Positive Periodic Solutions for Higher-order Functional Difference Equations*

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Abstract

In this paper, we apply a fixed point theorem to obtain sufficient conditions for the existence of positive periodic solutions for two classes of higher-order functional difference equations.


Keywords: Difference equations, periodic solutions, cone.

1. Introduction

The existence of positive periodic solutions of discrete mathematical models has been studied extensively in recent years, see [1, 5–12], for example,

(i) discrete model of blood cell production:

\[ \Delta x(n) = -a(n)x(n) + b(n) \frac{1}{1 + x^k(n - \tau(n))}, \quad k \in \mathbb{N}, \quad (1.1) \]

\[ \Delta x(n) = -a(n)x(n) + b(n) \frac{x(n - \tau(n))}{1 + x^k(n - \tau(n))}, \quad k \in \mathbb{N}, \quad (1.2) \]

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(ii) the periodic Michaelis–Menton discrete model:

\[ \Delta x(n) = a(n)x(n) \left[ 1 - \sum_{j=1}^{k} \frac{a_j(n)x(n - \tau_j(n))}{1 + c_j(n)x(n - \tau_j(n))} \right], \quad (1.3) \]

(iii) the single species discrete periodic population model:

\[ \Delta x(n) = x(n) \left[ a(n) - \sum_{j=1}^{k} b_j(n)x(n - \tau_j(n)) \right]. \quad (1.4) \]

Jiang, O'Regan and Agarwal in [4] have obtained the optimal existence theorem for

single and multiple positive periodic solutions to general functional difference equations

\[ \Delta x(n) = x(n)[a(n) - g(n, x(n - \tau_1(n)), \ldots, x(n - \tau_k(n))]], \quad (1.5) \]

\[ \Delta x(n) = -a(n)x(n) + g(n, x(n - \tau(n))). \quad (1.6) \]

Note that the equations (1.1)–(1.6) are first-order functional difference equations. Our aim of this paper is to study existence of positive periodic solutions for the higher-order difference equations

\[ x(n + m) = a(n)x(n) + f(n, x(n - \tau(n))), \quad (1.7) \]

\[ x(n + m) = a(n)x(n) - f(n, x(n - \tau(n))), \quad (1.8) \]

where \( a(n) = a(n + \omega), \quad f(n + \omega, u) = f(n, u), \quad \tau : \mathbb{Z} \to \mathbb{Z}, \quad \tau(n + \omega) = \tau(n) \) and \( \mathbb{Z} \) denotes the set of integers, \( \omega, m \in \mathbb{N} \). By using a fixed point theorem in a cone, we obtain existence results for single and multiple positive periodic solutions to the equation (1.7) and (1.8).

To prove our main results, we present an existence theorem.

**Theorem 1.1.** [2, 3] Let \( X \) be a Banach space and \( K \) be a cone in \( X \). Suppose \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( X \) such that \( 0 \in \Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2 \) and suppose that

\[ \Phi : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K \]

is a completely continuous operator such that

(i) \( \|\Phi u\| \leq \|u\| \) for \( u \in K \cap \partial \Omega_1 \), and there exists \( \psi \in \Omega_1 \setminus \{0\} \) such that \( u \neq \Phi u + \lambda \psi \) for \( u \in K \cap \partial \Omega_2 \) and \( \lambda > 0 \), or

(ii) \( \|\Phi u\| \leq \|u\| \) for \( u \in K \cap \partial \Omega_2 \), and there exists \( \psi \in \Omega_1 \setminus \{0\} \) such that \( u \neq \Phi u + \lambda \psi \) for \( u \in K \cap \partial \Omega_1 \) and \( \lambda > 0 \).

Then \( \Phi \) has a fixed point in \( K \cap (\bar{\Omega}_2 \setminus \Omega_1) \).
2. Positive Periodic Solutions of (1.7)

In this section we establish the existence of positive periodic solutions of equation (1.7). We always assume the following condition throughout this section:

\((H)\) 0 < \(a(n)\) < 1 for all \(n \in [0, \omega - 1]\) and \(f : \mathbb{Z} \times [0, \infty) \to [0, \infty)\) is continuous.

Let \(X\) be the set of all real \(\omega\)-periodic sequences. When endowed with the maximum norm \(\|x\| = \max_{n \in [0, \omega - 1]} |x(n)|\), \(X\) is a Banach space. Set \((m, \omega) = l, \omega / l = h\). From (1.7), we have that for any \(x \in X\)

\[
\frac{1}{a(n)} x(n + m) - x(n) = \frac{1}{a(n)} f(n, x(n - \tau(n)));
\]

\[
\frac{1}{a(n)a(n + m)} x(n + 2m) - \frac{1}{a(n)} x(n + m)
= \frac{1}{a(n)a(n + m)} f(n + m, x(n + m - \tau(n + m)));
\]

\[
\left(\prod_{i=0}^{h-1} \frac{1}{a(n + im)}\right) x(n + hm) - \left(\prod_{i=0}^{h-2} \frac{1}{a(n + im)}\right) x(n + (h - 1)m)
= \left(\prod_{i=0}^{h-1} \frac{1}{a(n + im)}\right) f(n + hm, x(n + (h - 1)m - \tau(n + (h - 1)m))).
\]

By summing the above equations and using periodicity of \(x\), we obtain the following result.

**Lemma 2.1.** \(x \in X\) is a solution of equation (1.7) if and only if

\[
x(n) = \left(\prod_{i=0}^{h-1} \frac{1}{a(n + im)} - 1\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n + jm)}\right) \times f(n + im, x(n + im - \tau(n + im)));
\]

(2.1)

Put

\[
M^* = \max \left\{ \prod_{i=0}^{h-1} a(n + im) : 0 \leq n \leq \omega - 1 \right\},
\]

\[
M_* = \min \left\{ \prod_{i=0}^{h-1} a(n + im) : 0 \leq n \leq \omega - 1 \right\},
\]

\[
\delta = \frac{M_*^2(1 - M^*)}{M^*(1 - M_*)}.
\]
Clearly, $\delta \in (0, 1)$. We define a cone by

$$P = \{ y \in X : y(n) \geq 0, n \in \mathbb{Z}, \ y(n) \geq \delta \|y\| \}. $$

and a mapping $T : X \to X$ by

$$(Tx)(n) = \left( \prod_{i=0}^{h-1} \frac{1}{a(n + im)} - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} \frac{1}{a(n + jm)} \right) \times f(n + im, x(n + im - \tau(n + im))).$$

By the nonnegativity of $f$ and $a$, $Tx(n) \geq 0$ on $[0, \omega - 1]$. It is clear that $(Tx)(n + \omega) = (Tx)(n)$ and $T$ is completely continuous on bounded subsets of $P$. Noting that

$$(Tx)(n) = \left( \prod_{i=0}^{h-1} \frac{1}{a(n + im)} - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} \frac{1}{a(n + jm)} \right) \times f(n + im, x(n + im - \tau(n + im)))$$

$$\leq \left( \frac{M_\ast}{M_\ast - M_*} \right)^{-1} \sum_{i=0}^{h-1} f(n + im, x(n + im - \tau(n + im))),$$

$$(Tx)(n) \geq \left( \frac{1}{M_\ast} - 1 \right)^{-1} \sum_{i=0}^{h-1} f(n + im, x(n + im - \tau(n + im))),$$

we easily obtain that $Tx(n) \geq \delta \|Tx\|$, that is, $T(P) \subset P$.

For convenience, we denote

$$\varphi(s) = \max \left\{ \frac{f(n, u)}{1 - a(n)} : n \in [0, \omega - 1], \ u \in [\delta s, s] \right\},$$

$$\psi(s) = \min \left\{ \frac{f(n, u)}{(1 - a(n))u} : n \in [0, \omega - 1], \ u \in [\delta s, s] \right\},$$

$$\varphi_0 = \lim_{s \to 0^+} \varphi(s), \ \varphi_\infty = \lim_{s \to \infty} \frac{\varphi(s)}{s},$$

$$\psi_0 = \lim_{s \to 0^+} \psi(s), \ \psi_\infty = \lim_{s \to \infty} \psi(s).$$

**Theorem 2.2.** Assume that $(H)$ holds, and there exist two positive constants $a, b$ with $a \neq b$ such that $\varphi(a) \leq a$ and $\psi(b) \geq 1$. Then the equation (1.7) has at least one positive solution $x \in X$ with $\min \{a, b\} \leq \|x\| \leq \max \{a, b\}$. 
Proof. Without loss of generality, we assume that $a < b$. Let $\Omega_1 = \{x \in X : \|x\| < a\}$ and $\Omega_2 = \{x \in X : \|x\| < b\}$. We claim that

(i) $\|Tx\| \leq \|x\|$, $x \in P \cap \partial \Omega_1$.

(ii) $x \neq Tx + \lambda$, $x \in P \cap \partial \Omega_2$ and $\lambda > 0$.

From $\varphi(a) \leq a$ and $\psi(b) \geq 1$, we have that

$$f(n, x) \leq (1-a(n))a, \ 0 \leq n \leq \omega - 1, \ a \delta \leq x \leq a,$$  \hspace{1cm} (2.2) $$f(n, x) \geq (1-a(n))x, \ 0 \leq n \leq \omega - 1, \ b \delta \leq x \leq b.$$ \hspace{1cm} (2.3)

To justify (i), let $x \in P \cap \partial \Omega_1$. Then $\|x\| = a$ and $\delta a \leq x(n) \leq a$ for $0 \leq n \leq \omega - 1$. It follows that

$$Tx(n) \leq \left( \prod_{i=0}^{h-1} a(n + im) - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} a(n + jm) \right) \times f(n + im, x(n + im - \tau(n + im)))$$

$$\leq \left( \prod_{i=0}^{h-1} a(n + im) - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} a(n + jm) \right) \times (1 - a(n + im))x(n + im - \tau(n + im))$$

$$\leq \left( \prod_{i=0}^{h-1} a(n + im) - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} a(n + jm) \right) (1 - a(n + im))a$$

$$\leq \|x\|.$$ 

This means that $\|Tx\| \leq \|x\|$ for all $x \in P \cap \partial \Omega_1$.

Next, we prove (ii). If not, there exist $x^* \in P \cap \partial \Omega_2$ and $\lambda_0 > 0$ such that $x^* = Tx^* + \lambda_0$.

Since $x^* \in P \cap \partial \Omega_2$, we have $\|x^*\| = b$ and $\delta b \leq x^*(n) \leq b$. Put $\chi = \min\{x(n), 0 \leq n \leq \omega - 1\}$. Then we have $\chi = x(n)$ for some $n \in [0, \omega - 1]$. Thus it follows that
\[ x^*(n) = (Tx^*)(n) + \lambda_0 = \left( \prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} \frac{1}{a(n+jm)} \right) \\
\times f(n+im, x(n+im - \tau(n+im))) + \lambda_0 \]
\[ \geq \left( \prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} \frac{1}{a(n+jm)} \right) \\
\times (1 - a(n+im))x(n+im - \tau(n+im)) + \lambda_0 \]
\[ \geq \chi \left( \prod_{i=0}^{h-1} \frac{1}{a(n+im)} - 1 \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} \frac{1}{a(n+jm)} \right) \\
\times (1 - a(n+im)) + \lambda_0 = \chi + \lambda_0, \]

and this implies \( \chi > \chi \), a contradiction.

Therefore, by Theorem 1.1, it follows that \( T \) has a fixed point \( x \in P \cap (\bar{\Omega}_2 \setminus \Omega_1) \).
Furthermore, \( a \leq \|x\| \leq b \) and \( x(n) \geq \delta a \), which means that \( x \) is a positive \( \omega \)-periodic solution of (1.7). The proof is complete. \( \blacksquare \)

**Corollary 2.3.** Assume that \((H)\) holds, and one of the following conditions holds:

(i) \( \varphi_0 < 1 \) and \( \psi_\infty > 1 \),

(ii) \( \varphi_\infty < 1 \) and \( \psi_0 > 1 \).

Then the equation (1.7) has at least one positive solution \( x \in X \) with \( \|x\| > 0 \).

**Theorem 2.4.** Assume that \((H)\) holds. There exist \( N + 1 \) positive constants \( p_1 < p_2 < \cdots < p_N < p_{N+1} \) such that one of the following conditions is satisfied:

(i) \( \varphi(p_{2k-1}) < p_{2k-1}, \ k = 1, 2, \ldots, [(N + 2)/2], \)
\( \psi(p_{2k}) > 1, \ k = 1, 2, \ldots, [(N + 1)/2], \)

(ii) \( \psi(p_{2k-1}) > 1, \ k = 1, 2, \ldots, [(N + 2)/2], \)
\( \psi(p_{2k}) < p_{2k}, \ k = 1, 2, \ldots, [(N + 1)/2], \)

where \([d]\) denotes the integer part of \( d \). Then the equation (1.7) has at least \( N \) positive solutions \( x_k \in X, k = 1, 2, \ldots, N \) with \( p_k < \|x_k\| < p_{k+1} \).

**Proof.** It is enough to prove case (i). Since \( \phi, \psi : (0, \infty) \to [0, \infty) \) are continuous, there exist \( p_k < a_k < b_k < p_{k+1}, \ k = 1, 2, \ldots, N \) such that
\[ \varphi(a_{2k-1}) \leq a_{2k-1}, \ \phi(b_{2k-1}) \geq 1, \ k = 1, 2, \ldots, [(N + 1)/2], \]
\[ \phi(a_{2k}) \geq 1, \ \varphi(b_{2k}) \leq b_{2k}, \ k = 1, 2, \ldots, \lfloor (N + 1)/2 \rfloor. \]

It follows by Theorem 2.2 that equation (1.7) has at least one positive periodic solution \( x_k \in X \) for every pair of numbers \( \{a_k, b_k\} \) with \( p_k < a_k \leq \|x_k\| \leq b_k < p_{k+1} \). The proof is complete. \[ \blacksquare \]

**Corollary 2.5.** Assume that \((H)\) holds, and the following conditions are satisfied:

(i) \( \varphi_0 < 1 \) and \( \varphi_\infty < 1 \),

(ii) there exists a positive constant \( b \) such that \( \psi(b) > 1 \).

Then the equation (1.7) has at least two positive solutions \( x_1, x_2 \in X \) with

\[ 0 < \|x_1\| < b < \|x_2\| < \infty. \]

**Corollary 2.6.** Assume that \((H)\) holds, and the following conditions are satisfied:

(i) \( \psi_0 > 1 \) and \( \psi_\infty > 1 \),

(ii) there exists a positive constant \( a \) such that \( \varphi(a) < a \).

Then the equation (1.7) has at least two positive solutions \( x_1, x_2 \in X \) with

\[ 0 < \|x_1\| < a < \|x_2\| < \infty. \]

### 3. Positive Periodic Solutions of (1.8)

In this section we establish the existence of positive periodic solutions of equation (1.8). We always assume the following condition throughout this section:

\((H^*)\) \( a(n) > 1 \) for all \( n \in [0, \omega - 1] \) and \( f: \mathbb{Z} \times [0, \infty) \to [0, \infty) \) is continuous.

The proofs of the results presented in this section are similar to those given in Section 2 and hence are omitted.

**Lemma 3.1.** \( x \in X \) is a solution of equation (1.8) if and only if

\[
x(n) = \left(1 - \prod_{i=0}^{h-1} \frac{1}{a(n+im)}\right)^{-1} \sum_{i=0}^{h-1} \left(\prod_{j=0}^{i} \frac{1}{a(n+jm)}\right) \\
\times f(n+im, x(n+im - \tau(n+im))),
\]

where \( X \) and \( h \) are defined in Section 2.

Put

\[
M^* = \max \left\{ \prod_{i=0}^{h-1} a(n+im) : 0 \leq n \leq \omega - 1 \right\},
\]
\[ M_\ast = \min \left\{ \prod_{i=0}^{h-1} a(n + im) : 0 \leq n \leq \omega - 1 \right\}, \]

\[ \delta^\ast = \frac{M_\ast - 1}{M_\ast(M^\ast - 1)}. \]

Clearly, \( \delta^\ast \in (0, 1) \). We define a cone by

\[ P = \{ y \in X : y(n) \geq 0, n \in \mathbb{Z}, y(n) \geq \delta^\ast \| y \| \}, \]

and a mapping \( T : X \to X \) by

\[ (Tx)(n) = \left( 1 - \prod_{i=0}^{h-1} \frac{1}{a(n + im)} \right)^{-1} \sum_{i=0}^{h-1} \left( \prod_{j=0}^{i} \frac{1}{a(n + jm)} \right) \times f(n + im, x(n + im - \tau(n + im))). \]

It is not difficult to verify that \( T(P) \subset P \) is completely continuous. Let

\[ \tilde{\varphi}(s) = \max \left\{ \frac{f(n, u)}{a(n) - 1} : n \in [0, \omega - 1], u \in [\delta^\ast s, s] \right\}, \]

\[ \tilde{\psi}(s) = \min \left\{ \frac{f(n, u)}{(a(n) - 1)u} : n \in [0, \omega - 1], u \in [\delta^\ast s, s] \right\}, \]

\[ \tilde{\varphi}_0 = \lim_{s \to 0^+} \frac{\tilde{\varphi}(s)}{s}, \quad \tilde{\varphi}_\infty = \lim_{s \to \infty} \frac{\tilde{\varphi}(s)}{s}, \]

\[ \tilde{\psi}_0 = \lim_{s \to 0^+} \tilde{\psi}(s), \quad \tilde{\psi}_\infty = \lim_{s \to \infty} \tilde{\psi}(s). \]

**Theorem 3.2.** Assume that \( (H^\ast) \) holds, and there exist two positive constants \( a, b \) with \( a \neq b \) such that \( \tilde{\varphi}(a) \leq a \) and \( \tilde{\psi}(b) \geq 1 \). Then the equation (1.8) has at least one positive solution \( x \in X \) with \( \min \{a, b\} \leq \| x \| \leq \max \{a, b\} \).

**Corollary 3.3.** Assume that \( (H^\ast) \) holds, and one of the following conditions holds:

(i) \( \tilde{\varphi}_0 < 1 \) and \( \tilde{\psi}_\infty > 1 \),

(ii) \( \tilde{\varphi}_\infty < 1 \) and \( \tilde{\psi}_0 > 1 \).

Then the equation (1.8) has at least one positive solution \( x \in X \) with \( \| x \| > 0 \).

**Theorem 3.4.** Assume that \( (H^\ast) \) holds, and there exist \( N + 1 \) positive constants \( p_1 < p_2 < \cdots < p_N < p_{N+1} \) such that one of the following conditions is satisfied:

(i) \( \tilde{\varphi}(p_{2k-1}) < p_{2k-1}, k = 1, 2, \ldots, [(N + 2)/2], \)
\[ \tilde{\psi}(p_{2k}) > 1, k = 1, 2, \ldots, [(N + 1)/2], \]
(ii) \( \tilde{\psi}(p_{2k-1}) > 1, \ k = 1, 2, \ldots, [(N + 2)/2], \)
\( \tilde{\varphi}(p_{2k}) < p_{2k}, \ k = 1, 2, \ldots, [(N + 1)/2], \)

where \([d]\) denotes the integer part of \(d\). Then the equation (1.8) has at least \(N\) positive solutions \(x_k \in X, k = 1, 2, \ldots, N\) with \(p_k < \|x_k\| < p_{k+1}\).

**Corollary 3.5.** Assume that \((H^*)\) holds, and the following conditions are satisfied:

(i) \( \tilde{\varphi}_0 < 1 \) and \( \tilde{\varphi}_\infty < 1 \),

(ii) there exists a positive constant \(b\) such that \(\tilde{\psi}(b) > 1\).

Then the equation (1.8) has at least two positive solutions \(x_1, x_2 \in X\) with

\[0 < \|x_1\| < b < \|x_2\| < \infty.\]

**Corollary 3.6.** Assume that \((H^*)\) holds, and the following conditions are satisfied:

(i) \( \tilde{\varphi}_0 > 1 \) and \( \tilde{\varphi}_\infty > 1 \),

(ii) there exists a positive constant \(a\) such that \(\tilde{\varphi}(a) < a\).

Then the equation (1.8) has at least two positive solutions \(x_1, x_2 \in X\) with

\[0 < \|x_1\| < a < \|x_2\| < \infty.\]

### 4. Some Examples

In this section, we apply the main results obtained in the previous sections to several examples.

**Example 4.1.** Consider the difference equation

\[x(n + 2) = a(n)x(n) + \frac{1}{1 + x(n - 2)},\]

(4.1)

where \(a\) is an \(\omega\)-periodic function with \(0 < a(n) < 1\) for all \(n \in [1, \omega]\). Obviously \(f(n, x) = 1/(x + 1)\) and \(\varphi_\infty = 0, \ \psi_0 = \infty\). By Corollary 2.3, (4.1) has at least one positive \(\omega\)-periodic solution.

**Example 4.2.** Consider the difference equation

\[x(n + 3) = a(n)x(n) + x^{100}(n - 5) + \frac{101}{100} \sin x(n - 5),\]

(4.2)

where \(a\) is an \(\omega\)-periodic function with \(0 < a(n) < 0.01\) for all \(n \in [1, \omega]\). It is clear that \(\psi_\infty = \infty, \ \psi_0 = 1.01 > 1\). Put \(a = \pi/6\). Then

\[f(n, x) = x^{100} + \frac{101}{100} \sin x < \frac{99}{100}a, \ 0 < x \leq a.\]
By Corollary 2.5, (4.2) has at least two positive $\omega$-periodic solutions.

**Example 4.3.** Consider the difference equation

$$x(n + 5) = a(n)x(n) - b(n)x^2(n + \tau(n)), \quad (4.3)$$

where $a, b, \tau$ are $\omega$-periodic functions with $a(n) > 1$, $b(n) > 0$ for all $n \in [1, \omega]$ and $\tau : \mathbb{Z} \to \mathbb{Z}$. By Corollary 3.3, (4.2) has at least one positive $\omega$-periodic solution.

**References**


