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A Nonlinear Sturm–Picone Comparison Theorem for Dynamic Equations on Time Scales

Boris Belinskiy
Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
E-mail: Boris-Belinskiy@utc.edu

John R. Graef*
Department of Mathematics, University of Tennessee at Chattanooga, Chattanooga, TN 37403, USA
E-mail: John-Graef@utc.edu

Sonja Petrović
Department of Mathematics, University of Kentucky, Lexington, KY 40506, USA
E-mail: petrovic@ms.uky.edu

Abstract

The authors derive an analog of the well-known Picone identity but for nonlinear dynamic equations on time scales. As a consequence, they obtain a nonlinear comparison theorem in the spirit of the classical Sturm–Picone comparison theorem. Comparison results yielding the nonoscillation of all solutions of nonlinear equations are also obtained.

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1. Introduction

In this paper, we consider dynamic equations on time scales of the form

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q\left(t, x^{\sigma}(t), x^{\Delta}(\sigma(t))\right) = r\left(t, x^{\sigma}(t), x^{\Delta}(\sigma(t))\right) \tag{1.1}$$

and give a nonlinear comparison theorem in the spirit of the classical Sturm–Picone comparison theorem. The literature on oscillation and nonoscillation of solutions of differential, difference, and now dynamic equations is a long and rich one stretching back to the work of Sturm in the 1830's. While we are especially interested here in results yielding the nonoscillation of all solutions, special cases of the theorems below include the well-known Sturm–Picone comparison theorem. We first derive a nonlinear Picone type identity, and as a consequence, we will be able to obtain some new oscillation and nonoscillation results for equations of the type (1.1). We will assume that the functions

$$p : \mathbb{T} \to \mathbb{R} \quad \text{and} \quad q,\ r : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$$

are rd-continuous, where $\mathbb{T}$ is a *time scale*, i.e., a closed subset of the real numbers $\mathbb{R}$. The *forward* and *backward* jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined in the usual way by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$

respectively, supplemented with $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. The function $\mu(t) = \sigma(t) - t$ is called the *graininess function*. We also use the usual notation that $x^{\sigma}(t) = x(\sigma(t))$. The point $t \in \mathbb{T}$ is *left-dense, left-scattered, right-dense,* or *right-scattered* if $\rho(t) = t$, $\rho(t) < t$, $\sigma(t) = t$, or $\sigma(t) > t$, respectively. The symbols $[a, b]$, $[a, b)$, etc. denote time scale intervals, for example,

$$[t_1, t_2] = \{t \in \mathbb{T} : t_1 \le t \le t_2\},$$

where $t_1, t_2 \in \mathbb{T}$ with $t_1 < \rho(t_2)$. We next define what is meant by a generalized zero of a solution.

Definition 1.1. A solution x of equation (1.1) has a *generalized zero at t* if either

$$x(t) = 0,$$

or t is left-scattered and

$$p(\rho(t))x(\rho(t))x(t) < 0.$$

It will be convenient to classify solutions according to their oscillatory behavior as follows.

Definition 1.2. A solution x of equation (1.1) will be called nonoscillatory if there exists $T \in \mathbb{T}$ such that x has no generalized zeros for $t \ge T$.

Definition 1.3. A solution x of equation (1.1) will be called oscillatory if for every $t_1 \in \mathbb{T}$ there exists $t_2 \in \mathbb{T}$ with $t_2 > t_1$ and

$$p(\rho(t_2))x(\rho(t_2))x(t_2) < 0.$$

Definition 1.4. A solution x of equation (1.1) will be said to be a Z-type solution if it has arbitrarily large generalized zeros but is ultimately nonnegative or nonpositive.

Remark 1.5. Notice that if $p > 0$ (or $p < 0$) and x is a Z-type solution, then eventually $x(t) = 0$ at each of its generalized zeros.

For additional basic facts about dynamic equations on time scales, we refer the reader to the monograph of Bohner and Peterson [2].

2. Comparison Results

We consider the pair of equations

$$\left(p(t)x^{\Delta}(t)\right)^{\Delta} + q\left(t, x^{\sigma}(t), x^{\Delta}(\sigma(t))\right) = r\left(t, x^{\sigma}(t), x^{\Delta}(\sigma(t))\right) \tag{2.1}$$

and

$$\left(P(t)y^{\Delta}(t)\right)^{\Delta} + Q\left(t, y^{\sigma}(t), y^{\Delta}(\sigma(t))\right) = 0, \tag{2.2}$$

where

$$p,\ P : \mathbb{T} \to \mathbb{R} \quad \text{and} \quad q,\ Q,\ r : \mathbb{T} \times \mathbb{R}^2 \to \mathbb{R}$$

are rd-continuous. We assume that

$$p(t) \geq P(t) > 0 \tag{2.3}$$

and

$$\frac{q(t, u_1, u_2)}{u_1} \leq \frac{Q(t, v_1, v_2)}{v_1} \quad \text{for all } t, u_1, u_2, v_1, v_2 \text{ with } u_1 \neq 0 \neq v_1. \tag{2.4}$$

Define $G : \mathbb{T} \to \mathbb{R}$ by

$$G(t) = \left\{ \frac{x(t)[p(t)x^{\Delta}(t)y(t) - P(t)x(t)y^{\Delta}(t)]}{y(t)} \right\}^{\Delta}.$$

Suppressing the argument t, we have

$$\begin{aligned}
G &= \left(\frac{x}{y}\right)^{\Delta}\left(px^{\Delta}y - Pxy^{\Delta}\right) + \frac{x^{\sigma}}{y^{\sigma}}\left(px^{\Delta}y - Pxy^{\Delta}\right)^{\Delta} \\
&= \frac{x^{\Delta}y - xy^{\Delta}}{yy^{\sigma}}\left(px^{\Delta}y - Pxy^{\Delta}\right) + \frac{x^{\sigma}}{y^{\sigma}}\big[(px^{\Delta})^{\Delta}y^{\sigma} + px^{\Delta}y^{\Delta} \\
&\quad - (Py^{\Delta})^{\Delta}x^{\sigma} - Px^{\Delta}y^{\Delta})\big] \\
&= p\frac{(x^{\Delta})^2y^2}{yy^{\sigma}} - P\frac{xyx^{\Delta}y^{\Delta}}{yy^{\sigma}} - p\frac{xx^{\Delta}yy^{\Delta}}{yy^{\sigma}} + P\frac{x^2(y^{\Delta})^2}{yy^{\sigma}} \\
&\quad + \frac{x^{\sigma}}{y^{\sigma}}\big[-q(t,x^{\sigma},x^{\Delta}(\sigma(t)))y^{\sigma} + r(t,x^{\sigma},x^{\Delta}(\sigma(t)))y^{\sigma} + px^{\Delta}y^{\Delta} \\
&\quad + Q(t,y^{\sigma},y^{\Delta}(\sigma(t)))x^{\sigma} - Px^{\Delta}y^{\Delta}\big] \\
&= p\frac{(x^{\Delta})^2y^2}{yy^{\sigma}} - P\frac{xyx^{\Delta}y^{\Delta}}{yy^{\sigma}} - p\frac{xx^{\Delta}yy^{\Delta}}{yy^{\sigma}} + P\frac{x^2(y^{\Delta})^2}{yy^{\sigma}} + \frac{x^{\sigma}}{y^{\sigma}}\big[px^{\Delta}y^{\Delta} - Px^{\Delta}y^{\Delta}\big] \\
&\quad + \frac{x^{\sigma}}{y^{\sigma}}\big[-q(t,x^{\sigma},x^{\Delta}(\sigma(t)))y^{\sigma} + Q(t,y^{\sigma},y^{\Delta}(\sigma(t)))x^{\sigma}\big] + r(t,x^{\sigma},x^{\Delta}(\sigma(t)))x^{\sigma}.
\end{aligned}$$

Now

$$\begin{aligned}
&\frac{x^{\sigma}}{y^{\sigma}}\big[-q(t,x^{\sigma},x^{\Delta}(\sigma(t)))y^{\sigma} + Q(t,y^{\sigma},y^{\Delta}(\sigma(t)))x^{\sigma}\big] \\
&\qquad\qquad = \frac{1}{yy^{\sigma}}[yy^{\sigma}(x^{\sigma})^2]\left[\frac{Q(t,y^{\sigma},y^{\Delta}(\sigma(t)))}{y^{\sigma}} - \frac{q(t,x^{\sigma},x^{\Delta}(\sigma(t)))}{x^{\sigma}}\right].
\end{aligned}$$

Using the fact that

$$u^{\sigma}(t) = u(t) + \mu(t)u^{\Delta}(t), \tag{2.5}$$

we obtain

$$\begin{aligned}
&p\frac{(x^{\Delta})^2y^2}{yy^{\sigma}} - P\frac{xyx^{\Delta}y^{\Delta}}{yy^{\sigma}} - p\frac{xx^{\Delta}yy^{\Delta}}{yy^{\sigma}} + P\frac{x^2(y^{\Delta})^2}{yy^{\sigma}} + \frac{x^{\sigma}}{y^{\sigma}}\big[px^{\Delta}y^{\Delta} - Px^{\Delta}y^{\Delta}\big] \\
&\quad = \frac{1}{yy^{\sigma}}\big[p(x^{\Delta})^2y^2 - pxx^{\Delta}yy^{\Delta} + px^{\Delta}yy^{\Delta}(x + \mu x^{\Delta})\big] \\
&\qquad + \frac{1}{yy^{\sigma}}\big[-Pxyx^{\Delta}y^{\Delta} + Px^2(y^{\Delta})^2 - Px^{\Delta}yy^{\Delta}(x + \mu x^{\Delta})\big] \\
&\quad = \frac{1}{yy^{\sigma}}\big[p(x^{\Delta})^2y^2 + p(x^{\Delta})^2yy^{\Delta}\mu - 2Pxyx^{\Delta}y^{\Delta} + Px^2(y^{\Delta})^2 - P(x^{\Delta})^2yy^{\Delta}\mu\big].
\end{aligned}$$

Since

$$(x^{\Delta}y - xy^{\Delta})^2 = (x^{\Delta})^2y^2 - 2xyx^{\Delta}y^{\Delta} + x^2(y^{\Delta})^2,$$

we can write

$$
\begin{aligned}
&-2Pxyx^{\Delta}y^{\Delta} + Px^2(y^{\Delta})^2 - P(x^{\Delta})^2yy^{\Delta}\mu \\
&\quad = P(x^{\Delta}y - xy^{\Delta})^2 - P(x^{\Delta})^2y^2 - Px^2(y^{\Delta})^2 + Px^2(y^{\Delta})^2 - P(x^{\Delta})^2yy^{\Delta}\mu.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{1}{yy^{\sigma}}\left[p(x^{\Delta})^2y^2 + p(x^{\Delta})^2yy^{\Delta}\mu - 2Pxyx^{\Delta}y^{\Delta} + Px^2(y^{\Delta})^2 - P(x^{\Delta})^2yy^{\Delta}\mu\right] \\
&\quad = \frac{P(x^{\Delta}y - xy^{\Delta})^2}{yy^{\sigma}} + \frac{1}{yy^{\sigma}}\left[(p-P)(x^{\Delta})^2y^2 + (p-P)(x^{\Delta})^2yy^{\Delta}\mu\right] \\
&\quad = \frac{P(x^{\Delta}y - xy^{\Delta})^2}{yy^{\sigma}} + \frac{1}{yy^{\sigma}}[(p-P)(x^{\Delta})^2y(y + y^{\Delta}\mu)] \\
&\quad = \frac{P(x^{\Delta}y - xy^{\Delta})^2}{yy^{\sigma}} + \frac{1}{yy^{\sigma}}(p-P)(x^{\Delta})^2yy^{\sigma} \\
&\quad = \frac{P(x^{\Delta}y - xy^{\Delta})^2}{yy^{\sigma}} + (p-P)(x^{\Delta})^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
G(t) = {} & \left[\frac{Q(t, y^{\sigma}, y^{\Delta}(\sigma(t)))}{y^{\sigma}} - \frac{q(t, x^{\sigma}, x^{\Delta}(\sigma(t)))}{x^{\sigma}}\right](x^{\sigma})^2 \\
& + \frac{P(x^{\Delta}y - xy^{\Delta})^2}{yy^{\sigma}} + (p-P)(x^{\Delta})^2 + r(t, x^{\sigma}, x^{\Delta}(\sigma(t)))x^{\sigma}. \qquad (2.6)
\end{aligned}
$$

We will make use of the nonlinear Picone type identity (2.6) in proving our comparison theorems.

Remark 2.1. The linear version of (2.6) was derived in [9].

In what follows, we will assume that not all inequalities in our hypotheses simultaneously become equalities on any open set.

Theorem 2.2. In addition to conditions (2.3)–(2.4), assume that

$$
r(t, u, v) \geq 0. \qquad (2.7)
$$

If equation (2.1) has a solution x with two generalized zeros at t_1 and t_2 in the interval $[a, \sigma^2(b)]$ with $x(t) > 0$ for $t \in (t_1, t_2)$, then every solution of (2.2) has a generalized zero in $[a, \sigma^2(b)]$.

Proof. Let x be a solution of (2.1) with two generalized zeros in $[a, \sigma^2(b))$ and let y be a solution of (2.2) that has no generalized zero in $[a, \sigma^2(b))$. Suppose that the two generalized zeros of $x(t)$ occur at the points $\sigma(c)$ and $\sigma(d)$, with

$$
a \leq c \leq \sigma(c) < d \leq \sigma(d) \leq \sigma^2(b)
$$

and $x(t) > 0$ for $t \in (c, d]$. For convenience, we define the auxiliary function $u : \mathbb{T} \to \mathbb{R}$ by

$$u(t) = \begin{cases} 0, & a \le t \le c, \\ x(t), & c < t \le d, \\ 0, & d < t \le \sigma^2(b). \end{cases}$$

Integrating the nonlinear Picone identity (2.6) with the function x replaced by u, we obtain

$$I = \int_a^{\sigma^2(b)} \left[\frac{u}{y}(pu^\Delta y - Puy^\Delta) \right]^\Delta (t)\Delta t.$$

Observing that

$$\int_a^{\sigma^2(b)} = \int_a^c + \int_c^{\sigma(c)} + \int_{\sigma(c)}^d + \int_d^{\sigma(d)} + \int_{\sigma(d)}^{\sigma^2(b)}$$

and using the fact that

$$\int_t^{\sigma(t)} f(s)\Delta s = \mu(t)f(t), \tag{2.8}$$

we have

$$I = \mu(c)G(c) + \int_{\sigma(c)}^d G(t)\Delta t + \mu(d)G(d). \tag{2.9}$$

We will consider three cases depending on the nature of the points c and d.

Case I: If both c and d are right-dense, then $u^\sigma(c) = u(c) = x(c) = 0 = x(d) = u(d)$ and $\mu(c) = \mu(d) = 0$, so

$$\begin{aligned} I = u(d)[p(d)u^\Delta(d)y(d) - P(d)u^\Delta(d)]/y(d) \\ - u(c)[p(c)u^\Delta(c)y(c) - P(c)u^\Delta(c)]/y(c) = 0. \end{aligned} \tag{2.10}$$

Case II: If c is right-dense and d is right-scattered, then $x(c) = x^\sigma(c) = \mu(c) = 0$, $x(d) > 0$, and $x^\sigma(d) < 0$. Since $u(d) = x(d)$, we see that

$$u^\Delta(d) = \frac{u^\sigma(d) - u(d)}{\mu(d)} = \frac{-x(d)}{\mu(d)},$$

and a simple calculation shows that

$$\left(\frac{u}{y}\right)^\Delta (d) = \frac{-x(d)}{\mu(d)y(d)}.$$

From (2.9), we have

$$
\begin{aligned}
I &= u(d)[p(d)u^{\Delta}(d)y(d) - P(d)u(d)y^{\Delta}(d)]/y(d) \\
&\quad + \mu(d)\left\{\left(\frac{u}{y}\right)^{\Delta}(d)\left[pu^{\Delta}y - Puy^{\Delta}\right](d) + \frac{u(\sigma(d))}{y(\sigma(d))}\left[pu^{\Delta}y - Puy^{\Delta}\right]^{\Delta}(d)\right\} \\
&= u(d)\Big[\big(p(d)u^{\Delta}(d)y(d) - P(d)u(d)y^{\Delta}(d)\big)/y(d) \\
&\quad - \frac{x(d)}{y(d)}(p(d)u^{\Delta}(d)y(d) - P(d)u(d)y^{\Delta}(d))\Big] \\
&= 0.
\end{aligned}
$$

Case III: If both c and d are right-scattered, then $u(c) = 0$, $u^{\sigma}(c) = x^{\sigma}(c)$, $u(d) = x(d)$, $u^{\sigma}(d) = 0$, $\mu(c) > 0$, and $\mu(d) > 0$. Hence,

$$
u^{\Delta}(c) = \frac{u^{\sigma}(c) - u(c)}{\mu(c)} = \frac{x^{\sigma}(c)}{\mu(c)}
$$

and

$$
u^{\Delta}(d) = -\frac{x(d)}{\mu(d)}.
$$

Direct calculations give

$$
\left[\frac{u}{y}\right]^{\Delta}(c) = \frac{x^{\sigma}(c)}{\mu(c)y^{\sigma}(c)}
$$

and

$$
\left[\frac{u}{y}\right]^{\Delta}(d) = \frac{-x(d)}{\mu(d)y(d)}.
$$

In this case, (2.9) becomes

$$
\begin{aligned}
&\left(\frac{u}{y}(pu^{\Delta}y - Puy^{\Delta})\right)(d) - \left(\frac{u}{y}(pu^{\Delta}y - Puy^{\Delta})\right)(\sigma(c)) \\
&\quad + \mu(c)\left[\left(\frac{u}{y}\right)^{\Delta}(c)(pu^{\Delta}y - Puy^{\Delta})^{\sigma}(c) + \left(\frac{u}{y}\right)(c)(pu^{\Delta}y - Puy^{\Delta})^{\Delta}(c)\right] \\
&\quad + \mu(d)\left[\left(\frac{u}{y}\right)^{\Delta}(d)(pu^{\Delta}y - Puy^{\Delta})(d) + \left(\frac{u}{y}\right)^{\sigma}(d)(pu^{\Delta}y - Puy^{\Delta})^{\Delta}(d)\right]
\end{aligned}
$$

$$\begin{aligned}
&= \left(\frac{u}{y}\right)(d)(pu^{\Delta}y - Puy^{\Delta})(d) - \left(\frac{u}{y}\right)^{\sigma}(c)(pu^{\Delta}y - Puy^{\Delta})^{\sigma}(c) \\
&\quad + \mu(c)\frac{x^{\sigma}(c)}{\mu(c)y^{\sigma}(c)}(pu^{\Delta}y - Puy^{\Delta})^{\sigma}(c) + \mu(d)\frac{-x(d)}{\mu(d)y(d)}(pu^{\Delta}y - Puy^{\Delta})(d) \\
&= \frac{x(d)}{y(d)}(pu^{\Delta}y - Puy^{\Delta})(d) - \frac{x^{\sigma}(c)}{y^{\sigma}(c)}(pu^{\Delta}y - Puy^{\Delta})^{\sigma}(c) \\
&\quad + \frac{x^{\sigma}(c)}{y^{\sigma}(c)}(pu^{\Delta}y - Puy^{\Delta})^{\sigma}(c) - \frac{x(d)}{y(d)}(pu^{\Delta}y - Puy^{\Delta})(d) = 0.
\end{aligned}$$

In each of the above three cases, we see that an integration of the left-hand side of the nonlinear Picone identity (2.6) from a to $\sigma^2(b)$ is 0, but in view of conditions (2.3) and (2.4), the integral of the right-hand side is positive. Therefore, y must have a generalized zero in the interval $[a, \sigma^2(b))$. ■

The proof of the following result should now be clear.

Theorem 2.3. In addition to conditions (2.3)–(2.4), assume that

$$r(t, u, v) \le 0. \tag{2.11}$$

If equation (2.1) has a solution x with two generalized zeros at t_1 and t_2 in the interval $[a, \sigma^2(b)]$ with $x(t) < 0$ for $t \in (t_1, t_2)$, then every solution of (2.4) has a generalized zero in $[a, \sigma^2(b)]$.

The following corollaries are immediate consequences of the above theorems.

Corollary 2.4. If conditions (2.3)–(2.4) and (2.7), ((2.11)) hold and equation (2.1) has an oscillatory or nonnegative (nonpositive) Z-type solution, then every solution of (2.2) is oscillatory or Z-type.

Corollary 2.5. Assume that conditions (2.3)–(2.4) and (2.7), ((2.11)) hold. If there is a solution of (2.2) with no generalized zeros in $[a, \sigma^2(b)]$, then no solution of (2.1) that is nonnegative (nonpositive) can vanish more than once there.

Of special interest here are nonoscillation results, so we have the following theorem.

Theorem 2.6. Suppose conditions (2.3)–(2.4) and (2.7) hold and

$$q(t, u, v) \le 0 \quad \text{for} \quad u \le 0. \tag{2.12}$$

If equation (2.2) has a nonoscillatory solution, then every solution of (2.1) is nonoscillatory.

Proof. It follows from Corollary 2.4 that no solution of (2.1) is oscillatory or nonnegative Z-type. Suppose x is a nonpositive Z-type solution of (2.1), say $x(t_1) = 0$ and $x(t) \le 0$

for $t \geq t_1$. From equation (2.1), we have

$$(p(t)x^{\Delta}(t))^{\Delta} = r(t, x^{\sigma}(t), x^{\Delta}(\sigma(t))) - q(t, x^{\sigma}(t), x^{\Delta}(\sigma(t))) \geq 0$$

for $t \geq t_1$. This implies that $p(t)x^{\Delta}(t)$ is eventually of one sign which is impossible for a Z-type solution. ■

Remark 2.7. A result analogous to Theorem 2.6 can be obtained by replacing (2.7) with (2.11) and (2.12) with

$$q(t, u, v) \geq 0 \quad \text{for} \quad u \geq 0. \tag{2.13}$$

We can obtain additional results in this same spirit by replacing conditions (2.7) and (2.11) with the single condition

$$u\, r(t, u, v) \geq 0 \quad \text{for} \quad u \neq 0. \tag{2.14}$$

This yields the following theorem.

Theorem 2.8. Assume that conditions (2.3), (2.4), and (2.14) hold.

(i) If equation (2.1) has an oscillatory or Z-type solution, then every solution of (2.2) is oscillatory.

(ii) If

$$u\, q(t, u, v) \geq 0 \quad \text{for} \quad u \neq 0$$

and equation (2.2) has a nonoscillatory solution, then every solution of (2.1) is nonoscillatory.

3. Consequences of the Main Results and Further Discussion

If, in equations (2.1) and (2.2), we have

$$r(t, u, v) \equiv r_1(t),$$

$$q(t, u, v) = q_1(t)u, \quad \text{and} \quad Q(t, u, v) = Q_1(t)u,$$

then our pair of equations become

$$(p(t)x^{\Delta}(t))^{\Delta} + q_1(t)x^{\sigma}(t) = r_1(t) \tag{3.1}$$

and

$$(P(t)y^{\Delta}(t))^{\Delta} + Q_1(t)y^{\sigma}(t) = 0. \tag{3.2}$$

As a consequence of the results here, we then have the following corollary.

Corollary 3.1. Suppose that

$$p(t) \geq P(t) \geq 0 \quad \text{and} \quad 0 \leq q_1(t) \leq Q_1(t), \tag{3.3}$$

and that equation (3.2) is nonoscillatory. If either $r_1(t) \geq 0$ or $r_1(t) \leq 0$, then every solution of equation (3.1) is nonoscillatory.

This corollary was first proved for differential equations by Švec [11] in 1963, rediscovered by Hammett [6] and Keener [7] and extended by Graef and Spikes [4].

If the time scale $\mathbb{T}$ is the set of real numbers $\mathbb{R}$, then Theorem 2.2 (or Theorem 2.3) becomes a nonlinear generalization of the well-known interlacing theorem. All the results here agree with the usual Sturm–Picone comparison results for unforced linear differential equations (see, for example, Swanson [12]). If the time scale $\mathbb{T}$ is the integers $\mathbb{Z}$, then we obtain the corresponding results for difference equations (see, for example, Agarwal [1] or Kelley and Peterson [8] in the linear case). Generalizations of the Sturm–Picone comparison theorem to higher order nonlinear difference equations can be found in Graef, Miciano–Cariño, and Qian [3], and of course it would be of interest to extend the results here in that direction as well.

We also wish to point out that for the nonlinear equations

$$(p(t)x^{\Delta}(t))^{\Delta} + q_1(t)f(x^{\sigma}(t)) = 0 \tag{3.4}$$

and

$$(P(t)y^{\Delta}(t))^{\Delta} + Q_1(t)F(y^{\sigma}(t)) = 0 \tag{3.5}$$

again with (3.3) holding and

$$0 < \frac{f(u)}{u} \leq \frac{F(v)}{v} \quad \text{for all } u, v \text{ with } u \neq 0 \neq v, \tag{3.6}$$

we have that if equation (3.5) has a nonoscillatory solution, then every solution of equation (3.4) is nonoscillatory. For example, if (3.3) holds and the linear equation

$$(P(t)y^{\Delta}(t))^{\Delta} + Q_1(t)y^{\sigma}(t) = 0 \tag{3.7}$$

is nonoscillatory, then every solution of the nonlinear equations

$$(p(t)x^{\Delta}(t))^{\Delta} + q_1(t)\frac{(x^{\sigma}(t))^3}{1 + (x^{\sigma}(t))^2} = 0, \tag{3.8}$$

$$(p(t)x^{\Delta}(t))^{\Delta} + q_1(t)\frac{(x^{\sigma}(t))^3}{1 + (x^{\sigma}(t))^2} = \pm(1 + \sin^2 t)\cos^2 x^{\sigma}(t), \tag{3.9}$$

and

$$(p(t)x^{\Delta}(t))^{\Delta} + q_1(t)\frac{(x^{\sigma}(t))^3}{1 + (x^{\sigma}(t))^2} = (1 - \cos t)(x^{\sigma}(t))^5 \tag{3.10}$$

is nonoscillatory.

We note that if $\mathbb{T} = \mathbb{R}$, then the results here include those of Graef and Spikes [5] for ordinary differential equations as a special case, and are new in the case $\mathbb{T} = \mathbb{Z}$ of difference equations.

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