+\nu)} \\ &= \frac{(b+2)!k!\Gamma\left(s_0+k+\nu+1\right)\Gamma\left(\nu+b-s_0\right)}{(s_0+k+2)!\,(b-s_0)!\Gamma(\nu+b+1)\Gamma(k+\nu)} \\ &= \frac{(b+2)\cdots(b-s_0+1)}{(k+2+s_0)\cdots(k+1)}\cdot\frac{(k+\nu+s_0)\cdots(k+\nu)}{(b+\nu)\cdots(b+\nu-s_0)} \\ &= \frac{b+2}{k+2+s_0}\cdot\left[\frac{(b+1)\cdots(b-s_0+1)}{(b+\nu)\cdots(b+\nu-s_0)}\right]\cdot\left[\frac{(k+\nu+s_0)\cdots(k+\nu)}{(k+1+s_0)\cdots(k+1)}\right]. \end{aligned} \tag{4.21}$$

But note that $\dfrac{b+2}{k+2+s_0}\geq 1$. As we can apply Lemma 2.5 to the remaining two factors on the right-hand side of (4.21), it follows that (4.20) holds, whence (4.18) holds.

Therefore, we conclude that $M'(\gamma) > 0$ on its domain. Consequently, we find that $G_1(t,s)$ is also an increasing function in γ.

Now, having established that $G_1(t,s)$ is increasing both in α and in γ, we observe that to establish the positivity of G_1, it suffices to establish this claim in case $\alpha = \gamma = 0$ on account of the monotonicity of G_1 in these two parameters. In particular, let us put

$$\begin{aligned}
&G_1^*(t,s)\\
&:= \delta(\nu-1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\left[-\beta(\nu-2)t^{\underline{\nu-1}} + \beta(\nu-1)t^{\underline{\nu-2}}\right]\\
&- \Gamma(t-s)(b-s+1)\\
&\times \left[-\beta\delta(\nu-2)(\nu-1)(\nu+b)^{\underline{\nu-2}} + \delta\beta(\nu-1)(\nu-2)(\nu+b)^{\underline{\nu-3}}\right],
\end{aligned} \tag{4.22}$$

where G_1^* is merely the left-hand side of (4.5) with α and γ put equal to zero. We claim that $G_1^*(t,s)$ is positive on its domain, which will establish the positivity of $G_1(t,s)$ on its domain.

To carry out this program, then, let us begin by observing that

$$\begin{aligned}
&\frac{1}{\beta\delta}G_1^*(t,s)\\
&= (\nu-1)(\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)\left[-(\nu-2)t^{\underline{\nu-1}} + (\nu-1)t^{\underline{\nu-2}}\right]\\
&- \Gamma(t-s)\left[-(\nu-2)(\nu-1)(\nu+b)^{\underline{\nu-2}} + (\nu-1)(\nu-2)(\nu+b)^{\underline{\nu-3}}\right](b-s+1)\\
&= \left[-(\nu-2)t^{\underline{\nu-1}} + (\nu-1)t^{\underline{\nu-2}}\right](\nu+b-s-1)^{\underline{\nu-1}}\Gamma(t-s-\nu+1)(\nu-1)\\
&- (\nu-1)(\nu-2)(\nu+b)^{\underline{\nu-3}}\Gamma(t-s)(b-s+1)\\
&+ (\nu-2)(\nu-1)(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1)\\
&= \left[-(\nu-2)(t-\nu+2)t^{\underline{\nu-2}} + (\nu-1)t^{\underline{\nu-2}}\right]\\
&\times (b-s+1)(\nu+b-s-1)^{\underline{\nu-2}}\Gamma(t-s-\nu+1)(\nu-1)\\
&- \frac{(\nu-1)(\nu-2)}{b+3}(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1)\\
&+ (\nu-2)(\nu-1)(\nu+b)^{\underline{\nu-2}}\Gamma(t-s)(b-s+1).
\end{aligned} \tag{4.23}$$

So, (4.23) shows that to establish the positivity of G_1^*, it suffices to show that

$$\begin{aligned}
&t^{\underline{\nu-2}}(b-s+1)(\nu+b-s-1)^{\underline{\nu-2}}\Gamma(t-s-\nu+1)\left[-(\nu-2)(t-\nu+2) + (\nu-1)\right]\\
&+ (\nu+b)^{\underline{\nu-2}}(\nu-2)\Gamma(t-s)\left[-\frac{1}{b+3} + 1\right](b-s+1)\\
&> 0
\end{aligned} \tag{4.24}$$

holds.

Evidently, (4.24) holds if and only if

$$\frac{t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}}\Gamma(t-s-\nu+1)(b+3)\left[(\nu-2)(t-\nu+2)-(\nu-1)\right]}{(\nu-2)(\nu+b)^{\underline{\nu-2}}(b+2)\Gamma(t-s)}>1 \tag{4.25}$$

is true. But notice that we can group the factors in (4.25) in the following way:

$$\begin{aligned}&\frac{t^{\underline{\nu-2}}(\nu+b-s-1)^{\underline{\nu-2}}\Gamma(t-s-\nu+1)(b+3)\left[(\nu-2)(t-\nu+2)-(\nu-1)\right]}{(\nu-2)(\nu+b)^{\underline{\nu-2}}(b+2)\Gamma(t-s)}\\&=\left[\frac{(\nu+b-s-1)^{\underline{\nu-2}}}{(\nu+b)^{\underline{\nu-2}}}\cdot\frac{t^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}}\cdot\frac{b+3}{b+2}\right]\\&\times\left[\frac{1}{t-\nu+2}\cdot\frac{(\nu-2)(t-\nu+2)-(\nu-1)}{\nu-2}\right].\end{aligned} \tag{4.26}$$

As a consequence of the grouping in (4.26), we can note the following. First of all, it is not difficult to see that each of the first three factors on the right-hand side of (4.26) is greater than unity – namely, we have that

$$\frac{(\nu+b-s-1)^{\underline{\nu-2}}}{(\nu+b)^{\underline{\nu-2}}}\cdot\frac{t^{\underline{\nu-1}}}{(t-s-1)^{\underline{\nu-1}}}\cdot\frac{b+3}{b+2}\geq 1. \tag{4.27}$$

On the other hand, we must obtain an estimate on the magnitude of the second factor – namely,

$$\frac{1}{t-\nu+2}\cdot\frac{(\nu-2)(t-\nu+2)-(\nu-1)}{\nu-2}. \tag{4.28}$$

To this end, observe that when $\nu=1$, we find that (4.28) implies that

$$\left[\frac{1}{t-\nu+2}\cdot\frac{(\nu-2)(t-\nu+2)-(\nu-1)}{\nu-2}\right]_{\nu=1}=1. \tag{4.29}$$

On the other hand, if we put, for each fixed t,

$$N(\nu):=\frac{1}{t-\nu+2}\cdot\frac{(\nu-2)(t-\nu+2)-(\nu-1)}{\nu-2}, \tag{4.30}$$

where $N : [1,2)\to\mathbb{R}$, then a routine calculation shows that

$$\frac{d}{d\nu}N(\nu)=\frac{t-\nu^2+2\nu}{(t-\nu+2)^2(\nu-2)^2}\geq\frac{-\nu^2+3\nu-2}{(t-\nu+2)^2(\nu-2)^2}>0, \tag{4.31}$$

for each $\nu\in[1,2)$. In particular, (4.31) shows that $N(\nu)$ is increasing in ν. But (4.29) shows that $N(1)=1$. Consequently, from (4.29)–(4.31) we conclude that

$$\frac{1}{t-\nu+2}\cdot\frac{(\nu-2)(t-\nu+2)-(\nu-1)}{\nu-2}>1,$$

whenever $\nu \in (1, 2)$. (Note that when $\nu = 2$, $G_1^*(t, s)$ is clearly nonnegative, so there is no loss of generality here.) Putting this fact together with (4.27), we deduce at once that the product on the right-hand side of (4.26) is greater than unity – in other words, (4.25) holds. As this is true, we conclude that $G_1^*(t, s) > 0$ for each admissible pair (t, s). But by the comment earlier in the proof, this implies at once that $G_1(t, s) > 0$ on T_1. Finally, then, it follows that $G(t, s) > 0$, and the proof is complete. □

Remark 4.2. As in Section 3, we observe that the results of both [6] and [14] regarding the positivity of their Green's functions are hereby subsumed by Theorem 4.1 above.

Remark 4.3. Let us make some general comments about Theorem 4.1 and its proof. First of all, one may observe the technical nature of the proof; the use of the simple but effective Lemma 2.5 is essential here. This is quite different than in the integer-order case where proving the positivity of the Green's function is substantially simpler. Even in the *continuous* fractional setting it is usually simpler to deduce the positivity of a Green's function (e.g., [13]).

In addition to the technical interest of this result and its proof, it ought to be noted, as was remarked in Section 1, that this result provides an important step in the direction of a full analysis of problem (1.1), a task we do not undertake in this work. Given the existing literature on this general problem in the integer-order setting, it may be interesting to investigate the fractional analogue, in which case Theorem 4.1 shall be of use.

5 An Application

As an application of the positivity of the Green's function, we shall prove a general comparison result for problem (1.1). To do this, however, we first need to consider the inhomogeneous boundary value problem given by

$$\begin{cases} -\Delta^{\nu} y(t) = 0 \\ \alpha y(\nu-2) - \beta \Delta y(\nu-2) = A \\ \gamma y(\nu+b) + \delta \Delta y(\nu+b) = B \end{cases} . \tag{5.1}$$

We first state and prove a preliminary lemma regarding this problem.

Lemma 5.1. *Suppose that $\beta \neq 0$. Then the unique solution to problem* (5.1) *is given by*

$$y(t) = C_1 t^{\underline{\nu-1}} + C_2 t^{\underline{\nu-2}},$$

where

$$C_1 := \frac{\alpha C_2 - A - \beta C_2(\nu-2)}{\beta(\nu-1)} \tag{5.2}$$

and

$$C_2 := \left[B + \frac{A\gamma}{\beta(\nu-1)}(\nu+b)^{\underline{\nu-1}} + \frac{A\delta}{\beta}(\nu+b)^{\underline{\nu-2}}\right]$$
$$\times \frac{1}{\frac{\phi_1(\alpha,\beta)}{\beta(\nu-1)}\left[\gamma(\nu+b)^{\underline{\nu-1}} + \delta(\nu-1)(\nu+b)^{\underline{\nu-2}}\right] + \gamma(\nu+b)^{\underline{\nu-2}} + \delta(\nu-2)(\nu+b)^{\underline{\nu-3}}}. \tag{5.3}$$

On the other hand, suppose that $\beta = 0$*. Then the unique solution to problem* (5.1) *is given by the function* $y(t) = C_1^* t^{\underline{\nu-1}} + C_2^* t^{\underline{\nu-2}}$*, where*

$$C_1^* := \frac{B - \frac{A}{\alpha\Gamma(\nu-1)}\left[\gamma(\nu+b)^{\underline{\nu-2}} + \delta(\nu-2)(\nu+b)^{\underline{\nu-3}}\right]}{\gamma(\nu+b)^{\underline{\nu-1}} + \delta(\nu-1)(\nu+b)^{\underline{\nu-2}}} \tag{5.4}$$

and

$$C_2^* := \frac{A}{\alpha\Gamma(\nu-1)}. \tag{5.5}$$

Proof. The proof of this lemma is nearly identical to the first half of the proof of Theorem 3.1, and so, is omitted. □

Now, in the sequel let us put $Ly := \Delta^\nu y(t)$. We wish to use Lemma 5.1 together with Theorem 4.1 to prove a comparison theorem regarding the fractional operator L. In order to do this, we shall need to make certain assumptions regarding the numbers A and B appearing in problem (5.1). These assumptions will yield a few diverse situations under which our comparison theorem will hold. With this in mind, we have the following lemmas.

Lemma 5.2. *Let* $\psi(t)$ *be the solution to problem* (5.1) *given by Lemma 5.1. Provided that* $A = 0$*,* $B \geq 0$*, and* $\delta = 0$*, it follows that* $\psi(t) \geq 0$ *for* $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$.

Proof. By requiring $A = 0$, $B \geq 0$, and $\delta \geq 0$, it is clear from the form of (5.2) and (5.3) that $\psi(t) \geq 0$, as claimed. □

Lemma 5.3. *Let* $\psi(t)$ *be the solution to problem* (5.1) *given by Lemma 5.1. Provided that* $\beta = \gamma = 0$ *and that* A*,* $B \geq 0$*, it follows that* $\psi(t) \geq 0$ *for* $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$.

Proof. Upon setting $\gamma = 0$ in expression (5.4) and by the form of (5.4) and (5.5), this is immediate □

Lemma 5.4. *Let* $\psi(t)$ *be the solution to problem* (5.1) *given by Lemma 5.1. Provided that* $\beta = \delta = 0$ *and that* A*,* $B \geq 0$*, it follows that* $\psi(t) \geq 0$ *for* $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$.

Proof. Routine, and so, omitted. □

Remark 5.5. Observe that the boundary conditions implied by Lemma 5.3 are right-focal boundary conditions, whereas the boundary conditions implied by Lemma 5.4 are Dirichlet boundary conditions.

We now prove a comparison result for the operator Δ^ν. For convenience in the sequel, let us put $L_1y := \alpha y(\nu-2) - \beta\Delta y(\nu-2)$ and $L_2y := \gamma y(\nu+b) + \delta\Delta y(\nu+b)$. Let us also call hypothesis **(H1)** the hypotheses of Lemma 5.2, hypothesis **(H2)** the hypotheses of Lemma 5.3, and hypothesis **(H3)** the hypotheses of Lemma 5.4. We then get the following comparison-type theorem.

Theorem 5.6. *Let L be defined as above – that is, $Ly := \Delta^\nu y$. Suppose that u and v satisfy $Lu \le Lv$, $L_1u \ge L_1v$, and $L_2u \ge L_2v$. In addition, suppose that one of conditions* **(H1)**, **(H2)**, *or* **(H3)** *holds. Then $u(t) \ge v(t)$, for each $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$.*

Proof. Put $w := u - v$. Then it follows that w is a solution of the problem

$$\begin{cases} Lw = h(t+\nu-1) \\ L_1w = A \\ L_2w = B \end{cases}, \tag{5.6}$$

where $A,\ B \ge 0$ since $A := L_1u - L_1v$ and $B := L_2u - L_2v$, and $h(t+\nu-1) := Lu - Lv \le 0$. In particular, we know that w has the form

$$w(t) = \psi(t) - \sum_{s=0}^{b} G(t,s)h(s+\nu-1), \tag{5.7}$$

where ψ is the solution given by Lemma 5.1 and $G(t,s)$ is the Green's function from Theorem 3.1. Indeed, from (5.6)–(5.7), the definition of L, and the derivation of G, we find that

$$Lw = L\psi - L\Delta^{-\nu}h = 0 - \left[L\Delta^{-\nu}h\right] = 0 - (-h(t+\nu-1)) = h(t+\nu-1).$$

But as one of **(H1)**, **(H2)**, and **(H3)** holds, we have from Lemmas 5.2–5.4 that $\psi(t) \ge 0$, for $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$. Moreover, Theorem 4.1 shows that $G(t,s) > 0$ on its domain. So as $h(t+\nu-1) \le 0$, it follows at once that $w(t) \ge 0$, whence $u(t) \ge v(t)$, for each $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$, as claimed. □

Let us make three remarks regarding both the applicability and implications of this result.

Remark 5.7. Observe that using Theorem 5.6 together with condition **(H3)** implies that the operator L together with the Dirichlet boundary conditions, which was considered in [6], for example, satisfies the usual comparison theorem as is well-known in the classical theory of differential equations (cf., [19]) and as is also well-known in the more general time scales case (cf., [11]).

Remark 5.8. In case $\alpha = \gamma = 1$ and $\beta = \delta = 0$, the result of Theorem 5.6 implies that the ν-th fractional difference satisfies a sort of classical concavity property. This is interesting since the operator L evidently has no geometric interpretation unlike its classical incarnation in case $\nu = 2$. In particular, given $\nu \in (1, 2]$, the result of Theorem 5.6 can be recast by asserting that if $\Delta^{\nu} u(t) \leq \Delta^{\nu} v(t)$ and if both $u(\nu-2) \geq v(\nu-2)$ and $u(\nu+b) \geq v(\nu+b)$, then $u(t) \geq v(t)$, for all $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$. Of course, when $\nu = 2$, this is a well-known result with a clear geometric interpretation. When $\nu \neq 2$, although the geometry is evidently lost, this result shows that the abstract geometric condition still holds. And this implies that the ν-th fractional sum satisfies an abstract concavity property, which is mathematically interesting. This is made particularly clear by taking $v \equiv 0$; similarly, a convexity result is implied if we take $u \equiv 0$. We elucidate this claim in Example 5.10 in the sequel.

Remark 5.9. As stated in Remark 5.8 above, we have presently established that the fractional difference operator satisfies a type of concavity result. It perhaps ought to be noted that this fact might not automatically be expected. Indeed, as is well known from the existing literature on fractional boundary value problems (particularly in the continuous case), certain very important properties that hold in the integer-order case fail to hold in the fractional case. So, it seems useful to know that this particular property does remain true in the fractional case.

We conclude with an example illustrating the comparison result of Theorem 5.6. In particular, this example is related to both Remark 5.8 and Remark 5.9.

Example 5.10. Suppose $w : [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}} \to \mathbb{R}$ is a function satisfying

$$\Delta^{\nu} w(t) \leq 0, \tag{5.8}$$

for each $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$, together with the boundary conditions

$$w(\nu-2) \geq 0 \tag{5.9}$$

and

$$w(\nu+b) \geq 0. \tag{5.10}$$

Then using (5.8)–(5.10) we may apply Theorem 5.6 (by putting $v \equiv 0$) to deduce that

$$w(t) \geq 0, \tag{5.11}$$

for each $t \in [\nu-2, \nu+b]_{\mathbb{N}_{\nu-2}}$. Of course, this is trivial in case $\nu = 2$, for it is just a standard concavity result. But in the fractional case it seems less immediately clear. Indeed, it can be shown (by means of Definition 2.2 together with a straightforward, discrete Leibniz rule) that, for admissible $t \in \mathbb{N}_0$,

$$\Delta^{\nu} w(t) = \frac{1}{\Gamma(-\nu)} \sum_{s=\nu-2}^{t+\nu-2} (t-s-1)^{\underline{-\nu-1}} w(s) + w(t+\nu) - \nu w(t+\nu-1). \tag{5.12}$$

Given (5.8)–(5.10), it does not seem obvious that (5.12) implies that $w(t)$ is nonnegative for each admissible t. Moreover, when $\nu \neq 2$, we have no apparent geometric intuition. Therefore, Theorem 5.6 is helpful in deciding that (5.11) indeed holds.

Similarly, suppose $z : [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}} \to \mathbb{R}$ is a function satisfying

$$\Delta^{\nu} z(t) \geq 0, \tag{5.13}$$

for each $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$, together with the boundary conditions

$$z(\nu - 2) \leq 0 \tag{5.14}$$

and

$$z(\nu + b) \leq 0. \tag{5.15}$$

Then using (5.13)–(5.15) we may apply Theorem 5.6 (by putting $u \equiv 0$) to deduce that

$$z(t) \leq 0, \tag{5.16}$$

for each $t \in [\nu - 2, \nu + b]_{\mathbb{N}_{\nu-2}}$. Once again, this is trivial in case $\nu = 2$, for it is just a standard convexity result. But when $\nu \neq 2$, we are once more left without recourse to any sort of geometric intuition. Consequently, the result of Theorem 5.6 helps to show that a sort of convexity condition remains true, as might be hoped.

References

[1] O. P. Agrawal, *Formulation of Euler-Lagrange equations for fractional variational problems*, J. Math. Anal. Appl. 272 (2002), 368–379.

[2] F. M. Atici and P. W. Eloe, *Fractional q-calculus on a time scale*, J. Nonlinear Math. Phys., 14, 3 (2007), 333–344.

[3] ———, *A transform method in discrete fractional calculus*, Int. J. Difference Equ., 2 (2007), 165–176.

[4] ———, *Initial value problems in discrete fractional calculus*, Proc. Amer. Math. Soc., 137 (2009), 981–989.

[5] ———, *Discrete fractional calculus with the nabla operator*, Electronic J. of Qualitative Theory of Differential Equation (Spec. Ed. I), 3 (2009), 1–12.

[6] ———, *Two-point boundary value problems for finite fractional difference equations*, J. Difference Equ. Appl., 17 (2011), 445–456.

[7] F. M. Atici and S. Şengül, *Modeling with fractional difference equations*, J. Math. Anal. Appl., 369 (2010), 1–9.

[8] A. Babakhani and V. Daftardar-Gejji, *Existence of positive solutions of nonlinear fractional differential equations*, J. Math. Anal. Appl., 278 (2003), 434–442.

[9] Z. Bai and H. Lü, *Positive solutions for boundary value problem of nonlinear fractional differential equation*, J. Math. Anal. Appl., 311 (2005), 495–505.

[10] N. R. O. Bastos, et al., *Discrete-time fractional variational problems*, Signal Process., 91 (2011), 513–524.

[11] M. Bohner and A. Peterson, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, 2001.

[12] C. S. Goodrich, *Continuity of solutions to discrete fractional initial value problems*, Comput. Math. Appl., 59 (2010), 3489–3499.

[13] ———, *Existence of a positive solution to a class of fractional differential equations*, Appl. Math. Lett., 23 (2010), 1050–1055.

[14] ———, *Solutions to a discrete right-focal fractional boundary value problem*, Int. J. Difference Equ., 5 (2010), 195–216.

[15] ———, *Existence and uniqueness of solutions to a fractional difference equation with nonlocal conditions*, Comput. Math. Appl., 61 (2011), 191–202.

[16] ———, *Some new existence results for fractional difference equations*, Int. J. Dyn. Syst. Differ. Equ., 3 (2011), 145–162.

[17] ———, *Existence of a positive solution to a system of discrete fractional boundary value problems*, Appl. Math. Comput., 217 (2011), 4740–4753.

[18] ———, *On positive solutions to nonlocal fractional and integer-order difference equations*, Appl. Anal. Discrete Math., 5 (2011), 122–132.

[19] W. Kelley and A. Peterson, *The Theory of Differential Equations: Classical and Qualitative, First Edition*, Prentice Hall, Upper Saddle River, 2002.

[20] V. Lakshmikantham and A. S. Vatsala, *Basic theory of fractional differential equations*, Nonlinear Anal., 69 (2008), 2677–2682.

[21] K. Oldham and J. Spanier, *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, Dover Publications, Inc., Mineola, New York, 2002.

[22] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, 1999.

[23] H. G. Schuster (ed.), *Reviews of Nonlinear Dynamics and Complexity*, Wiley-VCH, Weinheim, 2008.

[24] X. Xu, D. Jiang, C. Yuan, *Multiple positive solutions for the boundary value problem of a nonlinear fractional differential equation*, Nonlinear Anal., 71 (2009), 4676–4688.