Some New Families of Face Integer Cordial Labeling of Graph

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Abstract - In this paper, we have investigated the face integer cordial labeling of gear graph G_n, switching of any one vertex of cycle C_n, P_n  P_m and C_n  K_2.

Keywords - Integer cordial graph, face integer cordial labeling, face integer cordial graph.

I. INTRODUCTION

We begin with simple, finite, planar, undirected graph. A (p,q) planar graph G means a graph G=(V,E), where V is the set of vertices with |V| = p, E is the set of edges with |E| = q and F is the set of interior faces of G with |F| = number of interior faces of G. For standard terminology and notations related to graph theory we refer to Harary [3].

A graph labeling is the assignment of unique identifiers to the edges and vertices of a graph. For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian [2].

A mapping f : V(G) → {0,1} is called binary vertex labeling of G and f(v) is called the label of the vertex v of G under f. If for an edge e = uv, the induced edge labeling f* : E(G ) → {0,1} is given by f*(e ) = | f(u) – f(v)|. Then v_f(i) = number of vertices having label i under f and e_f(i) = number of edges having label i under f*. A binary vertex labeling f of a graph G is called a cordial labeling of G if |v_f(0) – v_f(1)| ≤ 1 and |e_f(0) – e_f(1) | ≤ 1. A graph G is cordial if it admits cordial labeling. In [1], Cahit introduced the concept of cordial labeling of graph.

Let G be a simple connected graph with p vertices. Let f:V→[-p2,…,p2]* or [-p2,…,p2] as p is even or odd be an injective map, which induces an edge labeling f* such that f(uv) = 1, if f(u)+f(v) ≥ 0 and f(uv) = 0 otherwise. Let e_f(i) = number of edges labeled with i, where i = 0 or 1. f is said to be integer cordial if |e_f(0) – e_f(1)| ≤ 1. A graph G is called integer cordial if it admits an integer cordial labeling. Here [-x,…,x] = {t / t is an integer and |t| ≤ x} and [-x,…,x]* = [-x, …, x] – {0}.

In [5], Nicholas et al. introduced the concept of integer cordial labeling of graphs and proved that some standard graphs such as cycle C_n, Path P_n, Wheel graph W_n; n > 3, Star graph K_1,n, Helm graph H_n, Closed helm graph CH_n are integer cordial, K_n is not integer cordial, K_n,n is integer cordial iff n is even and K_n,n\M is integer cordial for any n, where M is a perfect matching of K_n,n.

For a planar graph G, the vertex labeling function is defined as g : V → [-p2,…,p2]* or [-p2,…,p2] as p is even or odd be an injective map, which induces an edge labeling function g* : E(G)→{0,1} such that g*(uv) = 1, if g(u) + g(v) ≥ 0 and g*(uv) = 0 otherwise and face labeling function g** : F(G) → {0,1} such that g**(f) = 1, if g(v_1)+…+ g(v_n) ≥ 0
and $g^*(f) = 0$ otherwise, where $v_1, v_2, \ldots, v_n$ are the vertices of face $f$. $g$ is called face integer cordial labeling of graph $G$ if $|e_g(0) - e_g(1)| \leq 1$ and $|f_g(0) - f_g(1)| \leq 1$. $e_g(i)$ is the number of edges of $G$ having label $i$ under $g^*$ and $f_g(i)$ is the number of interior faces of $G$ having label $i$ under $g^*$ for $i = 1, 2$. A planar graph $G$ is face integer cordial if it admits face integer cordial labeling.

In [4], Mohamed Sheriff et al introduced the concept of face integer cordial labeling of graphs and proved the face integer cordial labeling of wheel $W_n$, fan $f_n$, triangular snake $T_n$, double triangular snake $DT_n$, star of cycle $C_n$ and $DS(B_{n,n})$. The brief summaries of definition which are necessary for the present investigation are provided below.

**Definition : 1.1**

A wheel $W_n$ is a graph with $n+1$ vertices, formed by connecting a single vertex to all the vertices of cycle $C_n$. It is denoted by $W_n = C_n + K_1$.

**Definition : 1.2**

The gear graph $G_n$ is obtained from the wheel $W_n$ by subdividing each of its rim edge.

**Definition : 1.3**

The corona $G_1 \bigcirc G_2$ of two graphs $G_1(p_1, q_1)$ and $G_2(p_2, q_2)$ is defined as the graph obtained by taking one copy of $G_1$ and $p_1$ copies of $G_2$ and then joining the $i^{th}$ vertex of $G_1$ to all the vertices in the $i^{th}$ copy of $G_2$.

**Definition : 1.4**

A vertex switching $G_v$ of a graph $G$ is obtained by taking a vertex $v$ of $G$, removing the entire edges incident with $v$ and adding edges joining $v$ to every vertex which are not adjacent to $v$ in $G$.

II. MAIN THEOREMS

**Theorem : 2.1**

The gear graph $G_n$ is a face integer cordial graph for $n \geq 3$.

**Proof.**

Let $v$ be the apex vertex, $v_1, v_2, \ldots, v_{2n}$ be rim vertices, $e_1, e_2, \ldots, e_{3n}$ be edges and $f_1, f_2, \ldots, f_6$ be interior faces of the gear graph $G_n$, where $e_i = vv_{2i-1}$, for $1 \leq i \leq n$, $e_{n+i} = v_i v_{i+1}$, for $1 \leq i \leq 2n-1$, $e_{3n} = v_{2n} v_1$, $f_i = vv_{2i-1}v_{2i}v_{2i+1}v$, for $1 \leq i \leq n-1$ and $f_6 = vv_{2n-1}v_{2n}v_1v$.

Let $G$ be the gear graph $G_n$. Then $|V(G)| = 2n+1$, $|E(G)| = 3n$ and $|F(G)| = n$.

**Case (i) :** $n$ is odd. Let $n = 2k+1$.

Define vertex labeling of $g : V(G) \to [-n, \ldots, n]$ as follows.

- $g(v) = 0$
- $g(v_i) = -i$ for $1 \leq i \leq n$
- $g(v_{n+i}) = n+1- i$ for $1 \leq i \leq n$

Then induced edge labels are

- $g^*(e_i) = 0$ for $1 \leq i \leq \frac{n+1}{2}$
- $g^*(e_i) = 1$ for $\frac{n+3}{2} \leq i \leq n$
- $g^*(e_{n+i}) = 0$ for $1 \leq i \leq n-1$
- $g^*(e_{n+i}) = 1$ for $n \leq i \leq 2n$
Also the induced face labels are
\[ g^{**}(f_i) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{n-1}{2} \]
\[ g^{**}(f_i) = 1 \quad \text{for} \quad \frac{n+1}{2} \leq i \leq n \]

In view of the above defined labeling pattern, we have \( e_g(1) = e_g(0) + 1 = \frac{3n+1}{2} \) and \( f_g(1) = f_g(0) + 1 = \frac{n+1}{2} \). Then \( |e_g(0) - e_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \).

Thus the wheel \( W_n \) is the face integer cordial for \( n \) is odd.

**Case 2:** \( n \) is even. Let \( n = 2k \).

Define vertex labeling of \( g : V(G) \to [-n, \ldots, n] \) as follows.
\[ g(v) = 0 \]
\[ g(v_i) = -i \quad \text{for} \quad 1 \leq i \leq n \]
\[ g(v_{n+i}) = n+1 - i \quad \text{for} \quad 1 \leq i \leq n \]

Then induced edge labels are
\[ g^*(e_i) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} \]
\[ g^*(e_i) = 1 \quad \text{for} \quad \frac{n+2}{2} \leq i \leq n \]
\[ g^*(e_{n+i}) = 0 \quad \text{for} \quad 1 \leq i \leq n \]
\[ g^*(e_{n+i}) = 1 \quad \text{for} \quad n+1 \leq i \leq 2n \]

Also the induced face labels are
\[ g^{**}(f_i) = 0 \quad \text{for} \quad 1 \leq i \leq \frac{n}{2} \]
\[ g^{**}(f_i) = 1 \quad \text{for} \quad \frac{n+2}{2} \leq i \leq n \]

In view of the above defined labeling pattern, we have \( e_g(0) = e_g(1) = \frac{3n}{2} \) and \( f_g(1) = f_g(0) = \frac{n}{2} \). Then \( |e_g(0) - e_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \).

Thus the gear graph \( G_n \) is the face integer cordial for \( n \) is even.

Hence the gear graph \( G_n \) is the face integer cordial graph for \( n \geq 3 \).

**Example : 2.1**

The gear graph \( G_5 \) and its face integer cordial labeling is shown in figure 2.1.
**Theorem : 2.2**

Switching of any one vertex in cycle $C_n$ admits face integer cordial labeling for $n \geq 4$.

**Proof.**

Let $v_1, v_2, \ldots, v_n$ be the successive vertices of $C_n$. $G_v$ denotes graph is obtained by switching of vertex $v$ of $C_n$. Without loss of generality let the switched vertex be $v_1$.

Let $G$ be a graph $G_{v_1}$. Then $v_1, v_2, \ldots, v_n$ are vertices of $G$, $e_1, e_2, \ldots, e_{2n-5}$ are edges of $G$ and $f_1, f_2, \ldots, f_{n-4}$ are the interior faces of $G$. $e_i = v_1v_{i+2}$, for $1 \leq i \leq n-3$, $e_{n-3+i} = v_{i+1}v_{i+2}$, for $1 \leq i \leq n-2$.

Then $|V(G)| = n$, $|E(G)| = 2n-5$ and $|F(G)| = n-4$.

**Case (i) :** $n$ is even and $n = 2k$.

Define a vertex labeling $g : V(G) \rightarrow [-k, \ldots, k]$ as follows

\[
g(v_i) = -i, \quad \text{for } 1 \leq i \leq \frac{n}{2}
\]

\[
g(v_i) = i - \frac{n}{2}, \quad \text{for } \frac{n+2}{2} \leq i \leq n
\]

Then induced edge labels are

\[
g^*(e_i) = 0, \quad \text{for } 1 \leq i \leq \frac{n-4}{2}
\]

\[
g^*(e_i) = 1, \quad \text{for } \frac{n-2}{2} \leq i \leq n-3
\]

\[
g^*(e_{n-3+i}) = 0, \quad \text{for } 1 \leq i \leq \frac{n-2}{2}
\]

\[
g^*(e_{n-3+i}) = 1, \quad \text{for } \frac{n}{2} \leq i \leq n-2
\]

Also the induced face labels are

\[
g^{**}(f_i) = 0, \quad \text{for } 1 \leq i \leq \frac{n-4}{2}
\]

\[
g^{**}(f_i) = 1, \quad \text{for } \frac{n-2}{2} \leq i \leq n-4
\]

In view of the above defined labeling pattern, we have $e^f(1) = e^g(0)+1 = \frac{2n-4}{2}$ and $f^g(0) = f^g(1) = \frac{n-4}{2}$.

Then $|e^g(0) - e^g(1)| \leq 1$ and $|f^g(0) - f^g(1)| \leq 1$

Hence $G$ is face integer cordial graph for $n$ is even.

**Case (ii) :** $n$ is odd and $n = 2k+1$.

Define a vertex labeling $g : V(G) \rightarrow [-k, \ldots, k]$ as follows

\[
g(v_1) = 0,
\]

\[
g(v_{i+1}) = -i, \quad \text{for } 1 \leq i \leq \frac{n-1}{2}
\]

\[
g(v_{i+1}) = i - \frac{n-1}{2}, \quad \text{for } \frac{n+1}{2} \leq i \leq n-1
\]

Then induced edge labels are

\[
g^*(e_i) = 0, \quad \text{for } 1 \leq i \leq \frac{n-3}{2}
\]
\( g^*(e_i) = 1, \quad \text{for } \frac{n-1}{2} \leq i \leq n-3 \)

\( g^*(e_{n-3+i}) = 0, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \)

\( g^*(e_{n-3+i}) = 1, \quad \text{for } \frac{n+1}{2} \leq i \leq n-2 \)

Also the induced face labels are

\( g^{**}(f_i) = 0, \quad \text{for } 1 \leq i \leq \frac{n-3}{2} \)

\( g^{**}(f_i) = 1, \quad \text{for } \frac{n-1}{2} \leq i \leq n-4 \)

In view of the above defined labeling pattern, we have \( e_{i}(0) = e_{i}(1)+1 = n-2 \) and \( f_{g}(0) = f_{g}(1)+1 = \frac{n-3}{2} \).

Then \( |e_{i}(0) - e_{i}(1)| \leq 1 \) and \( |f_{g}(0) - f_{g}(1)| \leq 1 \)

Hence \( G \) is face integer cordial graph for \( n \) is odd.

Therefore switching of any one vertex in cycle \( C_n \) is face integer cordial graph for \( n \geq 4 \).

**Example : 2.2**

Switching of a vertex \( v_1 \) in cycle \( C_8 \) and its face integer cordial labeling is shown in figure 2.2.

![Figure 2.2](image_url)

**Theorem : 2.3**

\( C_n \oplus K_2 \) is face integer cordial graph for \( n \geq 3 \).

**Proof.**

Let \( v_1, v_2, ..., v_n \) and \( e_1, e_2, ..., e_n \) be the vertices and edges of \( C_n \). Let \( G \) be a graph \( C_n \oplus K_2 \).

Let \( v_1, v_2, ..., v_n, v_{ij} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq 2 \), \( e_1, e_2, ..., e_n, e_{ij} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq 3 \) and \( f, f_1, f_2, ..., f_n \) be vertices, edges and an interior faces of \( G \), where \( e_i = v_i v_{i+1} \), for \( 1 \leq i \leq n-1 \), \( e_n = v_n v_1 \), \( e_{i1} = v_i v_{i+1} \), \( e_{i2} = v_i v_{i+2} \), \( e_{i3} = v_{i+2} v_i \), for \( 1 \leq i \leq n \), \( f = v_1 v_2 ... v_n \) and \( f_i = v_i v_{i+1} v_{i+2} \) for \( 1 \leq i \leq n \).

Then \( |V(G)| = 3n \), \( |E(G)| = 4n \) and \( |F(G)| = n+1 \).

**Case (i):** \( n \) is even.

Let \( 3n = 2k \)

Define a vertex labeling \( g : V(G) \to [-k, ..., k]^* \) as follows.

\( g(v_i) = i, \quad \text{for } 1 \leq i \leq \frac{n}{2} \)

\( g^*(e_i) = 1, \quad \text{for } \frac{n-1}{2} \leq i \leq n-3 \)

\( g^*(e_{n-3+i}) = 0, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \)

\( g^*(e_{n-3+i}) = 1, \quad \text{for } \frac{n+1}{2} \leq i \leq n-2 \)

Also the induced face labels are

\( g^{**}(f_i) = 0, \quad \text{for } 1 \leq i \leq \frac{n-3}{2} \)

\( g^{**}(f_i) = 1, \quad \text{for } \frac{n-1}{2} \leq i \leq n-4 \)

In view of the above defined labeling pattern, we have \( e_{i}(0) = e_{i}(1)+1 = n-2 \) and \( f_{g}(0) = f_{g}(1)+1 = \frac{n-3}{2} \).

Then \( |e_{i}(0) - e_{i}(1)| \leq 1 \) and \( |f_{g}(0) - f_{g}(1)| \leq 1 \)

Hence \( G \) is face integer cordial graph for \( n \) is odd.

Therefore switching of any one vertex in cycle \( C_n \) is face integer cordial graph for \( n \geq 4 \).
\[ g(v_i) = \frac{n}{2} - i, \quad \text{for } \frac{n+2}{2} \leq i \leq n \]

\[ g(v_{ij}) = \frac{n}{2} + 2(i-1) + j, \quad \text{for } 1 \leq i \leq \frac{n}{2} \text{ and } 1 \leq j \leq 2 \]

\[ g(v_{ij}) = -\frac{n}{2} - 2 \left[ i - \frac{n+2}{2} \right] - j, \quad \text{for } \frac{n+2}{2} \leq i \leq n \text{ and } 1 \leq j \leq 2 \]

Then induced edge labels are

\[ g^*(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} \]

\[ g^*(e_i) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n \]

\[ g^*(e_{ij}) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} \text{ and } 1 \leq j \leq 3 \]

\[ g^*(e_{ij}) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n \text{ and } 1 \leq j \leq 3 \]

Also the induced face labels are

\[ g^{**}(f) = 1, \]

\[ g^{**}(f_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} \]

\[ g^{**}(f_i) = 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n \]

In view of the above defined labeling pattern we have \( e_g(1) = e_g(0) = 2n \) and \( f_g(1) = f_g(0) + 1 = \frac{n+2}{2} \).

Then \( |e_g(0) - e_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \)

Therefore, the graph \( C_n \oplus K_2 \) is face integer product cordial graph for \( n \) is even.

**Case (ii): \( n \) is odd.**

Let \( 3n = 2k+1 \)

Define a vertex labeling \( g : V(G) \rightarrow [-k, \ldots, k] \) as follows.

\[ g(v_i) = 1+i, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \]

\[ g(v_{\frac{n+1}{2}}) = 0, \]

\[ g(v_i) = -1 - \left[ i - \frac{n+1}{2} \right], \quad \text{for } \frac{n+3}{2} \leq i \leq n-1 \]

\[ g(v_n) = -k \]

\[ g(v_{ij}) = 1+ \frac{n-1}{2} + 2(i-1) + j, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ and } 1 \leq j \leq 2 \]

\[ g(v_{i1}) = 1, \quad \text{for } i = \frac{n+1}{2} \]

\[ g(v_{i2}) = -1, \quad \text{for } i = \frac{n+1}{2} \]

\[ g(v_{ij}) = -1 - \frac{n-1}{2} - 2 \left[ i - \frac{n+3}{2} \right] - j, \quad \text{for } \frac{n+3}{2} \leq i \leq n-1 \text{ and } 1 \leq j \leq 2 \]
\[ g(v_{n1}) = -k+1, \]
\[ g(v_{n2}) = \left( \frac{n+1}{2} \right). \]

Then induced edge labels are
\[ g^*(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \]
\[ g^*(e_i) = 0, \quad \text{for } \frac{n+1}{2} \leq i \leq n \]
\[ g^*(e_{ij}) = 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ and } 1 \leq j \leq 3 \]
\[ g^*(e_{ij}) = 1, \quad \text{for } i = \frac{n+1}{2} \text{ and } 1 \leq j \leq 2 \]
\[ g^*(e_{ij}) = 0, \quad \text{for } i = \frac{n+1}{2} \]
\[ g^*(e_{ij}) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n \text{ and } 1 \leq j \leq 3 \]

Also the induced face labels are
\[ g^{**}(f) = 0, \]
\[ g^{**}(f_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n+1}{2} \]
\[ g^{**}(f_i) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n \]

In view of the above defined labeling pattern we have \( e_g(1) = e_g(0) = 2n \) and \( f_g(1) = f_g(0) = \frac{n+1}{2} \). Then \(|e_g(0) - e_g(1)| \leq 1\) and \(|f_g(0) - f_g(1)| \leq 1\)

Therefore, the graph \( C_n \circ K_2 \) is face integer product cordial graph for \( n \) is even and \( m \) is even. Therefore \( C_n \circ K_2 \) is face integer cordial graph for \( n \geq 3 \).

**Example : 2.3**

The graph \( C_3 \circ K_2 \) and its face integer cordial labeling of graph is shown in figure 2.3.
Theorem: 2.4

The graph \( P_n \square P_m \) is face integer product cordial graph for any \( n, m \geq 2 \).

Proof.

Let \( v_1, v_2, \ldots, v_n \) and \( e_1, e_2, \ldots, e_{n-1} \) be the vertices and edges of \( P_n \).

Let \( G \) be the graph \( P_n \square P_m \).

The vertex set \( V(G) = \{v_i, v_{ij} : 1 \leq i \leq n, 1 \leq j \leq m\} \), edge set \( E(G) = \{e_i, e_{jk} : 1 \leq i \leq n-1, 1 \leq j \leq n \text{ and } 1 \leq k \leq 2m-1\} \) and interior face set \( F(G) = \{f_i : 1 \leq i \leq n(m-1)\} \), where \( e_i = v_i v_{i+1} \) for \( 1 \leq i \leq n-1 \), \( e_{jk} = v_j v_{jk} \) for \( 1 \leq j \leq n \text{ and } 1 \leq k \leq m \), \( e_{(m+k)} = v_j v_{jk} v_{j(k+1)} \) for \( 1 \leq j \leq n \) and \( 1 \leq k \leq m-1 \).

Then \( |V(G)| = n(m+1) \), \( |E(G)| = 2nm-1 \) and \( |F(G)| = n(m-1) \).

Case (i): \( n \) is even and \( m \) is either odd or even.

Let \( n(m+1) = 2k \).

Define a vertex labeling \( g : V(G) \to [-k, \ldots, k] \) as follows.

\[
\begin{align*}
g(v_i) &= i, \quad \text{for } 1 \leq i \leq \frac{n}{2} \\
g(v_i) &= \frac{n}{2} - i, \quad \text{for } \frac{n+2}{2} \leq i \leq n \\
g(v_{ij}) &= \frac{n}{2} + m(i-1) + j, \quad \text{for } 1 \leq i \leq \frac{n}{2} \text{ and } 1 \leq j \leq m \\
g(v_{ij}) &= -\frac{n}{2} - m \left[i - \frac{n+2}{2}\right], \quad \text{for } \frac{n+2}{2} \leq i \leq n \text{ and } 1 \leq j \leq m.
\end{align*}
\]

Then induced edge labels are

\[
\begin{align*}
g^*(e_i) &= 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} \\
g^*(e_i) &= 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n-1 \\
g^*(e_{ij}) &= 1, \quad \text{for } 1 \leq i \leq \frac{n}{2} \text{ and } 1 \leq j \leq 2m-1 \\
g^*(e_{ij}) &= 0, \quad \text{for } \frac{n+2}{2} \leq i \leq n \text{ and } 1 \leq j \leq 2m-1
\end{align*}
\]

Also the induced face labels are

\[
\begin{align*}
g^{**}(f_{ij}) &= 1, \quad \text{for } 1 \leq i \leq \frac{n(m-1)}{2} \text{ and } 1 \leq j \leq m \\
g^{**}(f_{ij}) &= 0, \quad \text{for } \frac{n(m-1)+2}{2} \leq i \leq n(m-1) \text{ and } 1 \leq j \leq m
\end{align*}
\]

In view of the above defined labeling pattern we have \( e_g(1) = e_g(0)+1 = nm \) and \( f_g(0) = f_g(1) = \frac{n(m-1)}{2} \).

Then \( |e_g(0) - e_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \)

Therefore, the graph \( P_n \square P_m \) is face integer product cordial graph for \( n \) is even and \( m \) is either odd or even.

Case (ii): \( n \) is odd and \( m \) is even.

Let \( n(m+1) = 2k+1 \).

Define a vertex labeling \( g : V(G) \to [-k, \ldots, k] \) as follows.
\[ g(v_i) = \frac{m}{2} + i, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \]

\[ g(v_{n+1}) = 0, \]

\[ g(v_i) = -\frac{m}{2} - \left[ i - \frac{n+1}{2} \right], \quad \text{for } \frac{n+3}{2} \leq i \leq n \]

\[ g(v_i) = \frac{m}{2} + \frac{n-1}{2} + m(i-1) + j, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ and } 1 \leq j \leq m \]

\[ g(v_i) = j, \quad \text{for } i = \frac{n+1}{2} \text{ and } 1 \leq j \leq \frac{m}{2} \]

\[ g(v_i) = -\left[ j - \frac{m}{2} \right], \quad \text{for } i = \frac{n+1}{2} \text{ and } \frac{m+2}{2} \leq j \leq m \]

\[ g(v_i) = -\frac{m}{2} - \frac{n-1}{2} - m \left[ i - \frac{n+3}{2} \right] - j, \quad \text{for } \frac{n+3}{2} \leq i \leq n \text{ and } 1 \leq j \leq m \]

Then induced edge labels are

\[ g^*(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \]

\[ g^*(e_i) = 0, \quad \text{for } \frac{n+1}{2} \leq i \leq n-1 \]

\[ g^*(e_{ij}) = 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ and } 1 \leq j \leq 2m-1 \]

\[ g^*(e_{ij}) = 1, \quad \text{for } i = \frac{n+1}{2} \text{ and } 1 \leq j \leq \frac{m}{2} \]

\[ g^*(e_{ij}) = 0, \quad \text{for } i = \frac{n+1}{2} \text{ and } \frac{m+2}{2} \leq j \leq m \]

\[ g^*(e_{ij}) = 1, \quad \text{for } i = \frac{n+1}{2} \text{ and } m+1 \leq j \leq \frac{3m}{2} \]

\[ g^*(e_{ij}) = 0, \quad \text{for } i = \frac{n+1}{2} \text{ and } \frac{3m+2}{2} \leq j \leq 2m-1 \]

\[ g^*(e_{ij}) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n \text{ and } 1 \leq j \leq 2m-1 \]

Also the induced face labels are

\[ g^{**}(f_{ij}) = 1 \quad \text{for } 1 \leq i \leq \frac{n(m-1)+1}{2} \text{ and } 1 \leq j \leq m \]

\[ g^{**}(f_{ij}) = 0 \quad \text{for } \frac{n(m-1)+3}{2} \leq i \leq n(m-1) \text{ and } 1 \leq j \leq m \]

In view of the above defined labeling pattern we have

\[ e_g(1) = e_g(0)+1 = nm \text{ and } f_g(1) = f_g(0)+1 = \frac{n(m-1)+1}{2}. \]

Then \( |e_g(0) - e_g(1)| \leq 1 \) and \( |f_g(0) - f_g(1)| \leq 1 \)

Therefore, the graph \( P_n \Theta P_m \) is face integer product cordial graph for \( n \) is even and \( m \) is even.

**Case (iii):** \( n \) is odd and \( m \) is odd.

Let \( n(m+1) = 2k \)
Define a vertex labeling \( g : V(G) \rightarrow [-k,\ldots,k]^* \) as follows.

\[
g(v_i) = \frac{m+1}{2} + i, \quad \text{for } 1 \leq i \leq \frac{n-1}{2}
\]

\[
g(v_{n+1}) = 1,
\]

\[
g(v_i) = -\frac{m+1}{2} - \left[ \frac{i-n+1}{2} \right], \quad \text{for } \frac{n+3}{2} \leq i \leq n
\]

\[
g(v_{ij}) = \frac{m+1}{2} + \frac{n-1}{2} + m(i-1) + j, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ and } 1 \leq j \leq m
\]

\[
g(v_i) = 1+j, \quad \text{for } i = \frac{n+1}{2} \text{ and } 1 \leq j \leq \frac{m-1}{2}
\]

\[
g(v_i) = -\left[ j - \left( \frac{m-1}{2} \right) \right], \quad \text{for } i = \frac{n+1}{2} \text{ and } \frac{m+1}{2} \leq j \leq m
\]

\[
g(v_{ij}) = -\frac{m+1}{2} - \frac{n-1}{2} - m\left[ i - \frac{n+3}{2} \right] - j, \quad \text{for } \frac{n+3}{2} \leq i \leq n \text{ and } 1 \leq j \leq m
\]

Then induced edge labels are

\[
g^*(e_i) = 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2}
\]

\[
g^*(e_i) = 0, \quad \text{for } \frac{n+1}{2} \leq i \leq n-1
\]

\[
g^*(e_{ij}) = 1, \quad \text{for } 1 \leq i \leq \frac{n-1}{2} \text{ and } 1 \leq j \leq 2m-1
\]

\[
g^*(e_i) = 1, \quad \text{for } i = \frac{n+1}{2} \text{ and } 1 \leq j \leq \frac{m+1}{2}
\]

\[
g^*(e_{ij}) = 0, \quad \text{for } i = \frac{n+1}{2} \text{ and } \frac{m+3}{2} \leq j \leq m
\]

\[
g^*(e_i) = 1, \quad \text{for } i = \frac{n+1}{2} \text{ and } m+1 \leq j \leq \frac{3m-1}{2}
\]

\[
g^*(e_{ij}) = 0, \quad \text{for } i = \frac{n+1}{2} \text{ and } \frac{3m+1}{2} \leq j \leq 2m-1
\]

\[
g^*(e_i) = 0, \quad \text{for } \frac{n+3}{2} \leq i \leq n \text{ and } 1 \leq j \leq 2m-1
\]

Also the induced face labels are

\[
g^{**}(f_{ij}) = 1 \quad \text{for } 1 \leq i \leq \frac{n(m-1)}{2} \text{ and } 1 \leq j \leq m
\]

\[
g^{**}(f_{ij}) = 0 \quad \text{for } \frac{n(m-1)+2}{2} \leq i \leq n(m-1) \text{ and } 1 \leq j \leq m
\]

In view of the above defined labeling pattern we have \( e_g(1) = e_g(0) + 1 = nm \) and \( f_g(1) = f_g(0) = \frac{n(m-1)}{2} \).

Then \(|e_g(0) - e_g(1)| \leq 1 \) and \(|f_g(0) - f_g(1)| \leq 1 \). Therefore, the graph \( P_n \Theta P_m \) is face integer product cordial graph for \( n \) is even and \( m \) is even.

Hence \( P_n \Theta P_m \) is face integer product cordial graph for any \( n, m \geq 2 \).
Example : 2.4

The graph $P_3 \Theta P_3$ and its face product cordial labeling is given in figure 2.4.

III. CONCLUSION

In this paper, we prove the gear graph $G_n$, switching of any one vertex of cycle $C_n$, $P_n \Theta P_m$ and $C_n \Theta K_2$ are face integer cordial graph.

REFERENCES