Access Analysis of Secondary Users with an Opportunistic perspective in Cognitive Radio Networks

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Abstract
This paper analyses opportunistic access of secondary user in cognitive radio network (CRN) using analytical M/D/1 queuing model. Through using spectrum sharing, secondary user opportunistically share primary user spectrum when it empty. Through spectrum sharing primary user generates revenue and secondary user maximizes its satisfaction with spectrum utilization. Number of secondary users and their packet waiting time in the system is calculated. Numerical illustration for the same is presented using utility function upper bound and lower bound of primary arrival was determined. By using upper bound and lower bound the optimal secondary
arrival is calculated. Results indicate that optimal secondary arrival increases and decreases depending on reward and state of primary user’s arrivals.

**Keywords**: Cognitive radio Networks, secondary user, priority based, deterministic, rewards, profit

**INTRODUCTION**

Spectrum scarcity is more growing nowadays due to invention of portable mobile devices like Smart phones and Tablets, which have attracted more number of wireless network users into the spectrum usage. Spectrum is limited and scarce resource as the available spectrum is already allocated to licensed users, which is not fully utilized in both time and location perspective. Spectrum allocation and usage happens to be the thrust area for the research, this becomes more complex when spectrum is apportioned for both primary and secondary users. This demands application of operation research techniques specifically queuing models.

The major functionality of the transceiver in the CRN’s is to intelligently detecting which communication channels are in use and which are not, and instantly migrating the secondary users into a vacant channel while avoiding interference with occupied primary ones. This optimizes the use of available resources while maintaining the quality of transmitted packet.

Quality of service (QoS) is an important characteristic in designing the Cognitive Radio as it is chief criterion to select the best spectrum band. Spectrum management is addressed using four aspects which include spectrum sensing, spectrum decision, spectrum sharing and mobility. Spectrum management resources are reserved purposely to serve licensed user, using access method based on a fixed spectrum allocation known as Fixed Spectrum Access (FSA). These spectra allocated to licensed users are often underutilized in space and time as only small part of it is used at a certain time instant and location Salman A. AlQahtani and Hassan Ahmed [5]. In fixed spectrum much of its bands remain unused as licensed spectrum are exclusively owned operate by their network all the time whether under use it or not. The bands remain idle when not used by the licensed holder on the band.

The objective of the spectrum owner in CRNs who serves both Primary and Secondary users is to maximize the revenue while serving the regular Primary Users and attracting maximum number of Secondary Users. Through spectrum management, spectrum owner leases unused spectrum to unlicensed wireless networks (SU) through opportunistic spectrum access. Cognitive radio improves spectrum utilization and secondary market usage of spectrum and improves income of spectrum owner by striking the right balance between maximizing the spectrum
This paper mainly focuses on maximizing the spectrum allocation for the SUs which
in turn maximizes the profit. To arrive this objective the CRN is considered as M/D/1
queuing model [1] where PUs and SUs are arrived with a Poisson process and their
arrival rates are $\lambda_p$ and $\lambda_s$. The common deterministic service rate for both is $\mu$. The
objective of the paper is twofold. (i) to derive the utility function for finding the
optimum number of SUs for given $\lambda_p$, utilization factor $\theta$, reward for the completion
of service of each SU is $\alpha$ and operational cost of SU is $\beta$ (ii)To find the opportunistic
$\lambda_s^*$ for various values of $\lambda_p$, along with suitable numerical illustrations.

The remaining paper is organized as follows. In section II the review of the literature
is presented. The assumed CRN model is depicted in section III. The derived
expressions for the utility function and optimum $\lambda_s^*$ are presented in section IV. The
numerical illustration is given in Section V. Finally conclusions are drawn in Section
VI and also given future scope of research.

REVIEW OF THE LITERATURE

Several studies have been conducted on opportunistic spectrum access in cognitive
radio network, on how the SU user opportunistically access free channel to transmit
their packets. However no study conducted on secondary user optimization in the
given upper and lower bounds of primary arrivals.

Isameldin Suliman and Janne Lehtomaki [1] Analyzed opportunistic spectrum access
in cognitive radio network. In their analysis primary user is considered to have higher
priority user over secondary user and it does not care about the status of secondary
user in its transmission. Through using time slotted system, secondary user senses a
free time slot and start transmission and when it sense the presence of PU, it removes
its packet form the channel. Waiting time in a queue for PU and SU is determined.
Average waiting time was compared under different scenario to determine which has
minimum delay. However how spectrum accessed opportunistically in given bound is
not shown.

Opportunistic spectrum access channel determined according to targeted QoS before
making decision on which channel to use at the beginning of each slot on a channel. A
CR user desire to request information regarding available channel characteristics from
the base station based on information sensed on channel. The spectrum is selected
according to the QoS requirement and the spectrum characteristics. Data rate and
bandwidth are determined according to decision rule and appropriate spectrum band
chosen Aylin Turhan, Murat Alanyali, and David Starobinski [6].
Primary users have higher priority spectrum access and they don’t care about the state of SU and interruption can occur at any time even a time SU start transmission. SU leases a channel when it is free however it is preempted for a certain period on PU arrival and resume when PU finishes transmission. But during the channel handoff between PU and SU, the Primary User (PU) may face interference for a certain time of delay on rejoining the channel Isameldin Suliman and Janne Lehtomaki [1]. How the secondary user optimizes the channel in case it has almost to finish transmission, primary user interfere transmission, instead of tolerating its short until SU finish transmission.

Narvada Khedun and Vandana Bassoo [2] modelled the system based on the M/D/1 systems with priority queuing. Pre-emptive and non pre-emptive approach to determine which use minimum time. The M/D/1 was extended to M/D/S model, and then the performance of M/D/s non pre-emptive and pre-emptive priority queuing model was accessed. Waiting time using M/D/s modal was compared to waiting time of M/D/1

Yun Zhang, Tao Jiang, Lei Zhang, Daiming Qu, and We Peng [3] employed the pre-emptive resume priority (PRP) M/M/1 queuing model to analyze the transmission delay of priority based secondary users (SUs) in the opportunistic spectrum access (OSA) based cognitive radio network. The interrupted packet transmission time was reduced by assigning of interrupted user to higher priority than the newly arrived SUs. The expression of transmission delay was derived basing on the PRP M/M/1 queuing modal. An analytical result was compared with simulation results to determine SUs minimum delay.

Mina Fahima and Abdorasoul Ghasemi [4] formulated PRP M/G/1 model to investigate delivery time of SU in cognitive radio network. The rate of channel handoff was investigated using pre-emptive resume priority M/G/1 model on multiple handoff and data delivery time for SU. Stability region for arrival rate of secondary user was determined.

Salman A. AlQahtani and Hassan Ahmed [5] proposed and investigated efficient admission with eviction control for CRN to improve spectrum utilization efficiency. They considered a tolerance waiting time for the PU and enhanced eviction police. They proposed scheme which provides better balance between SU QoS and primary user’s protection. They compared call admission control (CAC) and delayed primary user (DPU) to determine the impact on transmission and profit maximization on admission control.

Aylin Turhan, Murat Alanyali, and David Starobinski [6] determined optimal admission control of secondary users (SUs) in cognitive radio (CR) considering as primary user arrives to the system and finds all channel occupied. SU user is preempted except if all channels are occupied by PU. They have applied admission
control on SU to determine the number of admitted SU. Through using dynamic programming, optimal admission control policy maximized long run average profit. They found optimal admission control policy, and they compared optimal admission policy with total number of user in the system and they determined the effect optimal admission control policy of SU to the dynamic programming to determine which is more efficient.

SYSTEM MODEL
The system model is adopted from M/D/1 preemptive priority model as suggested [2], wherein PU will have higher priority with an ability to preempt SU. The queuing model is presented in the following section.

M/D/1 Queuing Model
This paper considers the Model as suggested [1] and opportunistic spectrum access in which there is one primary user and one Secondary user on the channel. The secondary user accesses the channel at the beginning and transmits packet when channel is free. It is assumed that the channel has perfect sensing (noise and error free) available for transmission.

The spectrum allocation is based on the FCFS scheduling mechanism, giving the higher priority to PU. Time spent by the packet in the system is defined using M/D/1 priority queuing scheme, in which packet are transmitted according to Poisson distribution process and service time is deterministic.

The waiting time of each packet in the system has three components, the time until the beginning of the next slot, time spent in a queue waiting for the service to begin followed by the average service time (transmission time).

Reward is collected from each successful SU transmission, the successful transmission time of the SU packets depends on the three factors the packet arrival time and the number of packets both primary and secondary waiting in the system for transmission and the time each secondary packet spends in the queue waiting for free channel is determined. By using utility function optimum secondary arrival is determined in which opportunistic lower bounds for primary users when optimum access of secondary users for given primary users earnings and SU operation cost found and through using upper bounds for primary users when opportunistic access for SU given PU reward and operation cost.

Assumptions
- The arrival of the packets follows the Poisson process where arrival occurs at the rate $\lambda$.
- The Service rate is deterministic (constant) for all arrivals (primary and secondary users), the channel is divided into fixed time slots of the equal size, in which when a packet arrives it occupies one slot.
- The queue service discipline is FCFS
- The system is assumed to have an infinite buffer size
- Error free

**Notations**

- $\lambda_p$: Primary user arrival rate
- $\lambda_s$: Secondary user arrival rate
- $\mu$: Service time
- $\bar{X} = E[X] = \frac{1}{\mu}$: Average service time
- $N^p$: Average number of packets in the queue of primary user
- $N^s$: Average packet in the queue for the secondary user
- $W^p$: Waiting time in a queue for primary user
- $W^s$: Waiting time in a queue for secondary user
- $D^p$: Total time spent by the primary user in the system
- $D^s$: Total time spent by secondary user in the system
- $T_D$: Waiting time till the beginning of a new slot
- $\rho^p = \frac{\lambda_p}{\mu}$: Utilization factor for the primary queue
- $\rho^s = \frac{\lambda_s}{\mu}$: Utilization factor for secondary queue
- $\alpha$: Reward collected on successful Secondary Transmission
- $\beta$: Operational Cost pertaining to the channel when the same is offered to the SU.
- $\theta$: Utilization is a quantified measure of the channel usage over a period of time.
In this model the optimal number of secondary users is calculated analytically based on the M/D/1 model [1] as follows. The service times of PU and SU are same i.e. $\mu_p = \mu_s = \mu$.

The waiting time for any new primary user, includes $T_D$ and transmission time of the existing primary users.

$$W^p = T_D + \frac{1}{\mu} N^p$$

$$W^p = T_D + \frac{1}{\mu} \lambda_p W^p = T_D + \rho^p W^p$$ Since $\therefore N^p = \lambda_p W^p$

$$W^p = \frac{T_D}{1-\rho^p}$$

Waiting time for SU in queue is sum of waiting time until the beginning of a slot, the time which primary user spends in the queue, the time during which the existing secondary user transmits their respective packet, and the subsequent arrived PU service time.

$$W^s = T_D + \frac{1}{\mu} N^p + \frac{1}{\mu} N^s + \frac{1}{\mu} \lambda_p W^s$$

Apply little’s formula and substitute (1) into (3) then waiting time of secondary user in a queue is

$$W^s = T_D + \frac{1}{\mu} \lambda_p W^p + \frac{1}{\mu} \lambda_s W^s + \frac{1}{\mu} \lambda_p W^s$$

$$W^s = T_D + \rho^p W^p + \rho^s W^s + \rho^p W^s$$

$$W^s = \frac{T_D \rho^p W^p}{1 - \rho^p - \rho^s}$$

Substitute (2) into (5) then

$$W^s = \frac{T_D}{(1-\rho^p)(1-\rho^p-\rho^s)}$$

Average delay time/packet of the primary user.
\[ D^p = \bar{X} + \frac{T_D}{1 - \rho^p} \]  
(7)

Average delay time/packet of the secondary user.

\[ D^s = \bar{X} + \frac{T_D}{(1 - \rho^p)(1 - \rho^s - \rho^p)} \]  
(8)

From little’s formula, the expected number of packets of the secondary user in the system is given as

\[ N_S = \lambda_s W_S = \lambda_s (\bar{X} + \frac{T_D}{(1 - \rho^p)(1 - \rho^s - \rho^p)}) \]  
(9)

**Optimal number of Secondary Users**

In this section profit to consider for PU is used to derive the optimum number of secondary user on the channel. The utility function depends on reward (\(\alpha\)) paid by SU for using the spectrum and operation cost (\(\beta\)) for maintaining SU packets in the system.

\[ P = \alpha N_s - \beta W_s = W_s (\lambda_s \alpha - \beta) \]  
(10)

\(\alpha N_s\), is total reward charged on successful secondary transmission. \(\beta W_s\), is total operation cost for packet waiting in the system. The profit of spectrum owner on successful secondary user transmission is total revenue minus operation cost as shown below

Substitute (9) into (10) and applying little’s formula then profit function as follows

\[ P = \left[ \frac{1}{\mu} + \frac{\mu^2 T_D}{(\mu - \lambda_p) (\mu - \hat{\lambda}_s - \lambda_s)} \right] (\lambda_s \alpha - \beta) \]  
(11)

Analysis by obtain optimum number of secondary users is given in index I

\[ \lambda_s = (\mu - \lambda_p) + \sqrt{\mu T_D \left( \frac{u^2}{(\mu - \lambda_p) \alpha} - \frac{\mu^2}{(\mu - \lambda_p) (\mu - \hat{\lambda}_s)} \right)} \]  
(23)

By simplification to get the boundary equation

\[ \lambda_s = (\mu - \lambda_p) + \mu \sqrt{\mu T_D \left( \frac{\beta}{(\mu - \lambda_p) \alpha} - 1 \right)} \]
From the above equation, only $\lambda_p$ is varying and the remains are constant. So the bounds for $\lambda_p$ is as follows:

The upper bound for $\lambda_p$ is

$$1. \quad \mu - \lambda_p > 0 \quad \therefore \lambda_p < \mu \quad (24)$$

The lower bound of the $\lambda_p$ satisfy the following conditions

$$2. \quad \frac{\beta}{(\mu - \lambda_p)\alpha} - 1 > 0 \quad \therefore \mu - \frac{\beta}{\alpha} < \lambda_p \quad (25)$$

The stability region for the primary arrivals

$$3. \quad \mu - \frac{\beta}{\alpha} < \lambda_p < \mu \quad (26)$$

THE NUMERICAL ILLUSTRATION

For fixed values of $\alpha, \beta, \mu$ to study boundaries in IV and to calculate $\lambda^*_S$ for various $\lambda_p$.

Four scenarios are being considered for admitting secondary user.

1. $\lambda_p$ increasing
2. $\lambda_p$ and $\alpha$ increasing
3. $\alpha$ increasing
4. $\alpha$ decreasing

The service rate ($\mu$) is 4, Maximum bound reward ($\alpha$) is 5 and operation cost ($\beta$) is 1. Service rate and operation cost are constant. The lower and upper bound for primary user arrival as in the following range $4 - \frac{1}{5} < \lambda_p < 4 \Rightarrow 3.8 < \lambda_p < 4$. 
Values per scenario are as shown in tables below

<table>
<thead>
<tr>
<th>$\lambda_p$ increasing</th>
<th>$\lambda_p$ and $\alpha$ increasing</th>
<th>$\alpha$ increasing</th>
<th>$\alpha$ decreasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_p$</td>
<td>$\lambda'_p$</td>
<td>$\alpha$</td>
<td>$\lambda'_s$</td>
</tr>
<tr>
<td>3.81</td>
<td>2.79</td>
<td>3.82</td>
<td>0.56</td>
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<td>3.83</td>
<td>4.92</td>
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<td>0.83</td>
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<td>10.34</td>
<td>3.9</td>
<td>1.00</td>
</tr>
<tr>
<td>3.91</td>
<td>12.60</td>
<td>3.92</td>
<td>1.25</td>
</tr>
<tr>
<td>3.93</td>
<td>15.49</td>
<td>3.94</td>
<td>1.67</td>
</tr>
<tr>
<td>3.95</td>
<td>19.65</td>
<td>3.96</td>
<td>2.50</td>
</tr>
<tr>
<td>3.97</td>
<td>26.96</td>
<td>3.98</td>
<td>5.00</td>
</tr>
</tbody>
</table>

Table1 | Table2 | Table3 | Table4

**Scenario 1: $\lambda_p$ increasing**

![PU Arrival increasing](image)

Figure 1: $\lambda'_s$ for given value $\mu, \alpha, \beta, T_d$ are 4, 5, 1, and 2 respectively for various values of $\lambda_p$, which is increasing
For given values of $\mu, \alpha, \beta, T_a$ as $\lambda_p$ increases from lower bound, free spectrum decreases, decreasing of free spectrum causes $\lambda^*_S$ looking for free spectrum to increase. This idea is the same as decreasing free spectrum space so as to increase spectrum demand. From economic theory of demand and supply when the supply decrease the price increases and when supply increases price decreases.

**Scenario 2**: $\lambda_p$ and $\alpha$ increasing

![PU Arrival and Reward Increasing](image)

Figure 2: $\lambda^*_S$ for given value $\mu, \beta, T_a$ are 4,1, and 2 respectively for various $\lambda_p$ and $\alpha$ increasing

Increasing $\lambda_p$ decreases free spectrum. The decrement of spectrum forces the market to increase $\alpha$, these enable spectrum owner to get high profit due to increased $\alpha$ for the same service. However increasing $\alpha$ reduces demand as the secondary users are not willing to pay extra for the same service.

**Scenarios 3**: $\alpha$ is increasing

![Reward increasing](image)

Figure3: $\lambda^*_S$ for given value $\lambda_p, \mu, \beta, T_a$ are 3.9, 4,1, and 2 respectively for variation $\alpha_i$ increasing
For given value $\lambda_p, \mu, \beta, T_d$ as $\alpha$ increase from low to upper bound, $\lambda^*_S$ decreases. This is the same as the economics theory in which when price is increased product demand decreases. Low $\alpha$ increases $\lambda^*_S$ demand however spectrum owner gets low profit. High $\alpha$ decreases $\lambda^*_S$ demand as users are not willing to pay extra for same services but spectrum owners profit increases as extra is charged for same service.

**Scenario 4:** $\alpha$ is decreasing

![Figure 4: $\lambda^*_S$ for given value $\lambda_p, \mu, \beta, T_d$ are 3.9, 4.1, and 2 respectively for various $\alpha_i$ decreasing](image)

For given $\lambda_p, \mu, \beta, T_d$, as $\alpha$ decreases from upper bound to lower bound the $\lambda^*_S$ increases. From the economic theory when the price decreases the demand increases. Increasing $\lambda^*_S$ decreases the spectrum owner’s profits. If few $\lambda^*_S$ pays high $\alpha$ which enables spectrum owner to get super profit, as few $\lambda^*_S$ pays more for same $\mu$ and $\beta$ before and after increasing $\alpha$, until reaches at point of maximum satisfaction where few $\lambda^*_S$ afford proposed $\alpha$. This factor forces spectrum owner to reduce $\alpha$ so as to attract more $\lambda^*_S$.

**CONCLUSION**

This paper has determined $\lambda^*_S$ which maximizes single channel utilization through using utility function. Number of packets admitted into the system and their waiting time was calculated using M/D/1 model in which successful transmission of packets, the spectrum owner gets rewarded. Through utility function, upper and lower bound of $\lambda_p$ in which the optimal number of secondary users admitted is calculated. $\lambda_p$ and $\alpha$ have effect on $\lambda^*_S$ admitted. Increasing $\lambda_p$ with constant $\alpha$ increases $\lambda^*_S$. Increasing
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\(\lambda_p\) and \(\alpha\) decreases \(\lambda_s\). Keeping \(\lambda_p\) constant and decreasing the reward increases \(\lambda_s\). The study is important for maximizing the reward of primary users and also the maximum utilization of spectrum. The future scope of this work lies in determining the equilibrium point where spectrum supply and demand gets satisfaction for both the sides.

REFERENCES


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For easy simplification we let $X = \frac{\mu^2}{\mu - \lambda_p}$ and $Y = \mu - \lambda_s$

$$P = \left[ 1 + \frac{X}{(Y - \lambda_s)} \right] (\lambda_s \alpha - \beta)$$

(12)

Expand the terms

To determine optimal profit gain on secondary admission we differentiate with respect to $\lambda_s$

Apply different ion by part

From $UV = \frac{V \partial U - U \partial V}{\partial^2}$

$$\frac{\partial P}{\partial \lambda_s} = \frac{(Y - \lambda_s)(Y\alpha - \alpha \lambda_s^2 + \beta \lambda_s + X\alpha \lambda_s - Y\beta - X\beta) - (Y\alpha - \alpha \lambda_s^2 + \beta \lambda_s + X\alpha \lambda_s - Y\beta - X\beta)(Y - \lambda_s)}{(Y - \lambda_s)^2}$$

(13)

$$\frac{\partial P}{\partial \lambda_s} = \frac{(Y - \lambda_s)(Y\alpha - 2\alpha \lambda_s^2 + \beta + X\alpha) - (Y\alpha \lambda_s - \alpha \lambda_s^2 + \beta \lambda_s - Y\beta - X\beta)(-1)}{(Y - \lambda_s)^2} = 0$$

(14)

$$Y^2\alpha - 2Y\alpha \lambda_s + Y\beta + Y\alpha \lambda_s - \alpha \lambda_s^2 + 2\alpha \lambda_s^2 - \beta \lambda_s - X\alpha \lambda_s + Y\alpha \lambda_s - \alpha \lambda_s^2 + \beta \lambda_s - Y\beta - X\beta + X\alpha \lambda_s = 0$$

(15)

After Simplification

$$\alpha \lambda_s^2 - 2Y\alpha \lambda_s + Y\alpha \lambda_s + Y^2 \alpha - X\beta = 0$$

(16)

Secondary Arrival SU derivation

Solve for $\lambda_s$ using quadratic formula and SU arrival rate must be positive and negative value is neglected.
From

$$\lambda_s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$  \hspace{1cm} (17)

$$a = \alpha$$

$$b = -2Y\alpha$$

$$c = Y\alpha + Y^2\alpha - X\beta$$

Substitutes in the above formula

$$\lambda_s = \frac{(-2Y\alpha) \pm \sqrt{(-2Y\alpha)^2 - 4\alpha(Y\alpha + Y^2\alpha - X\beta)}}{2\alpha}$$  \hspace{1cm} (18)

$$\lambda_s = \frac{2Y\alpha \pm \sqrt{4Y^2\alpha^2 - 4\alpha^2Y\alpha - 4Y^2\alpha^2 + 4X\alpha\beta}}{2\alpha}$$  \hspace{1cm} (19)

$$\lambda_s = \frac{Y\alpha \pm \sqrt{X\alpha\beta - \alpha^2YX}}{\alpha} \Rightarrow Y \pm \sqrt{X\alpha\beta - \alpha^2YX} \alpha^2$$  \hspace{1cm} (20)

$$\lambda_s = Y \pm \sqrt{X \frac{\beta}{\alpha} - YX} \Rightarrow Y \pm \sqrt{X \left( \frac{\beta}{\alpha} - Y \right)}$$  \hspace{1cm} (21)

Substitute the value of Y and X in the equation above

$$\lambda_s = (\mu - \lambda_p) \pm \sqrt{\frac{u^2}{(\mu - \lambda_p)\alpha} - \frac{\mu^2}{(\mu - \lambda_p)}(\mu - \lambda_p)}$$

$$\lambda_s = (\mu - \lambda_p) \pm \mu \sqrt{\frac{\beta}{(\mu - \lambda_p)\alpha} - 1}$$  \hspace{1cm} (22)

Index 2

Second derivative proof

$$\frac{\partial^2 P}{\partial \lambda_s^2} \frac{\partial P}{\partial \lambda_s} = \frac{(Y - \lambda_s)\partial_s(Y - \lambda_s) + \beta\lambda_s + X\alpha\lambda_s - Y\beta - X\beta) - (Y - \alpha\lambda_s^2 + \beta\lambda_s + X\alpha\lambda_s - Y\beta - X\beta)\partial_s(Y - \lambda_s)}{(Y - \lambda_s)^2} \hspace{1cm} (28)$$
Using by part differentiation, we get

$$\frac{\partial^2 P}{\partial \lambda_s^2} = \frac{(Y - \lambda_s)^2 (2\alpha \lambda_s - 2Y\alpha) + 2(\alpha \lambda_s^2 - 2\alpha Y\lambda_s - X\beta + Y^2 \alpha)(Y - \lambda_s)}{(Y - \lambda_s)^4}$$

(29)

When we simplify the above equation we get

$$\frac{\partial^2 P}{\partial \lambda_s^2} = \frac{2X(Y\alpha - \beta)}{(Y - \lambda_s)^3}$$

(30)

Substitute the value of \(X\) and \(Y\) in the above equation, we get equation (31).