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Category of fuzzy metric spaces and an application

Mohd. Rafi

School of Applied Mathematics The University of Nottingham Malaysia Campus
Jalan Broga, 43500 Semenyih Selangor Darul Ehsan Malaysia
E-mail: Mohd.Rafi@nottingham.edu.my

Abstract

In this paper we introduce a category of fuzzy metric spaces $FMS^{\approx}$ with the objects are complete fuzzy metric spaces in the sense of Kramosil and Michalek and morphisms are ε-fuzzy adjoint pairs. As an application, we prove the existence of solution for fuzzy domain equation in category of fuzzy metric spaces.

Keywords: Category of fuzzy metric spaces, functors, fuzzy metric spaces, fixed point theorem

1. Introduction

Zadeh [19] introduced the notion of fuzzy sets that laid the foundation of fuzzy mathematics. The development and the rich growth of fuzzy mathematics in the past years were tremendous. Especially, Deng [5], Erceg [6], Kaleva and Seikkala [10], and Kramosil and Michalek [11] have introduced the concept of fuzzy metric spaces in different ways. It is well known that fuzzy metric space defined by Kramosil and Michalek is a generalization of classical metric space with special t-norm such as min norm. In [7] George and Veeramani modified the concept of fuzzy metric spaces introduced by Kramosil and Michalek and defined the Hausdorff topology of fuzzy metric spaces. Fixed point theorem in fuzzy metric spaces was introduced by Grabiec [9]. Recently, many authors observed that various contraction mappings in metric spaces may be exactly translated into fuzzy metric spaces [8, 12, 16, 17, 20].

In case of categorical theory, category of metric spaces has turned out to be very useful in giving denotational semantics to concurrent programming language [4]. In various papers [3, 14, 15], mathematical theories are developed for solving domain equations of the form $X = FX$, where F is a functor, in categories of complete metric spaces. In [1, 2], Alessi et al, have studied on solution of metric domain equation in the categories of complete metric spaces. In their papers, a new method for solving domain equations in categories of metric spaces is studied. This inspired us to introduce the category of fuzzy metric spaces with the objects are complete fuzzy metric spaces defined in the sense of Kramosil and Michalek. As an application, we investigate the existence of solution for domain equation in these fuzzy settings by defining a categorical contraction mapping in the sense of Grabiec, [9].

2. Preliminaries

We begin with some basic definition of categories and some related materials in fuzzy metric spaces.

Definition 2.1

A *category* $C := (C_0, C_1)$ is given by a collection C_0 of objects and a collection C_1 of morphisms which have the following structure.

- Each morphism has domain and codomain which are objects: one writes $f : X \to Y$ if $X = dom(f)$ and $Y = cod(f)$.
- Given two morphisms f and g, the composition of f and g, written $g \circ f$, is defined in the usual manner.
- Composition is associative, that is $f \circ (g \circ h) = (f \circ g) \circ h$.
- For every object X there is an identity morphism $id_X : X \to X$ satisfying $id_X \circ g = g$ for every $g : Y \to X$ and $f \circ id_X = f$ for every $f : X \to Y$

Definition 2.2

Given two categories $C := (C_0, C_1)$ and $D := (D_0, D_1)$, a *functor* $F : C \to D$ consists of operations $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$, such that

- for each $f : X \to Y$, $F_1(f) : F_0(X) \to F_0(Y)$;
- For $X \xrightarrow{f} Y \xrightarrow{g} Z$, $F_1(g \circ f) = F_1(g) \circ F_1(f)$;
- $F_1(id_X) = id_{F_0(X)}$ for each $X \in C_0$.

Usually, we write F instead of F_0, F_1.

Definition 2.3

A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous *t-norm* if $([0,1], *)$ is an Abelian (topological) monoid with the unit 1 such that $a * b \le c * d$ whenever $a \le c$ and $b \le d$ for all $a, b, c, d \in [0,1]$.

Definition 2.4

A fuzzy metric space is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$, satisfying the following properties: for all $x, y, z \in X$,

M-1: $M(x, y, 0) = 0$;

M-2: $M(x,y,t)=1$ for all $t>0$ if and only if $x=y$;
M-3: $M(x,y,t)=M(y,x,t)$;
M-4: $M(x,z,t+s)\geq M(x,y,t)*M(y,z,s)$ for all $t,s>0$;
M-5: $M(x,y,\cdot):[0,1)\to[0,1]$ is left-continuous;
M-6: $\lim_{t\to\infty}M(x,y,t)=1$

Definition 2.5
Let $(X,M,*)$ be a fuzzy metric space. A sequence $\{x_n\}_n$ in X is a *G-Cauchy* if $\lim_{n\to\infty}M(x_n,x_{n+p},t)=1$ for each $t>0$ and $p>0$, and $\{x_n\}_n$ is convergent to $x\in X$ if and only if $\lim_{n\to\infty}M(x_n,x,t)=1$ for each $t>0$.

The fuzzy metric space $(X,M,*)$ is called *G-complete* whenever each G-Cauchy sequence converges to an element of X.

3. Fuzzy metric adjoints pairs and isometrices

In this section, we define the basic notion of ε-fuzzy adjoint pair and ε-fuzzy isometry.

Definition 3.1
Let $(X,M,*)$ and $(Y,N,*)$ be fuzzy metric spaces under the same t-norm $*$. A mapping $f:X\to Y$ is called *fuzzy non-expansive* if for all $x,x'\in X$ and $t>0$ the following condition hold:

$$N(fx,fx',t)\geq M(x,x',t).$$

The set of fuzzy non-expansive mappings between two fuzzy metric spaces $(X,M,*)$ and $(Y,N,*)$,

$$F^{X,Y}=\{f:X\to Y:\ f\text{ is non-expansive}\},$$

can be supplied with a fuzzy metric

$$F^{X,Y}(f,g,t)=\inf_{x\in X}N(fx,gx,t).$$

I. A mapping $f\in F^{X,Y}$ is said to be *fuzzy contraction mapping* if there exists a $k\in(0,1)$ such that for all $x,x'\in X$ and for all $t>0$ (k is called the contractive constant of f)

$$N(fx,fx',kt)\geq M(x,x',t).$$

II. A mapping $f\in F^{X,Y}$ is said to be *fuzzy isometric embedding* if for all $x,x'\in X$ and for all $t>0$

$$N(fx,fx',t)=M(x,x',t).$$

If f is bijection then it is an *fuzzy isometry*.

Definition 3.2
For $\varepsilon, r, r' \in [0,1]$, let
$r \approx_{\varepsilon} r'$ if and only if $r * r' \geq \varepsilon$

$$N(fx, y, t) \approx_{\varepsilon} M(x, gy, t).$$

Definition 3.3.
Let (X,M,*) and (Y,N,*) be fuzzy metric spaces. Two fuzzy nonexpansive mappings
f : X------Y and g :Y-------X are said to be E-fuzzy adjoint if for all x ∈ X, y ∈ Y and t >0,
If f and g are 1-fuzzy adjoint, then $\langle f, g \rangle$ is called a proper fuzzy adjoint pair.
Consider the pair of fuzzy non-expansive mappings $f : X \to Y$ and $g : Y \to X$, and define

$$\Im(f, g, t) = Min\{F^{X,X}(id_X, g \circ f, t), F^{Y,Y}(f \circ g, id_Y, t)\}.$$

Definition 3.4
Let $(X, M, *)$ and $(Y, N, *)$ be fuzzy metric spaces. A pair of non-expansive mappings $f : X \to Y$ and $g : Y \to X$ with $\Im(f, g, t) \approx_{\varepsilon} 1$ is called an ε*-fuzzy isometry*.
Note that by definition, any pair $\langle f, g \rangle$ of fuzzy non-expansive mappings is an ε-*fuzzy isometry*, for $\varepsilon = \Im(f, g, t)$.

The above definition can be justified by observation that 1-fuzzy isometries satisfy $id_X = g \circ f$ and $f \circ g = id_Y$ for all $t > 0$ (due to M-2) and consequently f (and also g) is an isometry.

Under some strict condition on the t-norm, we have the following equivalence of mappings.

Theorem 3.1
Let $(X, M, *)$ and $(Y, N, *)$ be fuzzy metric spaces under the same t-norm $*$ such that $a * b = Min\{a, b\}$ and let $\varepsilon \in [0,1]$. For all non-expansive mappings $f : X \to Y$ and $g : Y \to X$, $\langle f, g \rangle$ is an ε-fuzzy adjoint $\Leftrightarrow$ $\langle f, g \rangle$ is an ε-fuzzy isometry.
Proof: Let f and g are ε-fuzzy adjoint. For any $x \in X$,

$$M(x, g \circ f(x), t) \approx_{\varepsilon} N(f(x), f(x), t) = 1.$$

Thus, $F^{X,X}(id_X, g \circ f, t) \approx_{\varepsilon} 1$. Similarly, for any $y \in Y$,

$$N(f \circ g(y), y, t) \approx_{\varepsilon} M(g(y), g(y), t) = 1,$$

which means $F^{Y,Y}(f\circ g, id_Y, t)\approx_\varepsilon 1$. Hence, $\Im(f,g,t)\approx_\varepsilon 1$ and we conclude that $\langle f,g\rangle$ is an ε-fuzzy isometry. Conversely, suppose that $\Im(f,g,t)\approx_\varepsilon 1$. For all $x\in X$, $y\in Y$, and $k>1$, we have

$$\begin{aligned} M(x,g(y),t) &\geq M(x, g\circ f(x),(1-k)t) * M(g\circ f(x), g(y), kt) \\ &\geq \varepsilon * M(g\circ f(x), g(y), kt) \\ &\geq Min\{\varepsilon, M(g\circ f(x), g(y), kt)\} \\ &\geq Min\{\varepsilon, N(f(x), y, kt)\}. \end{aligned}$$

Similarly,

$$\begin{aligned} N(f(x),y,t) &\geq N(f(x), f\circ g(y), kt) * N(f\circ g(y), y, (1-k)t) \\ &\geq N(f(x), f\circ g(y), kt) * \varepsilon \\ &\geq Min\{M(x, g(y), kt), \varepsilon\}. \end{aligned}$$

Hence,

$$Min\{M(x,g(y),t), N(f(x),y,t)\} \geq Min\{\varepsilon, M(x,g(y),kt), N(f(x),y,kt)\} \geq \varepsilon,$$

i.e., $M(x,g(y),t) * N(f(x),y,t) \geq \varepsilon$, which implies

$N(f\circ g(y), y, t)\approx_\varepsilon M(g(y), g(y), t)$ and $\langle f,g\rangle$ is an ε-fuzzy adjoint.

4. Category of fuzzy metric spaces ($FMS^{\approx}$)

In this section, we introduce the category of fuzzy metric spaces, Cauchy tower and prove the Initiality Lemma. Our definition of categories is follows Alessi et al [1, 2] except for notation and some modification to suits the fuzzy settings.

Definition 4.1

Let $FMS^{\approx}$ denote the category of fuzzy metric spaces that has nonempty G-complete fuzzy metric spaces as objects and ε-fuzzy adjoint pairs as morphisms.

The composition of a pair of morphisms $h_1 = \langle f_1, g_1\rangle : (X,M,*) \to (Y,N,*)$ and

$h_2 = \langle f_2, g_2\rangle : (Y,N,*) \to (Z,P,*)$ is defined as

$h_2\circ h_1 = \langle f_2\circ f_1, g_1\circ g_2\rangle : (X,M,*) \to (Z,P,*)$.

Definition 4.2. Let $FMS^{\approx}$ be a category.

(i) A *fuzzy tower* in $FMS^{\approx}$ is a sequence $\{(X_n, M_n,*), h_n\}_n$ of objects and morphisms such that for all $n\in N$, $h_n : (X_n, M_n,*) \to (X_{n+1}, M_{n+1},*)$.

(ii) A tower $\{(X_n, M_n,*), h_n\}_n$ in $FMS^{\approx}$, with $h_n = \langle f_n, g_n\rangle$, is called a $G^{\approx}-Cauchy$ if for

$$\lim_{n\to\infty} \Im(h_{n,n+p}, t) = 1,$$

for each and $p > 0$ $t > 0$, where $h_{n,n+p} = h_{n+p-1} \circ h_{n+p-2} \circ \cdots h_n$.

The direct limit construction of a $G^{\approx}$ – Cauchy tower is defined in the usual way, see […].

Definition 4.3

Let $\{(X_n, M_n, *), h_n\}_n$ be a $G^{\approx}$ – Cauchy tower in $FMS^{\approx}$, where $h_n = \langle f_n, g_n \rangle$. The *direct limit* of $\{(X_n, M_n, *), h_n\}_n$ is a *fuzzy cone* $((X, M, *), \{\delta_n\}_n)$, where $\delta_n = (\alpha_n, \beta_n)$, which defined as follows:

(a) The fuzzy metric space $(X, M, *)$ is given by

$$X = \{\{x_n\}_n : \forall n \in N.\ x_n \in X_n \ and \ x_n = g_n(x_{n+1})\}$$

and

$$M(\{x_n\}_n, \{x'_n\}_n, t) = \inf_{n \in N} M_n(x_n, x'_n, t).$$

(b) Morphisms δ_n are defined as

- $\alpha_n : X_n \to X$ where $\alpha_n(x) = \{x_k\}$ with $x_k = \lim_{r \to \infty} g_{kr} \circ f_{nr}(x)$
- $\beta_n : X \to X_n$ where $\beta_n(x) = x_k$.

The notion of initial object of a category is defined as follows.

Definition 4.4

An initial object of a category $FMS^{\approx}$ is an object $(\vartheta, M_{\vartheta}, *)$ in $FMS^{\approx}$ such that for every object $(X, M, *)$ in $FMS^{\approx}$, there exists a unique morphism $\tau : (\vartheta, M_{\vartheta}, *) \to (X, M, *)$.

Lemma 4.1. If $\lim_{n \to \infty} \Im(\delta_n, t) = 1$, then $((X, M, *), \{\delta_n\}_n)$ with $\delta_n = (\alpha_n, \beta_n)$ will be the initial cone of the $G^{\approx}$ – Cauchy tower $\{(X_n, M_n, *), h_n\}_n$.

Proof: Let $((X', M', *), \{\delta'_n\}_n)$ with $\delta'_n = (\alpha'_n, \beta'_n)$ be another cone for $\{(X_n, M_n, *), h_n\}_n$. We show that there exists a unique morphism $\tau : (X, M, *) \to (X', M', *)$ such that for all $n \in N$, $\delta'_n = \tau \circ \delta_n$. Note that $\{\alpha'_n \circ \beta_n\}$ and $\{\alpha_n \circ \beta'_n\}$ are $G^{\approx}$ – Cauchy sequence, since $\{(X_n, M_n, *), h_n\}_n$ is a $G^{\approx}$ – Cauchy sequence. Furthermore, the objects of $FMS^{\approx}$ are complete, so we can define $\lim_{n \to \infty}(\alpha'_n \circ \beta_n) = i$ and $\lim_{n \to \infty}(\alpha_n \circ \beta'_n) = j$. Obviously, this defines a morphism $\tau = (i, j) : (X, M, *) \to (X', M', *)$. It follows from the facts that $\lim_{n \to \infty} \Im(\delta_n, t) = 1$ that $\delta'_n = \tau \circ \delta_n$, τ is the unique morphism with this property. This proves that $((X, M, *), \{\delta_n\}_n)$ is the initial cone of the tower $\{(X_n, M_n, *), h_n\}_n$.

As a consequence of the above lemma, we have

Corollary 4.1. The direct limit of a $G^{\approx}$ – Cauchy tower is an initial cone for that tower.

Remark 4.1. We note that if $((X',M',*),\{\delta'_n\}_n)$ with $\delta'_n=(\alpha'_n,\beta'_n)$ be another initial cone for the Cauchy tower $\{(X_n,M_n,*),h_n\}_n$, then by the above corollary, we have $(X,M,*)\cong(X',M',*)$. Thus, we have $\lim_{n\to\infty}\Im(\delta_n,t)=\lim_{n\to\infty}\Im(\delta'_n,t)=1$.

From the above remark, we conclude that

Lemma 4.2(Initiality Lemma). Let $\{(X_n,M_n,*),h_n\}_n$ be a fuzzy Cauchy tower in $FMS^{\approx}$ and let $((X,M,*),\{\delta_n\}_n)$, with $\delta_n=(\alpha_n,\beta_n)$, be a cone. Then $((X,M,*),\{\delta_n\}_n)$ is an initial cone if and only if $\lim_{n\to\infty}\Im(\delta_n,t)=1$.

Proof: It follows by Lemma 4.1 and the Remark 4.1.

5. Fixed point theorem

In this section, we prove a fixed point theorem in category of complete fuzzy metric spaces. Before we prove the fixed point theorem in the category $FMS^{\approx}$, we state the following fuzzy Banach contraction theorem due to Grabiec, [9].

Theorem 5.1(Fuzzy Banach contraction theorem). Let $(X,M,*)$ be a G-complete fuzzy metric space and $f:X\to X$ be a fuzzy contractive mapping being k the contactive constant. Then, f has a unique fixed point, i.e., $x=f(x)$.

In category $FMS^{\approx}$, we have the following definition.

Definition 5.1

A functor $T:FMS^{\approx}\to FMS^{\approx}$ is called $CAT^{\approx}$*-contraction*, if there exists a $k\in(0,1)$ such that for each morphism $\tau:(X_1,M_1,*)\to(X_2,M_2,*)$,

$$\Im(T\tau,kt)\geq\Im(\tau,t),$$

where $T\tau=(Tf,Tg)$ for $\tau=(f,g)$.

By the Initiality Lemma, a $CAT^{\approx}$-contraction functor preserves $G^{\approx}-$Cauchy tower and the initial cones, in a similar way as fuzzy contracting functions preserves G-Cauchy sequence and their limits.

We prove the following theorem which shows the existence of fixed points for contracting functors on the category $FMS^{\approx}$.

Theorem 5.1. Let $T:FMS^{\approx}\to FMS^{\approx}$ be a $CAT^{\approx}$-contraction functor. Then, T has a fixed point, i.e., there exists a G-complete fuzzy metric space $(X,M,*)$ such that $(X,M,*)\cong(TX,M,*)$.

Proof: Let $(X_0,M_0,*)$ be a complete fuzzy metric space in $FMS^{\approx}$ and let $f_0:X_0\to\xi X_0$ be any morphism. Consider the fuzzy tower $\{(T^nX_0,M_0,*),T^nf_0\}_n$. Since T is a $CAT^{\approx}$-contraction functor, it is a $G^{\approx}-$Cauchy tower in $FMS^{\approx}$. Thus, it has a direct limit, $((X,M,*),\{\delta_n\}_n)$, which is an initial cone for the tower. Hence, we conclude that $(X,M,*)\cong(TX,M,*)$.

As given in the Remark 3.9 of [1], the contractiveness is not a necessary condition in order that a functor has fixed points. As an example, the identity functor is not contracting.

The uniqueness of fixed points in metric or probabilistic/fuzzy metric spaces are a direct consequence of contractions functions, but in the category of metric/fuzzy metric spaces required some additional conditions in the contractiveness on morphisms, see [1, 2]. We will consider the uniqueness of fixed points in the category of fuzzy metric spaces in our next paper.

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