Some New kinds of domination parameters in fuzzy graph

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Abstract

In this paper, the concept of fuzzy neighborhood clique domination number, fuzzy clique regular domination number, perfect disconnected domination number are introduced and its properties are investigated. Furthermore this new domination parameter is compare with other known domination parameters.

Keywords – Clique domination number, Neighborhood clique domination number, Clique regular domination number, perfect disconnected domination number.

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I. Introduction:

The study of domination set in graphs was begun by Ore and Berge\([1]\). Rosenfield \([2]\) introduced the notion of fuzzy graph and several fuzzy analogs of graph theoretic concepts such as path, cycles and connectedness. A Somasundram and S. Somasundaram \([3]\) discussed domination in fuzzy graphs. Selvaraju. P. et.al \([4]\) discussed the concept of neighborhood connected perfect domination in graphs. In this paper, we discuss the concept of neighborhood clique domination number, clique regular domination number and perfect disconnected domination number of fuzzy graphs and obtain the relationship with other known parameters.

II. PRELIMINARIES

Definition 2.1

Let \(V\) be a finite non empty set. Let \(E\) be the collection of all two element subsets of \(V\). A fuzzy graph \(G=(\sigma, \mu)\) is a set with two functions \(\sigma : V \rightarrow [0, 1]\) and \(\mu : E \rightarrow [0, 1]\) such that \(\mu(uv) \leq \sigma(u) \wedge \sigma(v)\) for all \(u, v \in V\).
Definition 2.2

Let $G=(\sigma, \mu)$ be a fuzzy graph on $V$ and $V_1 \subseteq V$. Define $\sigma_1$ on $V_1$ by $\sigma_1(u) = \sigma(u)$ for all $u \in V_1$ and $\mu_1$ on the collection $E_1$ of two element subsets of $V_1$ by $\mu_1(uv) = \mu(uv)$ for all $u, v \in V_1$, then $(\sigma_1, \mu_1)$ is called the fuzzy subgraph of $G$ induced by $V_1$ and is denoted by $<V_1>$.

Definition 2.3

The order $p$ and size $q$ of a fuzzy graph $G=(\sigma, \mu)$ are defined to be $p= \sum_{u \in V} \sigma(u)$ and $q= \sum_{uv \in E} \mu(uv)$.

Definition 2.4

Let $G=(\sigma, \mu)$ be a fuzzy graph on $V$ and $D \subseteq V$ then the fuzzy cardinality of $D$ is defined to be $\sum_{u \in D} \sigma(u)$.

Definition 2.5

An edge $e = uv$ of a fuzzy graph is called an effective edge if $\mu(uv) = \sigma(u) \land \sigma(v)$. $N(u) = \{ v \in V/ \mu(uv) = \sigma(u) \land \sigma(v)\}$ is called the neighborhood of $u$ and $N[u] = N(u) \cup \{u\}$ is the closed neighborhood of $u$.

The effective degree of a vertex $u$ is defined to be the sum of the weights of the effective edges incident at $u$ and is denoted by $d_E(u)$. $\sum_{v \in N(u)} \sigma(v)$ is called the neighborhood degree of $u$ and is denoted by $d_N(u)$. The minimum effective degree $\delta_E(G) = \min \{ d_E(u) | u \in V(G) \}$ and the maximum effective degree $\Delta_E(G) = \max \{ d_E(u) | u \in V(G) \}$.

Definition 2.6

The complement of a fuzzy graph $G$ denoted by $\bar{G}$ is defined to be $\bar{G} = (\sigma, \bar{\mu})$ where $\bar{\mu}(uv) = \sigma(u) \land \sigma(v) - \mu(uv)$.

Definition 2.7

Let $\sigma: V \rightarrow [0,1]$ be a fuzzy subset of $V$. Then the complete fuzzy graph on $\sigma$ is defined to be $(\sigma, \mu)$ where $\mu(uv) = \sigma(u) \land \sigma(v)$ for all $uv \in E$ and is denoted by $K_{\sigma}$.
A fuzzy graph \( G = (\sigma, \mu) \) is said to be connected if any two vertices in \( G \) are connected.

**Definition 2.9**

A fuzzy graph \( G = (\sigma, \mu) \) is said to be bipartite if the vertex \( V \) can be partitioned into two nonempty sets \( V_1 \) and \( V_2 \) such that \( \mu(v_1, v_2) = 0 \) if \( v_1, v_2 \in V_1 \) or \( v_1, v_2 \in V_2 \). Further if \( \mu(u, v) = \sigma(u) \land \sigma(v) \) for all \( u \in V_1 \) and \( v \in V_2 \) then \( G \) is called a complete bipartite graph and is denoted by \( K_{\sigma_1, \sigma_2} \) where \( \sigma_1 \) and \( \sigma_2 \) are, respectively, the restrictions of \( \sigma \) to \( V_1 \) and \( V_2 \).

**Definition 2.10**

An edge \( e \) of a fuzzy graph \( G \) is said to be an isolated edge if no effective edges incident with the vertices of \( e \). Thus an isolated edge does not dominate any other edge in \( G \).

**Definition 2.11**

Let \( G = (\sigma, \mu) \) be a fuzzy graph on \( V \). Let \( u, v \in V \). We say that \( u \) dominates \( v \) in \( G \) if \( \mu(\{u, v\}) = \sigma(u) \land \sigma(v) \). A subset \( D \) of \( V \) is called a dominating set in \( G \) if for every \( v \in D \), there exists \( u \in D \) such that \( u \) dominates \( v \). The minimum fuzzy cardinality of a dominating set in \( G \) is called the domination number of \( G \) and is denoted by \( \gamma(G) \).

**Definition 2.12**

A dominating set \( D \) of a fuzzy graph \( G = (\sigma, \mu) \) is connected dominating set if the induced fuzzy sub graph \( <D> \) is connected. The minimum fuzzy cardinality of a connected dominating set of \( G \) is called the connected dominating number of \( G \) and is denoted by \( \gamma_c(G) \).

**Definition 2.13**

A dominating set \( D \) of a fuzzy graph \( G = (\sigma, \mu) \) is a clique dominating set if the induced fuzzy subgraph \( <D> \) is a complete. The clique domination number \( \gamma_{cd}(G) \) is the minimum fuzzy cardinality of a clique dominating set.

**Definition: 2.14**

Let \( G = (\sigma, \mu) \) be a fuzzy graph without isolated vertices. A subset \( D \) of \( V \) is said to be a total dominating set if every vertex in \( V \) is dominated by a vertex in \( D \). The minimum fuzzy cardinality of a total dominating set is called the total domination number of \( G \) and is denoted by \( \gamma_t(G) \).
Definition 2.15

Let $G = (\sigma, \mu)$ be a fuzzy graph without isolated vertices. A subset $D$ of $V$ is said to be a perfect dominating set if every vertex in $V$ is dominated by exactly one vertex in $D$. The minimum fuzzy cardinality of a perfect dominating set is called the perfect domination number of $G$ and is denoted by $\gamma_p(G)$.

Definition 2.16

Let $G = (\sigma, \mu)$ be a regular fuzzy graph on $G^* = (V, E)$. If $d_G(v) = k$ for all $v \in V$, (i.e) if each vertex has same degree $k$, then $G$ is said to be a regular fuzzy graph of degree $k$ or $k$-regular fuzzy graph.

Remark 2.17

$G$ is $k$-regular graph iff $\delta = \Delta = k$.

Definition 2.18

Let $G = (\sigma, \mu)$ be a fuzzy graph on $G^*$. The total degree of a vertex $u \in V$ is defined by $td_G(u) = d_G(u) + \sigma(u) = \sum_{uv \in E} \mu(\overline{uv}) + \sigma(u)$. If each vertex of $G$ has the same total degree $k$ then $G$ is said to be a totally regular fuzzy graph of total degree $k$ or $k$-totally regular fuzzy graph.

III. Fuzzy neighborhood clique domination number

Definition 3.1

Let $G = (\sigma, \mu)$ be a fuzzy graph without isolated vertices. A subset $D_{nc}(G)$ of $V$ is said to be a neighborhood clique dominating set if $< N(D_{nc}(G))>$ is complete provided $N(D)$ contains the fuzzy vertices other than $D$. The fuzzy neighborhood clique domination number $\gamma_{nc}(G)$ is the minimum fuzzy cardinality taken over all minimal neighborhood clique dominating sets of $G$.

Example 3.2
D_{nc}(G) = \{v_5, v_6, v_7, v_8\}

\gamma_{nc}(G) = 1.4

<N (D_{nc}(G))> is complete

**Theorem 3.3**

If a fuzzy graph $G = (\sigma, \mu)$ is complete with $\sigma(v_i) = c$ (constant) , for every $v_i \in V$ then $\gamma_{nc}(G) = c$.

**Proof:**

Let $G = (\sigma, \mu)$ be the complete fuzzy graph with vertex set $V = \{v_1, v_2, \ldots, v_i-1, v_{i+1}, \ldots, v_n\}$, by the definition of fuzzy complete graph each $v_i$ is dominates the other vertices.

Let $D_{nc}(G)$ be the neighborhood clique dominating set of $G$. 

$D_{nc}(G) = \{v_i \text{ where } v_i \text{ is the vertex of minimum fuzzy cardinality}\}$. Therefore $N(D_{nc}(G)) = \{v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n\}$ and $<N(D_{nc}(G))>$ is complete. Since $\sigma(v_i)$’s are equal then the fuzzy neighborhood clique domination number $\gamma_{nc}(G) = \sigma(v_i) = c$.

**Theorem 3.4**

If a fuzzy graph $G = (\sigma, \mu)$ is complete and $D_{nc}(G)$ is the neighborhood clique dominating set then $<V – D_{nc}(G)>$ is complete.

**Proof:**
Let \( G = (\sigma, \mu) \) be the complete fuzzy graph with vertex set \( V = \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \), by the definition of fuzzy complete graph each \( v_i \) is dominates the other vertices in \( G \).

The neighborhood clique dominating set \( D_{nc}(G) = \{ v_i / \text{where } v_i \text{ is the vertex of minimum fuzzy cardinality} \} \). By definition of clique neighborhood dominating set \( < N(D_{nc}(G)) > \) is complete. \( < V-D_{nc}(G) > = \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \), also \( < V-D_{nc}(G) > \) is complete.

**Theorem 3.5**

If a fuzzy graph \( G = (\sigma, \mu) \) is complete and \( D_{nc}(G) \) is the neighborhood clique dominating set then \( \gamma_{nc}(G) = \min \{ \sigma(v_i) / v_i \in V \} \).

**Proof:**

Let \( G= (\sigma, \mu) \) be the complete fuzzy graph with vertex set \( V = \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \) having distinct fuzzy vertex cardinality.

Let \( D_{nc}(G) \) be the neighborhood clique dominating set of \( G \). That is \( D_{nc} = \{ v_i / \text{where } v_i \text{ is the vertex of minimum fuzzy cardinality} \} \). Therefore the neighborhood clique domination number \( \gamma_{nc} (K_\sigma) = \min \{ \sigma(v_i) / v_i \in V \} \).

**Theorem 3.6**

If a fuzzy graph \( G = (\sigma, \mu) \) is complete and \( D_{nc}(G) \) is the neighborhood clique dominating set then \( \gamma_{nc} (G) \leq \gamma_{nc} (G_1) \leq \gamma_{nc} (G_2) \leq \gamma_{nc} (G_3) \leq \ldots \leq \gamma_{nc} (G_{n-1}) \) where \( G_i \) is a fuzzy graph with \( V_i = \{ V_i - \{ v_i \} / \sigma(v_i) = \text{minimum fuzzy vertex cardinality} \} \).

**Proof:**

Let \( G= (\sigma, \mu) \) be the complete fuzzy graph with vertex set \( V = \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \). Let \( G_i \) be the fuzzy graph induced by the vertex set \( V_i = \{ V - D_{nc}(G) \} \). \( D_{nc}(G) \) is the neighborhood clique dominating set with respect to \( V_i \), by definition \( < N(D_{nc}(G)) > \) is complete. clear the neighborhood domination number of \( G_i \)'s are \( \gamma_{nc} (G), \gamma_{nc} (G_1), \gamma_{nc} (G_2), \gamma_{nc} (G_3), \ldots, \gamma_{nc} (G_{n-1}) \) such that \( \gamma_{nc} (G) \leq \gamma_{nc} (G_1) \leq \gamma_{nc} (G_2) \leq \gamma_{nc} (G_3) \leq \ldots \leq \gamma_{nc} (G_{n-1}) \).
Theorem 3.7

If \( G = (\sigma, \mu) \) is a fuzzy graph then \( \gamma(G) \leq \gamma_{nc}(G) \).

Proof:

Let \( G = (\sigma, \mu) \) be the complete fuzzy graph. Let \( D(G) \) and \( D_{nc}(G) \) be the dominating set and neighborhood clique dominating set of \( G \). Since \( D_{nc}(G) \) be the neighborhood clique dominating set. \( D_{nc}(G) \) is also dominating set, but need not be minimum fuzzy dominating set. Therefore \( \gamma(G) \leq \gamma_{nc}(G) \).

IV. Fuzzy clique regular domination number

Definition 4.1

Let \( G = (\sigma, \mu) \) be a fuzzy graph without isolated vertices. A subset \( D_{cr}(G) \) of \( V \) is said to be a clique regular dominating set if \( \langle N(D_{cr}(G)) \rangle \) is regular. The fuzzy clique regular domination number \( \gamma_{cr}(G) \) is the minimum fuzzy cardinality taken over all minimal clique regular dominating sets of \( G \).

Example 4.2

\[ D_{cr}(G) = \{v_1, v_2, v_3, v_4\} \]
\[ \gamma_{cr}(G) = 0.4 \]
\[ \langle N(D_{cr}(G)) \rangle \] is regular.

Theorem 4.3

If \( G = (\sigma, \mu) \) is a regular fuzzy graph then \( \gamma_{cr}(G) \) - set exists.
Proof:
Let \( G = (\sigma, \mu) \) be the complete fuzzy graph with \( n \) vertices set \( \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \). Let \( D_{cr}(G) \) be a clique regular dominating set which is a subset of a fuzzy vertex set of \( V \) of \( G \). Since \( G \) is a regular then \( d(v) = k \), for some \( k \in [0, 1] \), for every \( v_i \in V \). The induced graphs of \( N(D_{cr}(G)) \) other than \( D_{cr}(G) \) is regular. Therefore, \( \gamma_{cr}(G) \) - set exists.

**Theorem 4.4**

If a fuzzy graph \( G = (\sigma, \mu) \) is complete with \( \sigma(v_i) = c \) (constant), for every \( v_i \in V \), then \( \gamma_{cr}(G) = c \).

Proof:
Let \( G = (\sigma, \mu) \) be the complete fuzzy graph with vertex set \( V = \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \) and all vertices having constant fuzzy vertex cardinality. Therefore each vertex has degree \( (n-1) \sigma(v_i) \), clearly \( G \) is a regular fuzzy graph. Let \( D_{cr}(G) \) be a clique regular fuzzy graph, that is \( D_{cr}(G) = \{ v_i / \text{for any} \ v_i \in V \} \), by definition \( <N(D_{cr}(G))> \) is regular. Therefore \( \gamma_{cr}(G) \) is the fuzzy clique regular domination number of \( G \), which is equal to \( \sigma(v_i) = c \), that is \( \gamma_{cr}(G) = \sigma(v_i) = c \).

**Corollary: 4.5**

If \( G = (\sigma, \mu) \) is a totally regular fuzzy graph and \( \gamma_{cr}(G) \) - set exists then the alternate fuzzy edges having equal fuzzy cardinality (or) all the fuzzy vertices having equal fuzzy vertex cardinality.

**Theorem 4.6**

Every fuzzy complete graph \( G = K_\sigma, n \geq 2 \) has a \( \gamma_{cr}(G) \) -set with \( \sigma(v_i) = c \) (constant), for every \( v_i \in V \).

Proof:
Let \( G = (\sigma, \mu) \) be the complete fuzzy graph with vertex set \( V = \{ v_1, v_2, \ldots, v_i, v_{i+1}, \ldots, v_n \} \) and all vertices having constant fuzzy vertex cardinality. Moreover every vertex in \( V \) dominates the other and having \( \sigma(v_i) = c \), for every \( v_i \in V \). Let \( D_{cr}(G) \) be a clique regular fuzzy graph, that is \( D_{cr}(G) = \{ v_i / \text{for any} \ v_i \in V \} \), by definition \( V - D_{cr}(G) = \{ v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n \} \) and \( <V - D_{cr}(G)> \) is regular graph with degree \( (n-2)c \). Proceeding like this way, the complete fuzzy graph with \( n \geq 2 \) has a \( \gamma_{cr}(G) \) - set.
V. Fuzzy perfect disconnected domination number

Definition 5.1

Let $G = (\sigma, \mu)$ be a fuzzy graph without isolated vertices. A subset $D_{pd}(G)$ of $V$ is said to be a perfect disconnected dominating set if $D_{pd}(G)$ is perfect and $\langle D_{pd}(G) \rangle$ is disconnected. The fuzzy perfect disconnected domination number $\gamma_{pd}(G)$ is the minimum fuzzy cardinality taken over all minimal perfect disconnected dominating sets of $G$.

Example 5.2

![Diagram](0.1)v_1 -- 0.1v_5 (0.2) -- 0.1v_6 (0.1) -- 0.2v_8 (0.2) -- 0.2v_9 (0.2) -- 0.2v_6 (0.1) -- 0.1v_4 (0.1)

$D_{pd}(G) = \{v_5, v_6, v_7, v_8\}$, $\langle D_{pd}(G) \rangle$ is disconnected, $\gamma_{pd}(G) = 0$.

Theorem 5.3

If $G = (\sigma, \mu)$ is a fuzzy graph then $\gamma_{pd}(G) < \frac{p}{2}$.

Proof:

Let $G = (\sigma, \mu)$ be a fuzzy graph. The fuzzy perfect disconnected dominating set $D_{pd}(G) = \{v_i : v_i \in G\}$ such that every vertex in $V$ is exactly dominated by one vertex in $D$ and $\langle D_{pd}(G) \rangle$ is disconnected. Therefore, the fuzzy perfect disconnected domination number $\gamma_{pd}(G) < \frac{p}{2}$. The minimum cardinality of the fuzzy disconnected domination number $\gamma_{pd}(G) < \frac{p}{2}$. 
Corollary 5.4

If $G = (\sigma, \mu)$ be a fuzzy graph with $\sigma(v_i) = c$, for every $v_i \in V$ then $\gamma_{pd}(G) = \frac{p}{2}$.

Theorem 5.5

If $G = (\sigma, \mu)$ is a fuzzy graph and has a $\gamma_{pc}(G)$ - set and $\gamma_{pd}(G)$ - set then $\gamma_{pc}(G) + \gamma_{pd}(G) = p$ where $\gamma_{pc}(G)$ is a fuzzy connected perfect domination number of $G$.

Proof:

Let $G = (\sigma, \mu)$ be a fuzzy graph and $G$ has a $\gamma_{pc}(G)$ - set and $\gamma_{pd}(G)$ - set then by definition of perfect connected and perfect disconnected dominating set $\gamma_{pc}(G)$ has $\frac{n}{2}$ fuzzy vertices and $\gamma_{pd}(G)$ has $\frac{n}{2}$ fuzzy vertices, further $|D_{pc}(G)| = \frac{n}{2}$ and $|D_{pd}(G)| = \frac{n}{2}$, therefore $\gamma_{pc}(G) = \frac{p}{2}$, $\gamma_{pd}(G) = \frac{p}{2}$. Therefore, $\gamma_{pc}(G) + \gamma_{pd}(G) = p$.

Theorem 5.6

If $G = (\sigma, \mu)$ is a fuzzy graph and has $\gamma_{pd}(G)$ - set then the number of fuzzy vertices are even.

Proof:

If $G = (\sigma, \mu)$ is a fuzzy graph with $n$ vertices and let $D_{pd}(G)$ be the fuzzy perfect disconnected dominating set, then by definition of perfect dominating set, every $v$ in $V$ is dominated by exactly one vertex in $D_{pd}(G)$. Therefore $|D_{pd}(G)|$ has even or odd number of fuzzy vertices. By definition of perfect disconnected dominating set $V$ has even number of fuzzy vertices.

Theorem 5.7

If $G = (\sigma, \mu)$ is a fuzzy graph then $\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G) \leq \gamma_{pd}(G)$ where $\gamma_t(G)$ and $\gamma_p(G)$ are the fuzzy total and perfect domination numbers respectively.
Proof:

Let $G = (\sigma, \mu)$ be a fuzzy graph and $D, D_t, D_p$ and $D_{pd}$ be the fuzzy dominating, fuzzy total dominating, fuzzy perfect dominating and fuzzy perfect disconnected dominating sets of $G$ respectively. Since every total dominating set is a dominating set. Therefore $\gamma(G) \leq \gamma_t(G)$ and every perfect dominating set is total dominating set then $\gamma_t(G) \leq \gamma_p(G)$. Moreover, every perfect dominating set, $\gamma_p(G) \leq \gamma_{pd}(G)$. Finally, we have $\gamma(G) \leq \gamma_t(G) \leq \gamma_p(G) \leq \gamma_{pd}(G)$.

Theorem 5.8

If $G = (\sigma, \mu)$ is $K_n \circ K_1$ then $\gamma_{pc}(K_n \circ K_1) + \gamma_{pd}(K_n \circ K_1) = p$.

Corollary 5.9

If $G = (\sigma, \mu)$ be a fuzzy carona $K_n \circ K_1$ then $\gamma_{pd}(G)$ is the inverse domination number of $\gamma_{pc}(G)$, also $\gamma_{pc}(K_n \circ K_1) + \gamma_{pd}(K_n \circ K_1) = p$.

Theorem 5.10

If $G = (\sigma, \mu)$ is $K_n \circ K_1$ and $\sigma(v_i) = c$ then $\gamma_{cr}(G) + \gamma_{pd}(G) = 2 \sum_{i=1}^{n} \sigma(v_i)$, where $v_i \in V$ are the fuzzy pendent vertices of $G$.

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