Fuzzy Orbit $\mathcal{G}$-Structure Space

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Abstract

In this paper the concepts of fuzzy orbit open set, fuzzy orbit$^*$ continuous, fuzzy orbit$^{**}$ continuous, fuzzy $O^*$continuous, fuzzy$^*$ orbit continuous, fuzzy orbit$^*$ $\mathcal{G}$-structure space are introduced and studied. Some interesting properties and characterization are established. Fuzzy orbit $\mathcal{G}$-border, fuzzy orbit $\mathcal{G}$-frontier, fuzzy orbit $\mathcal{G}$-exterior and fuzzy orbit $\mathcal{G}$-irresolute are also discussed.

Keywords:
Fuzzy orbit open set, fuzzy orbit$^*$ continuous, fuzzy orbit$^{**}$ continuous, fuzzy $O^*$ continuous, fuzzy$^*$ orbit continuous, fuzzy orbit $\mathcal{G}$-structure space, fuzzy orbit $\mathcal{G}$-border, fuzzy orbit $\mathcal{G}$-frontier, fuzzy orbit $\mathcal{G}$-exterior and fuzzy orbit $\mathcal{G}$-irresolute.

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1 Introduction

Ever since the introduction of fuzzy set by Zadeh[1], the fuzzy concept has invaded almost all branches of Mathematics. Fuzzy sets have applications in many fields such as information[2] and control[3]. The concept of fuzzy topological spaces was introduced and developed by Chang[4]. Since then various notions in classical topology have been extended to fuzzy topological spaces[5-7]. The concepts of g-interior, g-border and g-frontier were studied by Caldas, Jafari and Noiri[8]. The concept of chaotic in general metric space was introduced by R. L. Devaney[9]. In this paper we introduced the concepts of fuzzy orbit open set, fuzzy orbit$^*$ continuous, fuzzy orbit$^{**}$ continuous, fuzzy $O^*$ continuous, fuzzy$^*$ orbit continuous, fuzzy orbit $\mathcal{G}$-structure space, fuzzy orbit $\mathcal{G}$-border, fuzzy orbit $\mathcal{G}$-frontier, fuzzy orbit $\mathcal{G}$-exterior and fuzzy orbit $\mathcal{G}$-irresolute and investigate some of their fundamental properties.

2 Preliminaries

Definition 2.1. [1] A fuzzy set in $X$ is a function with domain $X$ and values in $I$, that is an element of $I^X$.

Definition 2.2. [10] Let $(X, T)$ and $(Y, S)$ be any two fuzzy topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is said to be almost$^*$-fuzzy continuous, if for every fuzzy set $\alpha \in I^X$ and every fuzzy open set $\mu$ with $f(\alpha) \leq \mu$, there exists a fuzzy open set $\sigma$ with $\alpha \leq \sigma$ such that $f(\sigma) \leq \text{int cl} (\mu)$.

Definition 2.3. [10] Let $(X, T)$ and $(Y, S)$ be any two fuzzy topological spaces. A mapping $f : (X, T) \rightarrow (Y, S)$ is said to be $\theta^*$-fuzzy continuous, if for every fuzzy set...
α ∈ I^X and every fuzzy open set μ with f(α) ≤ μ, there exists a fuzzy open set σ with α ≤ σ such that f(cl σ) ≤ cl μ.

**Definition 2.4.** [10] Let (X, T) and (Y, S) be any two fuzzy topological spaces. A mapping f : (X, T) → (Y, S) is said to be fuzzy weakly∗continuous, if for every fuzzy set α ∈ I^X and every fuzzy open set μ with f(α) ≤ μ, there exists a fuzzy open set σ with α ≤ σ such that f(cl σ) ≤ cl μ.

**Definition 2.5.** [10] Let (X, T) and (Y, S) be any two fuzzy topological spaces. A mapping f : (X, T) → (Y, S) is said to be slightly fuzzy continuous, if for every fuzzy set α ∈ I^X and every fuzzy clopen set μ with f(α) ≤ μ, there exists a fuzzy open set σ with α ≤ σ such that f(σ) ≤ μ.

**Definition 2.6.** [9] Orbit of a point x in X under the mapping f is

O_f(x) = {x, f(x), f^2(x), ...}

**Definition 2.7.** [11] A function f from a fuzzy topological space (X, T) to a fuzzy topological space (Y, S) is said to be fuzzy irresolute if f^{-1}(λ) is fuzzy semi open in (X, T) for each fuzzy semi open set λ in (Y, S).

**Definition 2.8.** [12] Let X be a group and G be a fuzzy set in X with membership function µ_G. Then G is a fuzzy group in X iff the following conditions are satisfied:

(i) µ_G(xy) ≥ min { µ_G(x), µ_G(y) }, for all x, y ∈ X,

(ii) µ_G(x^{-1}) ≥ µ_G(x), for all x ∈ X.

**Definition 2.9.** [13] Let X be a nonempty set and let f : X → X be any mapping. Let λ be any fuzzy set in X. The fuzzy orbit O_f(λ) of λ under the mapping f is defined as O_f(λ) = {λ, f(λ), f^2(λ), ...}.

**Definition 2.10.** [13] Let X be a nonempty set and let f : X → X be any mapping. The fuzzy orbit set of λ under the mapping f is defined as F O_f(λ) = {λ ∧ f(λ) ∧ f^2(λ) ∧ ...} the intersection of all members of O_f(λ).

**Definition 2.11.** [13] Let (X, T) be a fuzzy topological space. Let f : X → X be any mapping. The fuzzy orbit set under the mapping f which is in fuzzy topology T is called fuzzy orbit open set under the mapping f.Its complement is called a fuzzy orbit closed set under the mapping f.

3 Characterization of fuzzy* orbit continuous mappings

**Definition 3.1.** Let (X, T) be a fuzzy topological space and λ ∈ I^X. Let f : X → X be a mapping.

(i) F O_f int (λ) = ∨{μ : μ ≤ λ, μ is a fuzzy orbit open set under the mapping f} is called fuzzy orbit interior of λ under the mapping f.

(ii) F O_f cl (λ) = ∧{μ : μ ≥ λ, μ is a fuzzy orbit closed set under the mapping f} is called fuzzy orbit closure of λ under the mapping f.
**Definition 3.2.** Let \((X, T)\) be a fuzzy topological space. Let \(f : X \rightarrow X\) be any mapping. The fuzzy orbit set under the mapping \(f\) in a fuzzy topological space \((X, T)\) is said to be fuzzy orbit clopen set under the mapping \(f\) if it is both fuzzy orbit open and fuzzy orbit closed under the mapping \(f\).

**Definition 3.3.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(f : X \rightarrow X\) be a mapping. A mapping \(g : (X, T) \rightarrow (Y, S)\) is said to be fuzzy orbit continuous if the inverse image of every fuzzy open set in \((Y, S)\) is fuzzy orbit open set under the mapping \(f\) in \((X, T)\).

**Definition 3.4.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(f_1 : X \rightarrow X\) and \(f_2 : Y \rightarrow Y\) be any two mappings. A mapping \(g : (X, T) \rightarrow (Y, S)\) is said to be fuzzy orbit *continuous*, if for every fuzzy set \(\alpha \in I^X\) and every fuzzy orbit open set \(\mu\) under the mapping \(f_2\) with \(g(\alpha) \leq \mu\), there exists a fuzzy orbit open set \(\lambda\) under the mapping \(f_1\) with \(\alpha \leq \lambda\) such that \(g(\lambda) \leq \mu\).

**Example 3.1.** Let \(X = \{a, b, c\} = Y\). Define \(T = \{0, 1, \lambda, \gamma_1, \gamma_2, \}\) and \(S = \{0, 1, \mu, \delta, \}\) where \(\lambda, \gamma_1, \gamma_2, \mu, \delta : X \rightarrow [0, 1]\) are such that \(\lambda(a) = 0.4, \lambda(b) = 0, \lambda(c) = 0, \gamma_1(a) = 0.5, \gamma_1(b) = 0.7, \gamma_1(c) = 0.6, \gamma_2(a) = 0.4, \gamma_2(b) = 0.5, \gamma_2(c) = 0.6, \mu(a) = 0.7, \mu(b) = 0.7, \mu(c) = 0.7, \delta(a) = 0.9, \delta(b) = 0.7, \delta(c) = 0.8\). Clearly \((X,T)\) and \((Y, S)\) are fuzzy topological spaces.

Define \(g : (X, T) \rightarrow (Y, S), f_1 : X \rightarrow X\) and \(f_2 : Y \rightarrow Y\) as \(g(a) = b, g(b) = c, g(c) = a, f_1(a) = a, f_1(b) = a, f_1(c) = a\) and \(f_2(a) = b, f_2(b) = c, f_2(c) = a\). Let \(\alpha : X \rightarrow [0, 1]\) be any fuzzy set such that \(\alpha(a) = 0.2, \alpha(b) = 0, \alpha(c) = 0\). For the fuzzy orbit open set \(\mu\) under the mapping \(f_2\) in \((Y, S)\), \(g(\alpha) \leq \mu\). Now, \(\lambda\) is a fuzzy orbit open set under the mapping \(f_1\) in \((X, T)\) with \(\alpha \leq \lambda\) such that \(g(\lambda) \leq \mu\). Hence \(f\) is fuzzy orbit continuous.

**Proposition 3.1.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(g : (X, T) \rightarrow (Y, S)\) be a mapping. Then the following statements are equivalent:

(i) \(g\) is fuzzy orbit *continuous*.

(ii) Inverse image of every fuzzy orbit open set of \((Y, S)\) is fuzzy orbit open set of \((X, T)\).

(iii) Inverse image of every fuzzy orbit clopen set of \((Y, S)\) is fuzzy orbit clopen set of \((X, T)\).

**Proof:**
To prove (i) ⇒ (ii)

Let \(f_1 : X \rightarrow X\) and \(f_2 : Y \rightarrow Y\) be any two mappings. Let \(\mu\) be a fuzzy orbit open set under the mapping \(f_2\) of \((Y, S)\) and any fuzzy set \(\alpha\) with \(g(\alpha) \leq \mu\). Since \(g\) is orbit *continuous*, there exists a fuzzy orbit open set \(\lambda\) under the mapping \(f_1\) of \((X, T)\) with \(\alpha \leq \lambda\) such that \(g(\lambda) \leq \mu\). Hence \(g^{-1}(\mu)\) is a fuzzy orbit open set. Hence (i) ⇒ (ii).

To prove (ii) ⇒ (iii)
Let \( \mu \) be a fuzzy orbit open set under the mapping \( f_2 \) of \((Y, S)\). By (ii) \( g^{-1}(\mu) \) is a fuzzy orbit open set under the mapping \( f_1 \) of \((X, T)\). Now, \( 1 - \mu \) is also fuzzy orbit clopen set. By (ii) \( g^{-1}(1 - \mu) \) is fuzzy orbit open set under the mapping \( f_1 \) in \((X, T)\). So \( 1 - g^{-1}(1 - \mu) \) is fuzzy orbit closed set under the mapping \( f_1 \) in \((X, T)\). This implies that \( g^{-1}(\mu) \) is fuzzy orbit closed. Therefore, \( g^{-1}(\mu) \) is a fuzzy orbit open set and fuzzy orbit closed set under the mapping \( f_1 \) in \((X, T)\). Hence, \( g^{-1}(\mu) \) is a fuzzy orbit clopen set in \((X, T)\).

To prove (iii) \( \Rightarrow \) (i)

Let \( \mu \) be a fuzzy orbit clopen set under the mapping \( f_2 \) and any fuzzy set \( \alpha \) with \( g(\alpha) \leq \mu \). Now, \( g^{-1}(\mu) \) is fuzzy orbit open set under the mapping \( f_1 \) of \((X, T)\) and \( g(g^{-1}(\mu)) \leq \mu \). Hence, \( g \) is fuzzy orbit** continuous. Hence (iii) \( \Rightarrow \) (i).

**Definition 3.5.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( f_1 : X \to X \) and \( f_2 : Y \to Y \) be any two mappings. A mapping \( g : (X, T) \to (Y, S) \) is said to be fuzzy orbit** continuous, if for every fuzzy set \( \alpha \in I^X \) and every fuzzy orbit open set \( \mu \) under the mapping \( f_2 \) with \( g(\alpha) \leq \mu \), there exists a fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) with \( \alpha \leq \lambda \) such that \( g(cl(\lambda)) \leq \mu \).

**Proposition 3.2.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( g : (X, T) \to (Y, S) \) be a mapping. If \( g \) is fuzzy orbit** continuous mapping then \( g \) is a fuzzy orbit** continuous mapping.

**Proof:** Let \( f_1 : X \to X \) and \( f_2 : Y \to Y \) be any two mappings. Since \( g \) is a fuzzy orbit** continuous function, \( \mu \) is a fuzzy orbit clopen set under the mapping \( f_2 \) in \((X, S)\) with \( g(\alpha) \leq \mu \), there exists a fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) with \( \alpha \leq \lambda \) such that \( g(\lambda) \leq \mu \). Then \( g^{-1}(\mu) \) is a fuzzy orbit clopen set under the mapping \( f_1 \) in \((X, T)\). Therefore \( \mu \) is a fuzzy orbit closed set under the mapping \( f_2 \) with \( g(\alpha) \leq \mu \) and \( g^{-1}(\mu) \) is a fuzzy orbit open set under the mapping \( f_1 \) such that \( g(g^{-1}(\mu)) \leq \mu \). Since \( g^{-1}(\mu) \) is a fuzzy orbit closed set, \( cl(g^{-1}(\mu)) = g^{-1}(\mu) \). This implies that, \( g(cl(g^{-1}(\mu))) = \mu \). Hence, \( g \) is a fuzzy orbit** continuous mapping.

**Remark 3.1.** The converse of Proposition 3.2 need not be true as shown in the following example.

**Example 3.2.** Let \( X = \{ a, b, c \} = Y \). Define \( T = \{ 0, 1, \lambda, \lambda_1 \} \) and \( S = \{ 0, 1, \mu, \mu_1, \mu_2, \mu_3 \} \) where \( \lambda, \lambda_1, \mu, \mu_1, \mu_2, \mu_3 : X \to [0, 1] \) are such that \( \lambda(a) = 0.4, \lambda(b) = 0.4, \lambda(c) = 0.4, \lambda_1(a) = 0.5, \lambda_1(b) = 0.6, \lambda_1(c) = 0.4, \mu(a) = 0.7, \mu(b) = 0.7, \mu(c) = 0.7, \mu_1(a) = 0.7, \mu_1(b) = 0.8, \mu_1(c) = 0.9, \mu_2(a) = 0.3, \mu_2(b) = 0.3, \mu_2(c) = 0.3, \mu_3(a) = 0.3, \mu_3(b) = 0.4, \mu_3(c) = 0.5 \). Clearly \((X, T)\) and \((Y, S)\) are fuzzy topological spaces.

Define \( g : (X, T) \to (Y, S) \), \( f_1 : X \to X \) and \( f_2 : Y \to Y \) as \( g(a) = b, g(b) = c, g(c) = a, f_1(a) = b, f_1(b) = c, f_1(c) = a \) and \( f_2(a) = b, f_2(b) = c, f_2(c) = a \). Let \( \alpha : X \to [0, 1] \) be any fuzzy set such that \( \alpha(a) = 0.2, \alpha(b) = 0.1, \alpha(c) = 0.2 \). For the fuzzy orbit open set \( \mu \) under the mapping \( f_2 \) in \((Y, S)\), \( g(\alpha) \leq \mu \). Now, \( \lambda \) is a fuzzy orbit open set under the mapping \( f_1 \) in \((X, T)\) with \( \alpha \leq \lambda \) such that \( g(cl(\lambda)) \leq \mu \). Hence \( g \) is fuzzy orbit** continuous.
Now, \( g(\alpha) \leq \mu_2 \). But there is no fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) such that \( \alpha \leq \lambda \) and \( g(\lambda) \not\leq \mu_2 \). Hence \( g \) is not fuzzy orbit* continuous.

**Definition 3.6.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( f_1 : X \to X \) and \( f_2 : Y \to Y \) be any two mappings. A mapping \( g : (X, T) \to (Y, S) \) is said to be fuzzy O* continuous, if for every fuzzy set \( \alpha \in I^X \) and every fuzzy orbit closed set \( \mu \) under the mapping \( f_2 \) with \( g(\alpha) \leq \mu \), there exists a fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) with \( \alpha \leq \lambda \) such that \( g(\lambda) \leq \text{int}(\mu) \).

**Proposition 3.3.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( g : (X, T) \to (Y, S) \) be a mapping. If \( g \) is a fuzzy orbit* continuous mapping then \( g \) is a fuzzy O* continuous mapping.

**Proof:** Let \( f_1 : X \to X \) and \( f_2 : Y \to Y \) be any two mappings. Since \( g \) is a fuzzy orbit* continuous mapping, \( \mu \) is a fuzzy orbit open set under the mapping \( f_2 \) in \((X, S)\) with \( g(\alpha) \leq \mu \), there exists a fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) with \( \alpha \leq \lambda \) such that \( g(\lambda) \leq \text{int}(\mu) \). Since \( \mu \) is a fuzzy orbit open set under the mapping \( f_2 \) in \((X, S)\). \( \mu = \text{int}(\mu) \). This implies that \( g(\lambda) \leq \text{int}(\mu) \). Hence, \( g \) is a fuzzy O* continuous mapping.

**Remark 3.2.** The converse of Proposition 3.3 need not be true as shown in the following example.

**Example 3.3.** Let \( X = \{ a, b, c \} = Y \). Define \( T = \{ 0, 1, \lambda, \lambda_1 \} \) and \( S = \{ 0, 1, \mu, \mu_1, \mu_2, \mu_3 \} \) where \( \lambda, \lambda_1, \mu, \mu_1, \mu_2, \mu_3 : X \to [0, 1] \) are such that \( \lambda (a) = 0.4, \lambda (b) = 0.4, \lambda (c) = 0.4, \lambda_1 (a) = 0.4, \lambda_1 (b) = 0.6, \lambda_1 (c) = 0.7, \mu (a) = 0.7, \mu (b) = 0.7, \mu (c) = 0.7, \mu_1 (a) = 0.7, \mu_1 (b) = 0.8, \mu_1 (c) = 0.9, \mu_2 (a) = 0.3, \mu_2 (b) = 0.3, \mu_2 (c) = 0.4, \mu_3 (a) = 0.3, \mu_3 (b) = 0.3, \mu_3 (c) = 0.3 \). Clearly \((X, T)\) and \((Y, S)\) are fuzzy topological spaces.

Define \( g : (X, T) \to (Y, S) \), \( f_1 : X \to X \) and \( f_2 : Y \to Y \) as \( g(a) = b, g(b) = c, g(c) = a, f_1(a) = b, f_1(b) = c, f_1(c) = a \) and \( f_2(a) = b, f_2(b) = c, f_2(c) = a \). Let \( \alpha : X \to [0, 1] \) be any fuzzy set such that \( \alpha (a) = 0.2, \alpha (b) = 0.1, \alpha (c) = 0.2 \). For the fuzzy orbit closed set \( \mu \) under the mapping \( f_2 \) in \((Y, S)\), \( g(\alpha) \leq \mu \). Now, \( \lambda \) is a fuzzy orbit open set under the mapping \( f_1 \) in \((X, T)\) with \( \alpha \leq \lambda \) such that \( g(\lambda) \leq \text{int}(\mu) \). Hence \( g \) is fuzzy O* continuous.

Now, \( g(\alpha) \leq \mu_3 \). But there is no fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) such that \( \alpha \leq \lambda \) and \( g(\lambda) \not\leq \mu_3 \). Hence \( g \) is not fuzzy orbit* continuous.

**Definition 3.7.** Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \( f_1 : X \to X \) and \( f_2 : Y \to Y \) be any two mappings. A mapping \( g : (X, T) \to (Y, S) \) is said to be fuzzy* orbit continuous, if for every fuzzy set \( \alpha \in I^X \) and every fuzzy orbit closed set \( \mu \) under the mapping \( f_2 \) with \( g(\alpha) \leq \mu \), there exists a fuzzy orbit open set \( \lambda \) under the mapping \( f_1 \) with \( \alpha \leq \lambda \) such that \( g(\text{int}(\lambda)) \leq \mu \).
Proposition 3.4. Let \((X, T)\) and \((Y, S)\) be any two fuzzy topological spaces. Let \(g: (X, T) \rightarrow (Y, S)\) be a mapping. Then \(g\) is a fuzzy orbit\(^*\) continuous mapping if and only if \(g\) is a fuzzy orbit continuous mapping.

**Proof:** Let \(f_1 : X \rightarrow X\) and \(f_2 : Y \rightarrow Y\) be any two mappings. Since \(g\) is a fuzzy orbit\(^*\) continuous mapping, \(\mu\) is a fuzzy orbit open set under the mapping \(f_2\) in \((Y, S)\) with \(g(\alpha) \leq \mu\), there exists a fuzzy orbit open set \(\lambda\) with \(\alpha \leq \lambda\) such that \(g(\lambda) \leq \mu\). This implies \(\lambda \leq g^{-1}(\mu)\). Then \(g^{-1}(\mu)\) is a fuzzy orbit clopen set in \((Y, S)\). Therefore \(\mu\) is a fuzzy orbit closed set under the mapping \(f_2\) with \(g(\lambda) \leq \mu\) and \(g^{-1}(\mu)\) is a fuzzy orbit open set such that \(g(g^{-1}(\mu)) \leq \mu\). Since \(g^{-1}(\mu)\) is a fuzzy orbit open set, \(\text{int}(g^{-1}(\mu)) = g^{-1}(\mu)\). This implies that, \(g\) is a fuzzy orbit\(^*\) continuous mapping.

Conversely, let \(g\) be a fuzzy orbit continuous mapping, \(\mu\) is a fuzzy orbit closed set under the mapping \(f_2\) in \((Y, S)\) with \(g(\lambda) \leq \mu\), there exists a fuzzy orbit clopen set \(\lambda\) with \(\alpha \leq \lambda\) such that \(g(\text{int}\lambda) \leq \mu\). Since \(\lambda\) is fuzzy orbit closed, \(\text{cl}(\lambda) = \lambda\). Since \(\lambda\) is fuzzy orbit open set under the mapping \(f_1\), \(\mu\) is a fuzzy orbit open set under the mapping \(f_2\). Since \(\mu\) is both fuzzy orbit open and fuzzy orbit closed, \(\mu\) is fuzzy orbit clopen. Hence \(g\) is a fuzzy orbit\(^*\) continuous mapping.

4 Fuzzy Orbit G-Border

**Definition 4.1.** Let \(X\) be a group and \(f_1 : X \rightarrow X\) be any mapping. Let \(G\) be a fuzzy orbit set under the mapping \(f_1\) in \(X\) with membership function \((\lambda_G)\). Then \(G\) is a fuzzy orbit group in \(X\) if and only if the following conditions are satisfied:

(i) \(\lambda_G(xy) \geq \min \{ \lambda_G(x), \lambda_G(y) \}\), for all \(x, y \in X\),

(ii) \(\lambda_G(x^{-1}) \geq \lambda_G(x)\), for all \(x \in X\).

**Definition 4.2.** Let \(A\) be a fuzzy orbit set in \(X\) and \(T\) be a fuzzy topology on \(X\). Then the fuzzy orbit subspace topology on \(A\) is the family of fuzzy subsets of \(A\) which are intersection with \(A\) of fuzzy orbit open sets in \(X\). The fuzzy orbit subspace topology is denoted by \(T_A\) pair \((A, T_A)\) is called a fuzzy orbit subspace of \((X, T)\).
Definition 4.3. Let \((A, T_A)\) and \((B, T_B)\) be any two fuzzy orbit subspaces of fuzzy topological spaces \((X, T)\) and \((Y, S)\) respectively. A mapping \(g : (A, T_A) \rightarrow (B, T_B)\) is said to be relatively fuzzy orbit continuous if and only if for each fuzzy orbit open \(V' = V \cap B\) in \(S_B\), the intersection \(g^{-1}(V') \cap A\) is fuzzy orbit open in \(T_A\).

Definition 4.4. Let \(X\) be a group and \(T\) be a fuzzy topology on \(X\). Let \(G\) be any fuzzy orbit group in \(X\) and let \(G\) be endowed with the fuzzy orbit subspace topology \(T_G\). Then \(G\) is a fuzzy orbit topological group in \(X\) if and only if

(i) The mapping \(\alpha : (xy) \rightarrow xy\) of \((G, T_G) \times (G, T_G)\) into \((G, T_G)\) is relatively fuzzy orbit continuous.

(ii) The mapping \(\beta : x \rightarrow x^{-1}\) of \((G, T_G)\) into \((G, T_G)\) is relatively fuzzy orbit continuous.

Definition 4.5. Let \(X\) be a nonempty set. A family \(\mathcal{G}\) is a fuzzy orbit topological groups in \(X\) satisfies the following conditions:

(i) \(0, 1 \in \mathcal{G}\),

(ii) If \(\lambda_1, \lambda_2 \in \mathcal{G}\), then \(\lambda_1 \land \lambda_2 \in \mathcal{G}\),

(iii) If \(\lambda_j \in \mathcal{G}\) for all \(j \in J\), then \(\bigvee_j \lambda_j \in \mathcal{G}\).

then \(\mathcal{G}\) is said to be a fuzzy orbit topological group structure on \(X\) and the pair \((X, \mathcal{G})\) is said to be fuzzy orbit topological group(in short fuzzy orbit \(\mathcal{G}\)) structure space. Any member of fuzzy orbit \(\mathcal{G}\)-structure space is called a fuzzy orbit open group. The complement of fuzzy orbit open group is a fuzzy orbit closed group.

Definition 4.6. Let \((X, \mathcal{G})\) be a fuzzy orbit \(\mathcal{G}\)-structure space and \(\lambda\) be any fuzzy orbit topological group.

(i) \(\text{FO}\mathcal{G}\text{ int}(\lambda) = \bigvee\{\mu : \mu \leq \lambda, \mu\ \text{is a fuzzy orbit open group}\}\) is called fuzzy orbit \(\mathcal{G}\)-interior of \(\lambda\).

(ii) \(\text{FO}\mathcal{G}\text{ cl}(\lambda) = \bigwedge\{\mu : \mu \geq \lambda, \mu\ \text{is a fuzzy orbit closed group}\}\) is called fuzzy orbit \(\mathcal{G}\)-closure of \(\lambda\).

Definition 4.7. Let \((X, \mathcal{G})\) be a fuzzy orbit \(\mathcal{G}\)-structure space and \(\lambda\) be a fuzzy orbit topological group. Then the fuzzy orbit \(\mathcal{G}\)-border of \(\lambda\) is denoted and defined as \((\lambda) \land \text{FO}\mathcal{G}\text{ cl}(\lambda)

Definition 4.8. Let \((X, \mathcal{G})\) be a fuzzy orbit \(\mathcal{G}\)-structure space and \(\lambda\) be a fuzzy orbit topological group. Then the fuzzy orbit \(\mathcal{G}\)-frontier of \(\lambda\) is denoted and defined as \(\text{FO}\mathcal{G}\text{ cl}(\lambda) \land \text{FO}\mathcal{G}\text{ cl}(\lambda')\).

Definition 4.9. Let \((X, \mathcal{G})\) be a fuzzy orbit \(\mathcal{G}\)-structure space and \(\lambda\) be a fuzzy orbit topological group. Then the fuzzy orbit \(\mathcal{G}\)-exterior of \(\lambda\) is denoted and defined as \(\text{FO}\mathcal{G}\text{ int}(\lambda')\).
**Proposition 4.1.** Let \((X, \mathcal{G})\) be a fuzzy orbit \(\mathcal{G}\)-structure space and \(\lambda\) be a fuzzy orbit topological group. Then the following statements hold:

(i) \(F_b(\lambda) \leq \text{FO}_{\mathcal{G}}b(\lambda)\)
(ii) \(\text{FO}_{\mathcal{G}}b(\lambda) \leq \text{cl}(\mathcal{L}')\)
(iii) \(\text{FO}_{\mathcal{G}}\text{int}((\text{FO}_{\mathcal{G}}b(\lambda))) \leq \lambda\)
(iv) \(\text{FO}_{\mathcal{G}}b(\lambda \lor \mu) \leq (\text{FO}_{\mathcal{G}}b(\lambda)) \lor (\text{FO}_{\mathcal{G}}b(\mu))\)
(v) \(\text{FO}_{\mathcal{G}}b(\lambda \land \mu) \leq (\text{FO}_{\mathcal{G}}b(\lambda)) \land (\text{FO}_{\mathcal{G}}b(\mu))\)

**Proof:**
To prove (i) Since \(\text{FO}_{\mathcal{G}}\text{int}(\lambda) \leq \text{Fint}(\lambda), \lambda - \text{int}(\lambda) \leq \lambda - \text{FO}_{\mathcal{G}}\text{int}(\lambda), \lambda \land \text{cl}(\lambda') \leq \lambda \land \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}'),\) Therefore, \(F_b(\lambda) \leq \text{FO}_{\mathcal{G}}b(\lambda).\)
To prove (ii)
\[
\text{FO}_{\mathcal{G}}b(\lambda) = \lambda \land \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}')
\]
\[
= \lambda - \text{FO}_{\mathcal{G}}\text{int}(\lambda)
\]
\[
= \lambda - (1 - \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}'))
\]
\[
\leq 1 - 1 + \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}')
\]
\[
= \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}')
\]
Therefore, \(\text{FO}_{\mathcal{G}}b(\lambda) \leq \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}')\) To prove (iii)
\[
\text{FO}_{\mathcal{G}}\text{int}(\text{FO}_{\mathcal{G}}b(\lambda)) = \text{FO}_{\mathcal{G}}\text{ int}(\lambda \land \text{FO}_{\mathcal{G}}\text{cl}(\mathcal{L}'))
\]
\[
= \text{FO}_{\mathcal{G}}\text{int}(\lambda - \text{FO}_{\mathcal{G}}\text{int}(\lambda))
\]
\[
\leq \lambda - \text{FO}_{\mathcal{G}}\text{int}(\lambda)
\]
\[
\leq \lambda
\]
Therefore, \(\text{FO}_{\mathcal{G}}\text{int}((\text{FO}_{\mathcal{G}}b(\lambda))) \leq \lambda.\) To prove (iv)
\[
\text{FO}_{\mathcal{G}}b(\lambda \lor \mu) = (\lambda \lor \mu) \land \text{FO}_{\mathcal{G}}\text{cl}((\lambda \lor \mu)')
\]
\[
= (\lambda \lor \mu) - \text{FO}_{\mathcal{G}}\text{int}((\lambda \lor \mu))
\]
\[
= (\lambda \lor \mu) - (\text{FO}_{\mathcal{G}}\text{int}(\lambda)) \lor (\text{FO}_{\mathcal{G}}\text{int}(\mu))
\]
\[
\leq (\lambda - \text{FO}_{\mathcal{G}}\text{int}(\lambda)) \lor (\mu - \text{FO}_{\mathcal{G}}\text{int}(\mu))
\]
\[
\leq (\lambda \land \text{FO}_{\mathcal{G}}\text{cl}(\lambda') \lor (\mu \land \text{FO}_{\mathcal{G}}\text{cl}(\mu'))
\]
\[
= (\text{FO}_{\mathcal{G}}b(\lambda)) \lor (\text{FO}_{\mathcal{G}}b(\mu))
\]
Therefore, \( \text{FO}\mathcal{G}b(\lambda \lor \mu) \leq (\text{FO}\mathcal{G}b(\lambda)) \lor (\text{FO}\mathcal{G}b(\mu)) \)

To prove (v)

\[
\text{FO}\mathcal{G}b(\lambda \land \mu) = (\lambda \land \mu) \land \text{FO}\mathcal{G}\text{cl}(\lambda \land \mu) \leq (\lambda \land \mu) - \text{FO}\mathcal{G}\text{int}(\lambda \land \mu) \\
\geq (\lambda - \text{FO}\mathcal{G}\text{int}(\lambda)) \land (\mu - \text{FO}\mathcal{G}\text{int}(\mu)) \\
\geq (\lambda \land \text{FO}\mathcal{G}\text{cl}(\lambda)) \land (\mu \land \text{FO}\mathcal{G}\text{cl}(\mu)) \\
= (\text{FO}\mathcal{G}b(\lambda)) \land (\text{FO}\mathcal{G}b(\mu))
\]

Therefore, \( \text{FO}\mathcal{G}b(\lambda \land \mu) \geq (\text{FO}\mathcal{G}b(\lambda)) \land (\text{FO}\mathcal{G}b(\mu)) \).

**Proposition 4.2.** Let \((X, \mathcal{G})\) be a fuzzy orbit \(\mathcal{G}\)-structure space and \(\lambda\) be a fuzzy orbit topological group. Then the following statements hold:

(i) \( \text{FFr}(\lambda) \leq \text{FO}\mathcal{G}\text{Fr}(\lambda) \)

(ii) \( \text{FO}\mathcal{G}b(\lambda) \leq \text{FO}\mathcal{G}\text{Fr}(\lambda) \)

(iii) \( \text{FO}\mathcal{G}\text{Fr}(\lambda^\prime) = \text{FO}\mathcal{G}\text{Fr}(\lambda) \)

(iv) \( \text{FO}\mathcal{G}\text{Fr}(\text{FO}\mathcal{G}\text{int}(\lambda)) \leq \text{FO}\mathcal{G}\text{Fr}(\lambda) \)

(v) \( \text{FO}\mathcal{G}\text{Fr}(\text{FO}\mathcal{G}\text{cl}(\lambda)) \leq \text{FO}\mathcal{G}\text{Fr}(\lambda) \)

(vi) \( \lambda - \text{FO}\mathcal{G}\text{Fr}(\lambda) \leq \text{FO}\mathcal{G}\text{int}(\lambda) \)

(vii) \( \text{FO}\mathcal{G}\text{Fr}(\lambda \lor \mu) \leq (\text{FO}\mathcal{G}\text{Fr}(\lambda)) \lor (\text{FO}\mathcal{G}\text{Fr}(\mu)) \)

(viii) \( \text{FO}\mathcal{G}\text{Fr}(\lambda \land \mu) \leq (\text{FO}\mathcal{G}\text{Fr}(\lambda)) \land (\text{FO}\mathcal{G}\text{Fr}(\mu)) \).

**Proof:**

To prove (i)

\[
\text{FFr}(\lambda) = \text{F cl}(\lambda) \land \text{F cl}(\lambda^\prime) = \text{F cl}(\lambda) - \text{F int}(\lambda) \\
\leq \text{FO}\mathcal{G}\text{cl}(\lambda) - \text{FO}\mathcal{G}\text{int}(\lambda) \\
\leq \text{FO}\mathcal{G}\text{cl}(\lambda) \land \text{FO}\mathcal{G}\text{cl}(\lambda^\prime) = \text{FO}\mathcal{G}\text{Fr}(\lambda)
\]

Therefore, \( \text{FFr}(\lambda) \leq \text{FO}\mathcal{G}\text{Fr}(\lambda) \)

To prove (ii)

\[
\text{FO}\mathcal{G}b(\lambda) = \lambda \land \text{FO}\mathcal{G}\text{cl}(\lambda) = \lambda - \text{FO}\mathcal{G}\text{int}(\lambda) \\
\leq \text{FO}\mathcal{G}\text{cl}(\lambda) - \text{FO}\mathcal{G}\text{int}(\lambda),
\]
\[
\leq \text{FOGF} \text{cl}(\lambda) \land \text{FOGF} \text{cl}(\lambda)'
\]

Therefore, \( \text{FOG} b(\lambda) \leq \text{FOG} Fr(\lambda) \).

To prove (iii)

\[
\text{FOG} Fr(\lambda) = \text{FOG} \text{cl}(\lambda) \land \text{FOG}(\lambda)'
\]

\[
= \text{FOG} \text{cl}(\lambda) - (\text{FOG} \text{int}(\lambda))
\]

\[
= \text{FOG} \text{cl}(\lambda) - (1 - \text{FOG} \text{cl}(\lambda)')
\]

\[
= \text{FOG} \text{cl}(\lambda) - 1 + \text{FOG} \text{cl}(\lambda)''
\]

\[
= -\text{FOG} \text{int}(1 - \lambda) + \text{FOG} \text{cl}(\lambda)''
\]

\[
= \text{FOG} \text{cl}(\lambda)'' \land \text{FOG} \text{cl}(\lambda)
\]

\[
= \text{FOF} r(\lambda)'
\]

Therefore, \( \text{FOG} Fr(\lambda'') = \text{FOG} Fr(\lambda) \).

To prove (iv)

\[
\text{FOG} Fr(\text{FOG} \text{int}(\lambda)) = \text{FOG} \text{cl}(\lambda) \land \text{FOG} \text{cl}(\lambda)''
\]

\[
= \text{FOG} \text{cl}(\lambda) - (\text{FOG} \text{int}(\lambda)) - (\text{FOG} \text{int}(\lambda))
\]

\[
\leq \text{FOG} \text{cl}(\lambda) - \text{FOG} \text{int}(\lambda)
\]

\[
= \text{FOG} Fr(\lambda)
\]

Therefore, \( \text{FOG} Fr(\text{FOG} \text{int}(\lambda)) \leq \text{FOG} Fr(\lambda) \).

To prove (v)

\[
\text{FOG} Fr(\text{FOG} \text{cl}(\lambda)) = \text{FOG} \text{cl}(\lambda) \land \text{FOG} \text{cl}(\lambda)''
\]

\[
= \text{FOG} \text{cl}(\lambda) - (\text{FOG} \text{int}(\lambda))
\]

\[
\geq \text{FOG} \text{cl}(\lambda) - \text{FOG} \text{int}(\lambda)
\]

\[
\geq \text{FOG} \text{cl}(\lambda) \land \text{FOG} \text{cl}(\lambda)''
\]

\[
= \text{FOG} Fr(\lambda)
\]

Therefore, \( \text{FOG} Fr(\text{FOG} \text{cl}(\lambda)) \leq \text{FOG} Fr(\lambda) \).

To prove (vi)

\[
\lambda - \text{FOG} Fr(\lambda) = \lambda - (\text{FOG} \text{cl}(\lambda) \land \text{FOG} \text{cl}((\lambda)'))
\]

\[
= \lambda - (\text{FOG} \text{cl}(\lambda) - \text{FOG} \text{int}(\lambda))
\]

\[
\leq \text{FOG} \text{cl}(\lambda) - \text{FOG} \text{cl}(\lambda) + \text{FOG} \text{int}(\lambda)
\]

\[
= \text{FOG} \text{int}(\lambda)
\]

Therefore, \( \text{FOG} Fr(\lambda \lor \mu) \leq \text{FOG} Fr(\lambda) \).

To prove (vii)

\[
\text{FOG} Fr(\lambda \lor \mu) = \text{FOG} \text{cl}(\lambda \lor \mu) \land \text{FOG} \text{cl}((\lambda \lor \mu)')
\]

\[
= \text{FOG} \text{cl}(\lambda \lor \mu) - \text{FOG} \text{int}((\lambda \lor \mu))
\]

\[
= \text{FOG} \text{cl}(\lambda \lor \mu) - (\text{FOG} \text{int}(\lambda)) \lor (\text{FOG} \text{int}(\mu))
\]

\[
= \text{FOG} \text{cl}(\lambda \lor \mu) - \text{FOG} \text{int}(\lambda) \lor \text{FOG} \text{int}(\mu)
\]

\[
\leq \text{FOG} \text{cl}(\lambda \lor \mu) \land \text{FOG} \text{cl}(\lambda \lor \mu)'
\]

\[
= \text{FOG} Fr(\lambda \lor \mu)
\]
\[
(F \text{ O}_G \text{cl}(\lambda)) \lor (F \text{ O}_G \text{cl}(\mu)) - ((F \text{ O}_G \text{int}(\lambda)) \lor ((F \text{ O}_G \text{int}(\mu)))
\]
\[
\leq (F \text{ O}_G \text{cl}(\lambda) - F \text{ O}_G \text{int}(\lambda)) \lor (F \text{ O}_G \text{cl}(\mu) - F \text{ O}_G \text{int}(\mu))
\]
\[
\leq (F \text{ O}_G \text{cl}(\lambda) \land F \text{ O}_G \text{cl}(\lambda')) \lor (F \text{ O}_G \text{cl}(\mu) \land F \text{ O}_G \text{cl}(\mu'))
\]
\[
= (F \text{ O}_G \text{Fr}(\lambda)) \lor (F \text{ O}_G \text{Fr}(\mu))
\]

Therefore, \( F \text{ O}_G \text{Fr}(\lambda \lor \mu) \leq (F \text{ O}_G \text{Fr}(\lambda)) \lor (F \text{ O}_G \text{Fr}(\mu)) \).

To prove (viii)
\[
F \text{ O}_G \text{Fr}(\lambda \land \mu) = F \text{ O}_G \text{cl}(\lambda \land \mu) \land F \text{ O}_G ((\lambda \land \mu)')
\]
\[
= F \text{ O}_G \text{cl}(\lambda \land \mu) - F \text{ O}_G \text{int}((\lambda \land \mu))
\]
\[
= F \text{ O}_G \text{cl}(\lambda \land \mu) - (F \text{ O}_G \text{int}(\lambda)) \land (F \text{ O}_G \text{int}(\mu))
\]
\[
= (F \text{ O}_G \text{cl}(\lambda)) \land (F \text{ O}_G \text{cl}(\mu)) - ((F \text{ O}_G \text{int}(\lambda)) \land (F \text{ O}_G \text{int}(\mu))
\]
\[
\geq (F \text{ O}_G \text{cl}(\lambda) - F \text{ O}_G \text{int}(\lambda)) \land (F \text{ O}_G \text{cl}(\mu) - F \text{ O}_G \text{int}(\mu))
\]
\[
\geq (F \text{ O}_G \text{cl}(\lambda) \land F \text{ O}_G \text{cl}(\lambda') \land (F \text{ O}_G \text{cl}(\mu) \land F \text{ O}_G \text{cl}(\mu'))
\]
\[
= (F \text{ O}_G \text{Fr}(\lambda)) \land (F \text{ O}_G \text{Fr}(\mu))
\]

Therefore, \( F \text{ O}_G \text{Fr}(\lambda \land \mu) \geq (F \text{ O}_G \text{Fr}(\lambda)) \land (F \text{ O}_G \text{Fr}(\mu)) \).

**Proposition 4.3.** Let \((X, \text{ g})\) be a fuzzy orbit \( g \)-structure space and \( \lambda \) be a fuzzy orbit topological group. Then the following statements hold: then

(i) \( F \text{ O}_G \text{b}(\lambda) = F \text{ O}_G \text{Fr}(\lambda) \)

(ii) \( F \text{ O}_G \text{Exr}(\lambda) = 1 - \lambda \).

**Proof:** To prove (i) Since \( \lambda \) is a fuzzy orbit closed group, \( F \text{ O}_G \text{cl}(\lambda) = \lambda \)
\[
F \text{ O}_G \text{b}(\lambda) = \lambda \land F \text{ O}_G \text{cl}(\lambda')
\]
\[
= \lambda - F \text{ O}_G \text{int}(\lambda)
\]
\[
= F \text{ O}_G \text{cl}(\lambda) - F \text{ O}_G \text{int}(\lambda)
\]
\[
= F \text{ O}_G \text{cl}(\lambda) \land F \text{ O}_G \text{cl}(\lambda')
\]
\[
= F \text{ O}_G \text{Fr}(\lambda)
\]

Therefore, \( F \text{ O}_G \text{b}(\lambda) = F \text{ O}_G \text{Fr}(\lambda) \).

To prove (ii)
\[
F \text{ O}_G \text{cl}(\lambda) = \lambda
\]
\[
\Rightarrow F \text{ O}_G \text{int}(1 - \lambda) = 1 - \lambda
\]
Therefore by def, \( \text{FO}G\text{Exr}(\lambda) = 1 - \lambda \).

**Definition 4.10.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy orbit \(G\)-structure spaces. Let \(g : (X, G_1) \to (Y, G_2)\) be a fuzzy orbit continuous if the inverse image of every fuzzy open group in \((Y, G_2)\) is fuzzy orbit open group in \((X, G_1)\).

**Definition 4.11.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy orbit \(G\)-structure spaces. Let \(g : (X, G_1) \to (Y, G_2)\) be a fuzzy orbit irresolute if the inverse image of every fuzzy orbit open group in \((Y, G_2)\) is fuzzy orbit open group in \((X, G_1)\).

**Proposition 4.4.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy orbit \(G\)-structure spaces. Let \(g : (X, G_1) \to (Y, G_2)\) be a fuzzy orbit continuous mapping. Then for any fuzzy closed group \(\lambda\) in \((Y, G_2)\), \(\text{FO}G\text{Fr}(g^{-1}(\lambda)) = \text{FO}G\text{cl}(g^{-1}(\lambda))\).

**Proof:** Let \(\lambda\) be a fuzzy closed group in \((Y, G_2)\). Then by hypothesis, \(g^{-1}(\lambda)\) is a fuzzy orbit closed group in \((X, G_1)\). Therefore, \(\text{FO}G_2\text{cl}(g^{-1}(\lambda)) = g^{-1}(\lambda)\). Hence,

\[
\text{FO}G\text{Fr}(g^{-1}(\lambda)) = \text{FO}G\text{cl}(g^{-1}(\lambda)) \land \text{FO}G\text{cl}(g^{-1}(\lambda))' = \text{FO}G\text{cl}(g^{-1}(\lambda)) - \text{FO}G\text{int}(g^{-1}(\lambda)) = g^{-1}(\lambda) - \text{FO}G\text{int}(g^{-1}(\lambda)) = g^{-1}(\lambda) \land \text{FO}G\text{cl}(g^{-1}(\lambda))' = \text{FO}G\text{b}(g^{-1}(\lambda)).
\]

Therefore, \(\text{FO}G\text{Fr}(g^{-1}(\lambda)) = \text{FO}G\text{b}(g^{-1}(\lambda))\).

**Proposition 4.5.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy orbit \(G\)-structure spaces. Let \(g : (X, G_1) \to (Y, G_2)\) be any mapping. Then for any fuzzy group \(\lambda\) in \((Y, G_2)\), \(\text{FO}G\text{Ext}(g^{-1}(\lambda)) \leq \text{FO}G\text{cl}(g^{-1}(\lambda))\).

**Proof:** Let \(\lambda\) be any fuzzy group in \((Y, G_2)\). By def,

\[
\text{FO}G\text{Ext}(g^{-1}(\lambda)) = \text{FO}G\text{int}(g^{-1}(\lambda))' \leq (g^{-1}(\lambda))'.
\]

Again by the same result,

\[
\text{FO}G\text{Ext}(g^{-1}(\lambda)) = 1 - \text{FO}G\text{int}(g^{-1}(\lambda))' = \text{FO}G\text{cl}(g^{-1}(\lambda))'.
\]

Therefore, \(\text{FO}G\text{Ext}(g^{-1}(\lambda)) \leq \text{FO}G\text{cl}(g^{-1}(\lambda))'\).

**Proposition 4.6.** Let \((X, G_1)\) and \((Y, G_2)\) be any two fuzzy orbit \(G\)-structure spaces. Let \(g : (X, G_1) \to (Y, G_2)\) be a fuzzy orbit irresolute. Then for any fuzzy orbit closed group \(\lambda\) in \((Y, G_2)\),

1. \(\text{FO}G\text{b}(g^{-1}(\lambda)) = \text{FO}G\text{Fr}(g^{-1}(\lambda))\) and
2. \(\text{FO}G\text{Ext}(g^{-1}(\lambda)) = (g^{-1}(\lambda))'\).

**Proof:** (i)
Let $\lambda$ be a fuzzy orbit closed group in $(Y, \mathcal{G}_2)$. Then by hypothesis, $g^{-1}(\lambda)$ is a fuzzy orbit closed group in $(X, \mathcal{G}_1)$. Therefore, $\text{FO}\mathcal{G}\text{cl}(g^{-1}(\lambda)) = g^{-1}(\lambda)$. By def,

\[
\text{FO}\mathcal{G}\text{b}(g^{-1}(\lambda)) = g^{-1}(\lambda) \land \text{FO}\mathcal{G}\text{cl}(g^{-1}(\lambda))
\]

\[
= g^{-1}(\lambda) - \text{FO}\mathcal{G}\text{int}(g^{-1}(\lambda))
\]

\[
= \text{FO}\mathcal{G}\text{cl}(g^{-1}(\lambda)) - \text{FO}\mathcal{G}\text{int}(g^{-1}(\lambda))
\]

\[
= \text{FO}\mathcal{G}\text{cl}(g^{-1}(\lambda)) \land \text{FO}\mathcal{G}\text{cl}(1 - g^{-1}(\lambda))
\]

\[
= \text{FO}\mathcal{G}\text{Fr}(g^{-1}(\lambda))
\]

Therefore, $\text{FO}\mathcal{G}\text{b}(g^{-1}(\lambda)) = \text{FO}\mathcal{G}\text{Fr}(g^{-1}(\lambda))$.

(ii)

$\text{FO}\mathcal{G}\text{Ext}(g^{-1}(\lambda)) = \text{FO}\mathcal{G}\text{int}(g^{-1}(\lambda))$

By Prop 4.4, $\text{FO}\mathcal{G}\text{int}(g^{-1}(\lambda)) = (g^{-1}(\lambda))$.

$\text{FO}\mathcal{G}\text{Ext}(g^{-1}(\lambda)) = (g^{-1}(\lambda))'$.

**Proposition 4.7.** Let $(X, \mathcal{G}_1), (Y, \mathcal{G}_2)$ and $(Z, \mathcal{G}_3)$ be any three fuzzy orbit $\mathcal{G}$-structure spaces. Let $g : (X, \mathcal{G}_1) \rightarrow (Y, \mathcal{G}_2)$ and $h : (Y, \mathcal{G}_2) \rightarrow (Z, \mathcal{G}_3)$ be fuzzy orbit irresolute mappings. For every fuzzy orbit closed group in $(Z, \mathcal{G}_3)$, then the following statements hold:

(i) $\text{FO}\mathcal{G}\text{b}((h \circ g)^{-1}(\lambda)) = \text{FO}\mathcal{G}\text{Fr}((h \circ g)^{-1}(\lambda))$ and

(ii) $\text{FO}\mathcal{G}\text{Ext}((h \circ g)^{-1}(\lambda)) = ((h \circ g)^{-1}(\lambda))'$.

**Proof:**

To prove (i)

Let $\lambda$ be any fuzzy orbit closed group in $(Z, \mathcal{G}_3)$. Then, since $h$ is fuzzy orbit irresolute, $h^{-1}(\lambda)$ is fuzzy orbit closed group in $(Y, \mathcal{G}_2)$. And since $g$ is fuzzy orbit irresolute, $g^{-1}(h^{-1}(\lambda))$ is fuzzy orbit closed group in $(X, \mathcal{G}_1)$. That is, $(h \circ g)^{-1}(\lambda)$ is fuzzy orbit closed group in $(X, \mathcal{G}_1)$. (h \cdot g)^{-1}(\lambda)$ is fuzzy orbit closed group in $(X, \mathcal{G}_1)$. By definition,

\[
\text{FO}\mathcal{G}\text{b}((h \cdot g)^{-1}(\lambda)) = (h \cdot g)^{-1}(\lambda) \land \text{FO}\mathcal{G}\text{cl}((h \cdot g)^{-1}(\lambda))'
\]

\[
= (h \cdot g)^{-1}(\lambda) - \text{FO}\mathcal{G}\text{int}((h \cdot g)^{-1}(\lambda))
\]

\[
= \text{FO}\mathcal{G}\text{cl}((h \cdot g)^{-1}(\lambda)) - \text{FO}\mathcal{G}\text{int}((h \cdot g)^{-1}(\lambda))
\]

\[
= \text{FO}\mathcal{G}\text{cl}((h \cdot g)^{-1}(\lambda)) \land \text{FO}\mathcal{G}\text{cl}((h \cdot g)^{-1}(\lambda))'
\]

\[
= \text{FO}\mathcal{G}\text{Fr}((h \cdot g)^{-1}(\lambda))
\]

Therefore, $\text{FO}\mathcal{G}\text{b}((h \cdot g)^{-1}(\lambda)) = \text{FO}\mathcal{G}\text{Fr}((h \cdot g)^{-1}(\lambda))$.

(ii)

By def,

\[
\text{FO}\mathcal{G}\text{Ext}((h \cdot g)^{-1}(\lambda)) = \text{FO}\mathcal{G}\text{int}((h \cdot g)^{-1}(\lambda))'
\]

\[
= 1 - \text{FO}\mathcal{G}\text{cl}((h \cdot g)^{-1}(\lambda))
\]

\[
= 1 - (h \cdot g)^{-1}(\lambda)
\]

since $(h \cdot g)^{-1}(\lambda)$ is fuzzy orbit closed. Therefore, $\text{FO}\mathcal{G}\text{Ext}((h \cdot g)^{-1}(\lambda)) = ((h \cdot g)^{-1}(\lambda))'$. 

References