On Quasi sg-open and Quasi sg-closed Functions

O. Ravi, G. Ramkumar and S. Chandrasekar

1Department of Mathematics, P.M. Thevar College, Usilampatti, Madurai, Tamil Nadu, India. E-mail: siingam@yahoo.com
2Department of Mathematics, Rajapalayam Rajus’ College, Rajapalayam, Virudhunagar, Tamil Nadu, India E-mail: ramanujam_1729@yahoo.com.
3Department of Mathematics, Muthayammal Engineering College, Rasipuram, Namakkal, Tamil Nadu, India E-mail: chandrumat@gmail.com

Abstract

The purpose of this paper is to give a new type of open function called quasi sg-open function. Also, we obtain its characterizations and its basic properties.

2000 Mathematics Subject Classification: 54C10, 54C08, 54C05

Key words and phrases: Topological spaces, sg-open set, sg-closed set, sg-interior, sg-closure, quasi sg-open function.

1. Introduction and Preliminaries

Functions and of course open functions stand among the most important notions in the whole of mathematical science. Many different forms of open functions have been introduced over the years. Various interesting problems arise when one considers openness. Its importance is significant in various areas of mathematics and related sciences.

As a generalization of closed sets, the notion of sg-closed sets were introduced and studied by Bhattacharyya and Lahiri [1]. In this paper, we will continue the study of related functions by involving sg-open sets. We introduce and characterize the concept of quasi sg-open functions.

Throughout this paper, spaces mean topological spaces on which no separation axioms are assumed unless otherwise mentioned and \( f : (X, \tau) \rightarrow (Y, \sigma) \) (or simply \( f \)
: X → Y) denotes a function f of a space (X, τ) into a space (Y, σ). Let A be a subset of a space X. The closure and the interior of A are denoted by Cl(A) and Int(A), respectively.

**Definition 1.1.** A subset A of a space (X, τ) is called semi-open [5] if A ⊂ Cl(Int(A)).

The complement of a semi-open set is called semi-closed.

The semi-closure [3] of a subset A of X, denoted by sCl(A), is defined to be the intersection of all semi-closed sets containing A in X.

**Definition 1.2.** A subset A of a space X is called:

sg-closed [1] if sCl(A) ⊂ U whenever A ⊂ U and U is semi-open in X.

The complement of sg-closed set is called sg-open.

The union of all sg-open sets, each contained in a set A in a space X is called the sg-interior of A and is denoted by sg-Int(A) [7].

The intersection of all sg-closed sets containing a set A in a space X is called the sg-closure of A and is denoted by sg-Cl(A) [6].

**Definition 1.3.** A function f : (X, τ) → (Y, σ) is called:

(i) sg-irresolute [9] (resp. sg-continuous [9]) if f⁻¹(V ) is sg-closed in X for every sg-closed (resp. closed) subset V of Y ;

(ii) sg-open (resp. sg-closed [4]) if f(V ) is sg-open (resp. sg-closed) in Y for every open (resp. closed) subset of X.

### 2. Quasi sg-open Functions

We introduce a new definition as follows:

**Definition 2.1.** A function f : X → Y is said to be quasi sg-open if the image of every sg-open set in X is open in Y.

It is evident that, the concepts quasi sg-openness and sg-continuity coincide if the function is a bijection.

**Theorem 2.1.** A function f : X → Y is quasi sg-open if and only if for every subset U of X, f(sg-Int(U)) ⊂ Int(f(U)).

**Proof.** Let f be a quasi sg-open function. Now, we have Int(U) ⊂ U and sg-Int(U) is a sg-open set. Hence, we obtain that f(sg-Int(U)) ⊂ f(U). As f(sg-Int(U)) is open, f(sg-Int(U)) ⊂ Int(f(U)). Conversely, assume that U is a sg-open set in X. Then, f(U) = f(sg-Int(U)) ⊂ Int(f(U)) but Int(f(U)) ⊂ f(U). Consequently, f(U) = Int(f(U)) and hence f is quasi sg-open.

**Lemma 2.1.** If a function f : X → Y is quasi sg-open, then sg-Int(f⁻¹(G)) ⊂ f⁻¹(Int(G)) for every subset G of Y.
Proof. Let $G$ be any arbitrary subset of $Y$. Then, $\text{sg-Int}(f^{-1}(G))$ is a sg-open set in $X$ and $f$ is quasi sg-open, then $f(\text{sg-Int}(f^{-1}(G))) \subset \text{Int}(f^{-1}(G))) \subset \text{Int}(G)$. Thus, $\text{sg-Int}(f^{-1}(G)) \subset f^{-1}(\text{Int}(G))$.

Recall that a subset $S$ is called a sg-neighbourhood [7] of a point $x$ of $X$ if there exists a sg-open set $U$ such that $x \in U \subset S$.

**Theorem 2.2.** For a function $f : X \to Y$, the following are equivalent:

(i) $f$ is quasi sg-open;
(ii) For each subset $U$ of $X$, $f(\text{sg-Int}(U)) \subset \text{Int}(f(U))$;
(iii) For each $x \in X$ and each sg-neighbourhood $U$ of $x$ in $X$, there exists a neighbourhood $f(U)$ of $f(x)$ in $Y$ such that $V \subset f(U)$.

Proof. (i) $\Rightarrow$ (ii): It follows from Theorem 2.1.

(ii) $\Rightarrow$ (iii): Let $x \in X$ and $U$ be an arbitrary sg-neighbourhood of $x$ in $X$. Then there exists a sg-open set $V$ in $X$ such that $x \in V \subset U$. Then by (ii), we have $f(V) = f(\text{sg-Int}(V)) \subset \text{Int}(f(V))$ and hence $f(V) = \text{Int}(f(V))$. Therefore, it follows that $f(V)$ is open in $Y$ such that $f(x) \in f(V) \subset f(U)$.

(iii) $\Rightarrow$ (i): Let $U$ be an arbitrary sg-open set in $X$. Then for each $y \in f(U)$, by (iii) there exists a neighbourhood $V_y$ of $y$ in $Y$ such that $V_y \subset f(U)$. As $V_y$ is a neighbourhood of $y$, there exists an open set $W_y$ in $Y$ such that $y \in W_y \subset V_y$. Thus $f(U) = \bigcup \{W_y : y \in f(U)\}$ which is an open set in $Y$. This implies that $f$ is quasi sg-open function.

**Theorem 2.3.** A function $f : X \to Y$ is quasi sg-open if and only if for any subset $B$ of $Y$ and for any sg-closed set $F$ of $X$ containing $f^{-1}(B)$, there exists a closed set $G$ of $Y$ containing $B$ such that $f^{-1}(G) \subset F$.

Proof. Suppose $f$ is quasi sg-open. Let $B \subset Y$ and $F$ be a sg-closed set of $X$ containing $f^{-1}(B)$. Now, put $G = Y \setminus f(X \setminus F)$. It is clear that $f^{-1}(B) \subset F$ implies $B \subset G$. Since $f$ is quasi sg-open, we obtain $G$ as a closed set of $Y$. Moreover, we have $f^{-1}(G) \subset F$.

Conversely, let $U$ be a sg-open set of $X$ and put $B = Y \setminus f(U)$. Then $X \setminus U$ is a sg-closed set in $X$ containing $f^{-1}(B)$. By hypothesis, there exists a closed set $F$ of $Y$ such that $B \subset F$ and $f^{-1}(F) \subset X \setminus U$. Hence, we obtain $f(U) \subset Y \setminus F$. On the other hand, it follows that $B \subset F$, $Y \setminus F \subset Y \setminus B = f(U)$. Thus, we obtain $f(U) = Y \setminus F$ which is open and hence $f$ is a quasi sg-open function.

**Theorem 2.4.** A function $f : X \to Y$ is quasi sg-open if and only if $f^{-1}(\text{Cl}(B)) \subset \text{sg-Cl}(f^{-1}(B))$ for every subset $B$ of $Y$.

Proof. Suppose that $f$ is quasi sg-open. For any subset $B$ of $Y$, $f^{-1}(B) \subset \text{sg-Cl}(f^{-1}(B))$. Therefore by Theorem 2.3, there exists a closed set $F$ in $Y$ such that $B \subset F$ and $f^{-1}(F) \subset \text{sg-Cl}(f^{-1}(B))$. Therefore, we obtain $f^{-1}(\text{Cl}(B)) \subset f^{-1}(F) \subset \text{sg-Cl}(f^{-1}(B))$. 


Conversely, let $B \subseteq Y$ and $F$ be a sg-closed of $X$ containing $f^{-1}(B)$. Put $W = \text{Cl}_Y(B)$, then we have $B \subseteq W$ and $W$ is closed and $f^{-1}(W) \subseteq \text{sg-Cl}(f^{-1}(B)) \subseteq F$. Then by Theorem 2.3, $f$ is quasi sg-open.

**Lemma 2.2.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions and $g \circ f : X \to Z$ is quasi sg-open. If $g$ is continuous injective, then $f$ is quasi sg-open.

**Proof.** Let $U$ be a sg-open set in $X$. Then $(g \circ f)(U)$ is open in $Z$ since $g \circ f$ is quasi sg-open. Again $g$ is an injective continuous function, $f(U) = g^{-1}(g \circ f(U))$ is open in $Y$. This shows that $f$ is quasi sg-open.

### 3. Quasi sg-closed Functions

**Definition 3.1.** A function $f : X \to Y$ is said to be quasi sg-closed if the image of each sg-closed set in $X$ is closed in $Y$.

Clearly, every quasi sg-closed function is closed as well as sg-closed.

**Remark 3.1.** Every sg-closed (resp. closed) function need not be quasi sg-closed as shown by the following example.

**Example 3.1.** Let $X = Y = \{a, b, c\}$, $\tau = \{\emptyset, \{a, b\}, X\}$ and $\sigma = \{\emptyset, \{a\}, \{b, c\}, Y\}$. Define a function $f : (X, \tau) \to (Y, \sigma)$ by $f(a) = b$, $f(b) = c$ and $f(c) = a$. Then clearly $f$ is sg-closed as well as closed but not quasi sg-closed.

**Lemma 3.1.** If a function $f : X \to Y$ is quasi sg-closed, then $(f^{-1}(\text{Int}(B)) \subseteq \text{sg-Int}(f^{-1}(B)))$ for every subset $B$ of $Y$.

**Proof.** This proof is similar to the proof of Lemma 2.1.

**Theorem 3.1.** A function $f : X \to Y$ is quasi sg-closed if and only if for any subset $B$ of $Y$ and for any sg-open set $G$ of $X$ containing $f^{-1}(B)$, there exists an open set $U$ of $Y$ containing $B$ such that $f^{-1}(U) \subseteq G$.

**Proof.** This proof is similar to that of Theorem 2.3.

**Definition 3.2.** A function $f : X \to Y$ is called sg*-closed if the image of every sg-closed subset of $X$ is sg-closed in $Y$.

**Theorem 3.2.** If $f : X \to Y$ and $g : Y \to Z$ are two quasi sg-closed function, then $g \circ f : X \to Z$ is a quasi sg-closed function.

**Proof.** Obvious.

Furthermore, we have the following theorem:

**Theorem 3.3.** Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then:
(i) if $f$ is sg-closed and $g$ is quasi sg-closed, then $g \circ f$ is closed;
(ii) if $f$ is quasi sg-closed and $g$ is sg-closed, then $g \circ f$ is $sg^*$-closed;
(iii) if $f$ is $sg^*$-closed and $g$ is quasi sg-closed, then $g \circ f$ is quasi sg-closed.

**Proof.** Obvious.

**Theorem 3.4.** Let $f : X \to Y$ and $g : Y \to Z$ be two functions such that $g \circ f : X \to Z$ is quasi sg-closed. Then:

(i) if $f$ is sg-irresolute surjective, then $g$ is closed.
(ii) if $g$ is sg-continuous injective, then $f$ is $sg^*$-closed.

**Proof.** (i) Suppose that $F$ is an arbitrary closed set in $Y$. As $f$ is sg-irresolute, $f^{-1}(F)$ is sg-closed in $X$. Since $g \circ f$ is quasi sg-closed and $f$ is surjective, $(g \circ f)(f^{-1}(F)) = g(F)$, which is closed in $Z$. This implies that $g$ is a closed function.

(ii) Suppose $F$ is any sg-closed set in $X$. Since $g \circ f$ is quasi sg-closed, $(g \circ f)(F)$ is closed in $Z$. Again $g$ is a sg-continuous injective function, $g^{-1}(g \circ f(F)) = f(F)$, which is sg-closed in $Y$. This shows that $f$ is $sg^*$-closed.

**Theorem 3.5.** Let $X$ and $Y$ be topological spaces. Then the function $g : X \to Y$ is a quasi sg-closed if and only if $g(X)$ is closed in $Y$ and $g(V) \setminus g(X \setminus V)$ is open in $g(X)$ whenever $V$ is sg-open in $X$.

**Proof.** Necessity: Suppose $g : X \to Y$ is a quasi sg-closed function. Since $X$ is sg-closed, $g(X)$ is closed in $Y$ and $g(V) \setminus g(X \setminus V) = g(V) \cap g(X \setminus V)$ is open in $g(X)$ when $V$ is sg-open in $X$.

Sufficiency: Suppose $g(X)$ is closed in $Y$, $g(V) \setminus g(X \setminus V)$ is open in $g(X)$ when $V$ is sg-open in $X$, and let $C$ be closed in $X$. Then $g(C) = g(X) \setminus (g(X \setminus C) \setminus g(C))$ is closed in $g(X)$ and hence, closed in $Y$.

**Corollary 3.1.** Let $X$ and $Y$ be topological spaces. Then a surjective function $g : X \to Y$ is quasi sg-closed if and only if $g(V) \setminus g(X \setminus V)$ is open in $g(X)$ whenever $V$ is sg-open in $X$.

**Proof.** Obvious.

**Corollary 3.2.** Let $X$ and $Y$ be topological spaces and let $g : X \to Y$ be a sg-continuous quasi sg-closed surjective function. Then the topology on $X$ is $\{g(V) \setminus g(X \setminus V) : V \text{ is sg-open in } X\}$.

**Proof.** Let $W$ be open in $Y$. Then $g^{-1}(W)$ is sg-open in $X$, and $g(g^{-1}(W)) \setminus g(X \setminus g^{-1}(W)) = W$. Hence, all open sets in $Y$ are of the form $g(V) \setminus g(X \setminus V)$, $V$ is sg-open in $X$. On the other hand, all sets of the form $g(V) \setminus g(X \setminus V)$, $V$ is sg-open in $X$, are open in $Y$ from Corollary 3.1.
Definition 3.3. A topological space $(X, \tau)$ is said to be $sg^*$-normal if for any pair of disjoint $sg$-closed subsets $F_1$ and $F_2$ of $X$, there exist disjoint open sets $U$ and $V$ such that $F_1 \subset U$ and $F_2 \subset V$.

Theorem 3.6. Let $X$ and $Y$ be topological spaces with $X$ is $sg^*$-normal. If $g : X \to Y$ is a $sg$-continuous quasi $sg$-closed surjective function, then $Y$ is normal.

Proof. Let $K$ and $M$ be disjoint closed subsets of $Y$. Then $g^{-1}(K)$, $g^{-1}(M)$ are disjoint $sg$-closed subsets of $X$. Since $X$ is $sg^*$-normal, there exist disjoint open sets $V$ and $W$ such that $g^{-1}(K) \subset V$ and $g^{-1}(M) \subset W$. Then $K \subset g(V) \setminus g(X \setminus V)$ and $M \subset g(W) \setminus g(X \setminus W)$. Further by Corollary 3.1, $g(V) \setminus g(X \setminus V)$ and $g(W) \setminus g(X \setminus W)$ are open sets in $Y$ and clearly $(g(V) \setminus g(X \setminus V)) \cap (g(W) \setminus g(X \setminus W)) = \emptyset$. This shows that $Y$ is normal.

References