Symmetric Functions and Difference Equations with Asymptotically Period-two Solutions

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Abstract

This paper introduces easily verified conditions which guarantee that all solutions to the equation
\( y_n = f(y_{n-k}, y_{n-m}) \), with \( k, m \geq 1 \) and \( \gcd(k, m) = 1 \) are asymptotically periodic with period two. A recent result of Sun and Xi is employed. Several examples are included.

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1 Introduction

In this paper, we consider recursive equations of the form
\[ y_n = f(y_{n-k}, y_{n-m}), \]  
for \( n \geq 0 \), where \( k, m \geq 1 \), \( \gcd(k, m) = 1 \), \( s = \max\{k, m\} \) and \( y_{-s}, y_{-s+1}, \ldots, y_{-1} \in (0, \infty) \).

Recently, Sun and Xi [6] and Stević [4] proved the following interesting result regarding criteria for asymptotically two-periodic behavior of solutions to (1.1).

**Theorem 1.1.** Suppose that \( \{y_n\} \) satisfies (1.1), and in addition

1. \( f \in C((0, \infty)^2, (a, \infty)) \) with
\[ a = \inf_{(u,v) \in (0,\infty)^2} f(u,v) \geq 0, \]  
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(ii) \( f(u, v) \) is increasing in \( u \) and decreasing in \( v \),

(iii) there exists a decreasing function \( g \in C((a, \infty), (a, \infty)) \) such that

\[
g(g(x)) = x, \text{ for } x > a, \tag{1.3}
\]

\[
x = f(x, g(x)), \text{ for } x > a, \tag{1.4}
\]

and

\[
\lim_{x \to a^+} g(x) = \infty \text{ and } \lim_{x \to \infty} g(x) = a. \tag{1.5}
\]

Then, if \( k \) is even and \( m \) is odd, every positive solution to (1.1) converges to a (not necessarily prime) two-periodic solution. Otherwise, every solution converges to the unique equilibrium of (1.1).

For earlier related results and motivation, see [1, 2, 5].

Remark 1.2. Note that the self-inverse property in (1.3) gives that the range of \( g \) must comprise all of \((a, \infty)\). Hence under the assumption that \( g \) is decreasing, (1.5) is immediately satisfied and may be removed from the statement of Theorem 1.1.

2 Main Result and Examples

In this short note, we will give easily verified conditions for existence of a function \( g \) satisfying the requirements of Theorem 1.1. In particular, we will prove the following.

Theorem 2.1. Suppose that \( \{y_i\} \) satisfies (1.1) with \( f(u, v) = h(u, v)/v \) for some function \( h \) where

(i) \( h \in C((0, \infty)^2, (0, \infty)) \) is such that \( h(u, v) \) is symmetric in \( u \) and \( v \) and increasing in \( u \),

(ii) the function \( f \) is decreasing in \( v \),

(iii) with \( a \) as in (1.2), for all \( v > a \), there exist \( C_v \) and \( D_v \) (possibly infinite) such that

\[
\lim_{u \to a^+} f(u, v)/u = C_v > 1 \text{ and } \lim_{u \to \infty} f(u, v)/u = D_v < 1. \tag{2.1}
\]

Then, there exists a continuous function \( g \) satisfying (1.3), (1.4) and (1.5). Hence by Theorem 1.1, if \( k \) is even and \( m \) is odd, every positive solution to (1.1) converges to a (not necessarily prime) two-periodic solution, and otherwise, every positive solution converges to the unique equilibrium of (1.1).
Now, note that under assumptions (1) and (2) in Theorem 2.1, we have that, for fixed \( v > a \),
\[
\frac{f(u, v)}{u} = \frac{h(u, v)}{uv} = \frac{h(v, u)}{uv} = \frac{f(u, v)}{v} = \frac{f(v, u)}{v}
\] (2.2)
is decreasing in \( u \) and hence the (possibly infinite) limits in (2.1) exist. A key point here is that there is no need to have a closed form for \( g \) to verify the hypotheses of Theorem 1.1. In fact, finding such a closed form may not be very practical in practice (see for instance Example 2.6 below).

Before turning to a proof of Theorem 2.1, we give several examples of difference equations satisfying the requirements of the theorem.

**Example 2.2.** Consider the equation
\[
x_n = 1 + \frac{x_{n-k}}{x_{n-m}}.
\] (2.3)
Here we have \( h(u, v) = u + v, f(u, v) = 1 + u/v, a = 1 \), and for \( v > 1 \),
\[
\lim_{u \to 1^+} \frac{f(u, v)}{u} = \lim_{u \to 1^+} \frac{1}{u} + \frac{1}{v} = 1 + \frac{1}{v} > 1
\] (2.4)
and
\[
\lim_{u \to \infty} \frac{f(u, v)}{u} = \lim_{u \to \infty} \frac{1}{u} + \frac{1}{v} = \frac{1}{v} < 1.
\] (2.5)
Hence Theorem 2.1 is applicable and all positive solutions are asymptotically two-periodic whenever \( k \) is even and \( m \) is odd, and otherwise all positive solutions converge to the unique equilibrium. See [1–3] and the references therein for further discussion of Eq. (2.3).

**Example 2.3.** Consider the equation
\[
x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + \sqrt{\frac{x_{n-k}}{x_{n-m}}}.
\] (2.6)
Here we have \( h(u, v) = u + v + \sqrt{uv}, f(u, v) = 1 + u/v + \sqrt{u/v}, a = 1 \), and for \( v > 1 \),
\[
\lim_{u \to 1^+} \frac{f(u, v)}{u} = \lim_{u \to 1^+} \frac{1}{u} + \frac{1}{v} + \frac{1}{\sqrt{uv}} = 1 + \frac{1}{v} + \frac{1}{\sqrt{v}} > 1
\] (2.7)
and
\[
\lim_{u \to \infty} \frac{f(u, v)}{u} = \lim_{u \to \infty} \frac{1}{u} + \frac{1}{v} + \frac{1}{\sqrt{uv}} = \frac{1}{v} < 1.
\] (2.8)
By Theorem 2.1, we have the required asymptotic two-periodic behavior.
Example 2.4. Consider the equation
\[
x_n = \frac{1}{x_{n-m}} \left( \frac{x_{n-k}}{x_{n-k} + 1} + \frac{x_{n-m}}{x_{n-m} + 1} \right).
\] (2.9)
Here we have \(h(u, v) = u/(u + 1) + v/(v + 1)\), \(f(u, v) = (u/(u + 1) + v/(v + 1))/v\), \(a = 0\), and for \(v > 0\),
\[
\lim_{u \to 0^+} \frac{f(u, v)}{u} = \lim_{u \to 0^+} \frac{1}{v(u+1)} + \frac{1}{u(v+1)} = \infty
\] (2.10)
and
\[
\lim_{u \to \infty} \frac{f(u, v)}{u} = \lim_{u \to \infty} \frac{1}{v(u+1)} + \frac{1}{u(v+1)} = 0 < 1.
\] (2.11)
By Theorem 2.1, we have the required asymptotic two-periodic behavior.

Example 2.5. Consider the equation
\[
x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + \frac{\log(x_{n-k}x_{n-m})}{x_{n-m}}.
\] (2.12)
Here we have \(h(u, v) = u + v + \log(uv)\), \(f(u, v) = 1 + u/v + \log(uv)/v\), \(a = 1\), and for \(v > 1\),
\[
\lim_{u \to 1^+} \frac{f(u, v)}{u} = \lim_{u \to 1^+} \frac{1}{u} + \frac{1}{v} + \frac{\log(uv)}{uv} = 1 + \frac{1}{v} + \frac{\log(v)}{v} > 1
\] (2.13)
and
\[
\lim_{u \to \infty} \frac{f(u, v)}{u} = \lim_{u \to \infty} \frac{1}{u} + \frac{1}{v} + \frac{\log(uv)}{uv} = \frac{1}{v} < 1.
\] (2.14)
By Theorem 2.1, we have the required asymptotic two-periodic behavior.

Example 2.6. Consider the equation
\[
x_n = \frac{x_{n-k}^\alpha + x_{n-k}^\beta + x_{n-m}^\alpha + x_{n-m}^\beta}{x_{n-m}}.
\] (2.15)
Here, for \(0 < \alpha, \beta < 1\), we have \(h(u, v) = u^\alpha + u^\beta + v^\alpha + v^\beta\), \(f(u, v) = u^\alpha/v + u^\beta/v + v^{-(1-\alpha)} + v^{-(1-\beta)}\), \(a = 0\), and for \(v > 0\),
\[
\lim_{u \to 0^+} \frac{f(u, v)}{u} = \lim_{u \to 0^+} \frac{1}{vu^{1-\alpha}} + \frac{1}{vu^{1-\beta}} + \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} = \infty
\] (2.16)
and
\[
\lim_{u \to \infty} \frac{f(u, v)}{u} = \lim_{u \to \infty} \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} + \frac{1}{uv^{1-\alpha}} + \frac{1}{uv^{1-\beta}} = 0 < 1.
\] (2.17)
By Theorem 2.1, we have the required asymptotic two-periodic behavior.
Remark 2.7. To see that the requirement in (3) cannot be removed, consider the equation
\[ x_n = 1 + \frac{x_{n-k}}{x_{n-m}} + x_{n-k}. \] (2.18)
Here, we have \( h(u, v) = u + v + uv, \) \( f(u, v) = 1 + u/v + u \) and \( a = 1 \), yet the equation possesses no equilibrium. Indeed, for \( v > 1 \),
\[ \lim_{u \to 1^+} \frac{f(u, v)}{u} = \lim_{u \to 1^+} 1 + \frac{1}{u} + \frac{1}{v} = 2 + \frac{1}{v} > 1, \] (2.19)
but
\[ \lim_{u \to \infty} \frac{f(u, v)}{u} = \lim_{u \to \infty} 1 + \frac{1}{u} + \frac{1}{v} = 1 + \frac{1}{v} > 1. \] (2.20)
Condition (3) is not satisfied and Theorem 2.1 is not applicable.

3 Proof of the Main Result

We now turn to a proof of Theorem 2.1.

Proof of Theorem 2.1. By Theorem 1.1 and Remark 1.2, we need only show that there exists a decreasing continuous function \( g \) which satisfies (1.3) and (1.4). To that end, note that by (2.2), \( f(u, v)/u \) is decreasing in \( u \) and hence for fixed \( v > a \), by (3), there exists a unique \( u = g(v) \), say, which satisfies
\[ f(g(v), v)/g(v) = 1. \] (3.1)
To see that the function \( g \) is decreasing, note that for \( x > a \) and \( \epsilon > 0 \), by (2) and the definition of \( g \),
\[ \frac{f(g(x), x + \epsilon)}{g(x)} < \frac{f(g(x), x)}{g(x)} = 1 = \frac{f(g(x + \epsilon), x + \epsilon)}{g(x + \epsilon)}, \] (3.2)
and hence since \( f(u, v)/u \) is decreasing in \( u \), \( g(x + \epsilon) < g(x) \), as required.

Now, note that for \( x > a \), by the definition of \( g \) and the assumptions on \( f \) and \( h \), we have
\[ f(x, g(x)) = h(x, g(x))/g(x) = h(g(x), x) = f(g(x), x)x = x, \] (3.3)
and (1.4) is satisfied. Since \( u = g(g(x)) \) is the unique value satisfying \( f(u, g(x)) = u \), (3.3) gives that (1.3) also holds.

The continuity of \( g \) follows from its monotonicity and the fact that the range of \( g \) is \((a, \infty)\) (see Remark 1.2), and the theorem is proven. \( \square \)

Remark 3.1. Expanding on ideas in [1], Stević [4, 5] showed that similar asymptotically two-periodic behavior can occur for multivariable functions \( f \) in (1.1), with varying delays. The interested reader may verify that the ideas introduced here are applicable in that case as well.
References


[5] S. Stević, On the recursive sequence $x_n = 1 + \sum_{i=1}^{k} \alpha_i x_{n-p_i} / \sum_{j=1}^{m} \beta_j x_{n-q_j}$. Discrete Dynamics in Nature and Society Volume 2007 (2007), Article ID 39404, 7 pages, 2007.