INTUITIONISTIC FUZZY DIGITAL CONVEXITY

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Abstract

In the present article, the intuitionistic fuzzy digital subset of the lattice points in the Euclidean plane is defined and some of their interesting properties are studied. Further, their intuitionistic fuzzy digital convexity is established. Moreover, the notion of intuitionistic fuzzy digital upper and lower cut sets is introduced and their convexity and concavity are studied.

Keywords:
IFD subset, IFD convex set, IFD concave set, IFD regular set, IFD level sets, IFD upper cut sets, IFD lower cut sets.

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1. Introduction

In 1965, Lofti A. Zadeh [1] introduced the notion of a fuzzy subset of a set as a method for representing uncertainty in real physical world. In 1983, K. T. Atanassaoov [2] published the concept of “Intuitionistic Fuzzy Sets” and many works by him and his colleagues appeared in the literature [3,4]. An introduction to Intuitionistic fuzzy topological space was introduced by Dogan Coker [5]. Convex and concave fuzzy sets play important roles in Optimization theory, Operations research and Applied Mathematics. A significant definition of convex fuzzy sets was introduced by Zadeh [1]. Subsequently, works on convex (concave) fuzzy sets in theories and applications has been studied by Katsaras and Liu [6]. Fuzzy digital convexity was established by Janos and Rosenfeld [7] in 1982. Also the concepts of p-cuts (level sets) play a principal role in the relationship between fuzzy sets and crisp sets. They can be viewed as a bridge by which fuzzy sets and crisp sets are connected. The Cut sets of intuitionistic fuzzy sets were introduced by Li. M [8].
Rosenfeld and Kak [9] analyzed the geometrical properties and relationships among the parts of a digital picture by the subsets of the picture which are extracted from it by segmentation process of various types. To weaken this strong process Rosenfeld and Kak [10] extended the concepts of digital picture geometry to fuzzy subsets. Classical digital topology primarily concerns itself in the study of black white images in the digital plane[11,12]. A.Rosenfeld[13] represented the gray scale level images by the concept of fuzzy sets.

Intuitionistic fuzzy sets(IFS)are defined by two characteristic functions, namely the membership and the non-membership, describing the belongingness or non-belongingness of an element respectively. It is observed that in image processing the results using IFS is better than the fuzzy/non fuzzy set theory.

In this manner, this paper develops the concepts of intuitionistic fuzzy digital subsets of the lattice points in the Euclidean plane and establishes its convexity property.

2. Preliminaries

In this section, some well-known definitions are recalled. It will be necessary in order to understand the new concepts and theorems introduced in this paper.Throughout this paper $E$ denotes the Euclidean plane and $I$ denotes the unit interval $[0,1]$.

Definition 2.1[10]

Let $\Omega$ be a rectangular array of integer-coordinate points or lattice points in the Euclidean plane. Thus the point $P = (x, y)$ of $\Omega$ has four horizontal and vertical neighbors, namely $(x \pm 1, y)$ and $(x, y \pm 1)$; and it also has four diagonal neighbors, namely $(x \pm 1, y \pm 1)$ and $(x \pm 1, y \mp 1)$. We say that former points are 4-adjacent to, or 4-neighbors of $P$ and we say that both types of neighbors are 8-adjacent to, or 8-neighbors of $P$. Note that if $P$ is on the border of $\Omega$, some of these neighbors may not exist.

Definition 2.2 [10]

Let $P, Q$ be the points of $\Omega$. Then a path $\rho$ from $P$to $Q$ is a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that $P_i$ is adjacent (either 4-adjacent 8-adjacent) to $P_{i-1}$, for $1 \leq i \leq n$.

Remark 2.1[10]

Let $R$ be a subset of the plane such that $(R^0) = R$ (R is the closure of its interior); we call
an R regular. We regard each lattice point P as the center of an open unit square \( P^* \). We call such a square a cell. The set \( I(R) = \{ P \mid R \cap P^* \neq \emptyset \} \) is called the digital image of R. Note that the digital image is defined only for regular sets.

By the definition of I(R), R meets \( Q^* \) if and only if \( Q \in I(R) \). If R meets any \( \overline{Q}^* \) on its boundary, it meets the interior of at least one of the cells that share that boundary.

**Definition 2.3**

Let \( X \) be a nonempty fixed set and \( I \) be the closed interval \([0,1]\). An intuitionistic fuzzy set (IFS) \( A \) is an object of the following form \( A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \} \), where the mappings \( \mu_A : X \to I \) and \( \gamma_A : X \to I \) denote the degree of membership (namely, \( \mu_A(x) \)) and the degree of non membership (namely, \( \gamma_A(x) \)) for each element \( x \in X \) to the set \( A \), respectively, and \( 0 \leq \mu_A(x) + \gamma_A(x) \leq 1 \) for each \( x \in X \). For the sake of simplicity, we shall use the symbol \( A = \langle x, \mu_A, \gamma_A \rangle \) for the intuitionistic fuzzy set \( A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \} \).

**Definition 2.4**

Let \( X \) be a nonempty fixed set and the IFSs \( A \) and \( B \) be in the form \( A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \} \), \( B = \{ (x, \mu_B(x), \gamma_B(x)) : x \in X \} \). Then,

i. \( A \subseteq B \) iff \( \mu_A(x) \leq \mu_B(x) \) and \( \gamma_A(x) \geq \gamma_B(x) \) for all \( x \in X \);

ii. \( A = B \) iff \( A \subseteq B \) and \( B \subseteq A \);

iii. The complement of \( A \), \( \overline{A} = \{ (x, \gamma_A(x), \mu_A(x)) : x \in X \} \);

iv. \( A \cup B = \{ (x, \mu_A(x) \vee \mu_B(x), \gamma_A(x) \wedge \gamma_B(x)) : x \in X \} \);

v. \( A \cap B = \{ (x, \mu_A(x) \wedge \mu_B(x), \gamma_A(x) \vee \gamma_B(x)) : x \in X \} \);

vi. \( 0_\sim = \{ (x,0,1) : x \in X \} \) and \( 1_\sim = \{ (x,1,0) : x \in X \} \).

**Definition 2.5**

An intuitionistic fuzzy set \( A = \{ (x, \mu_A(x), \gamma_A(x)) : x \in X \} \), is an intuitionistic fuzzy topological space \((X, \tau)\) is said to be an intuitionistic fuzzy regular open set if \( A = \text{int}(\text{cl}(A)) \).
3. Intuitionistic Fuzzy Digital Subset of the Lattice Points

Definition 3.1
Let $A_\mu = \langle \mu_{A_\mu}, \nu_{A_\mu} \rangle$ be the intuitionistic fuzzy subset of $E$. Then the intuitionistic fuzzy digital subset (or IFD subset) of the points of $\Sigma$ is denoted by $A'_\mu$, whose degree of membership and the degree of non membership are defined by,

\[
\mu_{A'_\mu}(P) = \max\{\mu_{A_\mu}(R) \mid R \in P^*\} \quad \text{and} \quad \nu_{A'_\mu}(P) = \min\{\nu_{A_\mu}(R) \mid R \in P^*\}.
\]

Definition 3.2
Let $A_\mu = \langle \mu_{A_\mu}, \nu_{A_\mu} \rangle$ be the intuitionistic fuzzy subset of $E$. Then $A'_\mu$ is said to be intuitionistic fuzzy digital regular (or IFD regular) set if the sets $A_{r_\mu}$, defined by

\[
A_{r_\mu} = \{ R : \mu_{A_{r_\mu}}(R) > r \ \text{and} \ \nu_{A_{r_\mu}}(R) < r \}
\]

are intuitionistic fuzzy regular for all $r \in [0,1)$.

Definition 3.3
Let $A_{r_\mu} = \{ R : \mu_{A_{r_\mu}}(R) > r \ \text{and} \ \nu_{A_{r_\mu}}(R) < r \}$ be the intuitionistic fuzzy regular subset of $E$. Then the digital image of $A_{r_\mu}$ is an IFD set $D(A_{r_\mu})$, which is defined as

\[
D(A_{r_\mu}) = \{ R : A_{r_\mu} \cap R^* \text{ is non empty} \}.
\]

Proposition 3.1
(i) The IFD subset $A'_{r_\mu} = \langle \mu_{A'_{r_\mu}}, \nu_{A'_{r_\mu}} \rangle$ of the points of $\Sigma$ represents the digital image of the intuitionistic fuzzy subset $A_{r_\mu} = \langle \mu_{A_{r_\mu}}, \nu_{A_{r_\mu}} \rangle$ of $E$, if $A_{r_\mu}$ is intuitionistic fuzzy regular.

(ii) The IFD subset $A_\mu' = \langle \mu_{A_\mu'}, \nu_{A_\mu'} \rangle$ of the points of $\Sigma$ represents the digital image of the intuitionistic fuzzy subset $A_\mu = \langle \mu_{A_\mu}, \nu_{A_\mu} \rangle$ of $E$, if $A_\mu$ is IFD regular.

Proof: (i) Let $A_{r_\mu}$ is intuitionistic fuzzy regular and $A'_{r_\mu} = \langle \mu_{A'_{r_\mu}}, \nu_{A'_{r_\mu}} \rangle$ is the IFD subset of the points $\Sigma$. A lattice point $R$ will be in the digital image $D(A_{r_\mu})$ if $A_{r_\mu} \cap R^*$ is non empty. But $A_{r_\mu} \cap R^*$ is non empty iff $\max\{\mu_{A_\mu}(R) \mid R \in P^*\} > r$ and $\min\{\nu_{A_\mu}(R) \mid R \in P^*\} < r$. This is true
iff \( \mu_{A_r}(P) > r \) and \( \nu_{A_r}(P) < r \). Hence the lattice point \( R \) must be an element of \( A'_r \). Thus all the lattice points of \( \mathcal{D}(A_r) \) are the elements of \( A'_r \), and hence \( A'_r \) is the digital image of \( A_r \).

(ii) From the Definition 3.2, \( A_r \) is said to be IFD regular, if the sets \( \{ R : \mu_{A_r}(R) > r \ \text{and} \ \nu_{A_r}(R) < r \} \) are intuitionistic fuzzy regular for all \( r \in [0,1) \).

Hence if \( A_\alpha \) is IFD regular, from the above proof, the IFD subset \( A' = \{ \mu_{A_\alpha}, \nu_{A_\alpha} \} \) of the points \( \Sigma \) represents the digital image of \( A_\alpha \).

### 4. Intuitionistic Fuzzy Digital Convexity

**Definition 4.1**

Let \( A_\alpha = \{ \mu_{A_\alpha}, \nu_{A_\alpha} \} \) be the intuitionistic fuzzy subset of \( E \), then \( A_\alpha \) is said to be a convex set (or IFC set) of \( E \), if for every pair \( P, Q \) of points in \( E \) and for all \( P_i, 0 \leq i \leq n \) on the line segment, \( P = P_0, P_1, \ldots, P_n = Q \) from \( P \) to \( Q \),

\[
\mu_{A_\alpha}(P_i) \geq \min(\mu_{A_\alpha}(P), \mu_{A_\alpha}(Q)); \nu_{A_\alpha}(P_i) \leq \max(\nu_{A_\alpha}(P), \nu_{A_\alpha}(Q)).
\]

The complement \( 1 - A_\alpha \) of the IFC set \( A_\alpha \) is said to be an intuitionistic fuzzy concaveset of \( E \).

**Proposition 4.1**

An intuitionistic fuzzy subset \( A_\alpha = \{ \mu_{A_\alpha}, \nu_{A_\alpha} \} \) of \( E \) is an IFC set if and only if its level sets are all convex.

**Definition 4.2**

Let \( A_\alpha = \{ \mu_{A_\alpha}, \nu_{A_\alpha} \} \) be an intuitionistic fuzzy subset of \( E \), then the level sets of \( A_\alpha \) are defined as \( A^\alpha = \{ P \in E, \mu_{A_\alpha}(P) \geq \alpha, \nu_{A_\alpha}(P) \leq \alpha \} \), \( \alpha \in I \).

**Proposition 4.2**

Let \( A_\alpha = \{ \mu_{A_\alpha}, \nu_{A_\alpha} \} \) be the intuitionistic fuzzy subset of \( E \), then \( A_\alpha \) is convex if and only if its level sets are all convex.
Definition 4.3

Let $A_\infty = (\mu_{A_\infty}, \nu_{A_\infty})$ be the intuitionistic fuzzy subset of $E$ which is IFD regular and convex, then $A'_\infty$ is said to be an intuitionistic fuzzy digital convex (or IFD convex) set, if the digital image of $A_\infty$ is $A'_\infty$.

The complement $1 - A'_\infty$ of the IFD convex set $A'_\infty$ is said to be an intuitionistic fuzzy digital concave (or IFD concave) set.

Proposition 4.3

Let $A_\infty = (\mu_{A_\infty}, \nu_{A_\infty})$ be the intuitionistic fuzzy subset of $E$ which is IFD regular and let $A'_\infty$ be an IFD convex set, then the sets $A_{-r} = \{ R : \mu_{A_{-r}}(R) > r \text{ and } \nu_{A_{-r}}(R) < r \}$ are IFD convex for all $r \in I$.

Proof:

The proof follows from the Propositions 4.1 and 4.2.

Proposition 4.4

Let $A_\infty = (\mu_{A_\infty}, \nu_{A_\infty})$ be the intuitionistic fuzzy subset of $E$ which is IFD regular and let $A'_\infty$ be an IFD convex set, then for any line ‘l’ and any three points $P, Q, R \in l$ of $\Sigma$, where $Q$ lies in between $P$ and $R$,

$$\mu_{A'_\infty}(Q) \geq \min(\mu_{A'_\infty}(P), \mu_{A'_\infty}(R)) \text{ and } \nu_{A'_\infty}(Q) \leq \max(\nu_{A'_\infty}(P), \nu_{A'_\infty}(R)).$$

Proof:

Let $P' \in P^*$ and $R' \in R^*$ be the interior points. From the Definition 3.1,

$$\mu_{A'_\infty}(P) \leq \mu_{A_\infty}(P') + \varepsilon, \quad \nu_{A'_\infty}(P) \geq \nu_{A_\infty}(P') + \varepsilon \quad \text{and} \quad \mu_{A'_\infty}(Q) \leq \mu_{A_\infty}(Q') + \varepsilon,$$

$$\nu_{A'_\infty}(Q) \geq \nu_{A_\infty}(Q') + \varepsilon \quad \text{for } \varepsilon > 0.$$ Let ‘l’ be the line joining $P'$ and $R'$. Since $Q$ lies inbetween $P$ and $R$, ‘l’ will meet $Q^*$ at some of its interior points, say $Q'$. From Definition 3.1,

$$\mu_{A'_\infty}(Q) = \max\{ \mu_{A'_\infty}(T) \mid T \in Q^* \}$$

$$\geq \mu_{A_\infty}(Q')$$

$$\geq \min(\mu_{A_\infty}(P'), \mu_{A_\infty}(R')) \text{, since } A_\infty \text{ is convex}$$

$$> \min(\mu_{A_\infty}(P) - \varepsilon, \mu_{A_\infty}(R) - \varepsilon)$$
\[ \mu_{\mathcal{A}^c}(Q) = \min(\mu_{\mathcal{A}^c}(P), \mu_{\mathcal{A}^c}(R)) - \varepsilon \]

Thus, \( \mu_{\mathcal{A}^c}(Q) \geq \min(\mu_{\mathcal{A}^c}(P), \mu_{\mathcal{A}^c}(R)) \) since \( \varepsilon \) is arbitrary.

Similarly it can be proved that \( \nu_{\mathcal{A}^c}(Q) \leq \max(\nu_{\mathcal{A}^c}(P), \nu_{\mathcal{A}^c}(R)) \).

**Proposition 4.5**

Let \( P_0, P_1, \ldots, P_r \) be a set of lattice points, \( A'_i(P_i) \) be the IFD convex subset of those points, then \( C_\omega \subseteq A'_i \), where \( C_\omega = \sum_{i=0}^{r} a_{i} P_{i} \), \( C_\omega = \left\{ \mu_{C_\omega}, \nu_{C_\omega} \right\} \), \( a_i = \left( a_{i\mu}, a_{i\nu} \right) \),

\[ P_i = \left\{ \mu_{A'_{i}}(P_i), \nu_{A'_{i}}(P_i) \right\} \]

with \( \mu_{C_\omega} = \sum_{i=0}^{r} a_{i\mu} \mu_{A'_{i}}(P_i) \), \( \mu_{C_\omega} = \sum_{i=0}^{r} a_{i\nu} \nu_{A'_{i}}(P_i) \), and

\[ \sum_{i=0}^{r} (a_{i\mu} + a_{i\nu}) = 1 \] and \( 0 \leq a_{i\mu} + a_{i\nu} \leq 1 \) for each \( i \).

**Proof:**

This proposition can be proved by mathematical induction on \( r \). The result is obvious for \( r=1 \) and from the definition 4.1, it is true for \( r=1 \). By induction hypothesis, assume that the proposition is true for \( r=k \). Now to prove for \( r=k+1 \), consider \( \sum_{i=0}^{k} a_{i} P_{i} \). Let \( \sum_{i=0}^{k} a_{i} = \alpha \). Hence \( a_{k+1} = 1 - \alpha \).

Now, \( \sum_{i=0}^{k} a_{i} P_{i} + a_{k+1} P_{k+1} = \alpha \left( \sum_{i=0}^{k} \frac{a_{i}}{\alpha} P_{i} \right) + (1 - \alpha) P_{k+1} \) and \( \sum_{i=0}^{k} \frac{a_{i}}{\alpha} = 1 \). Hence by induction hypothesis, \( \sum_{i=0}^{k} \frac{a_{i}}{\alpha} P_{i} \subseteq C_{\omega} \). Since \( A'_i \) is an IFD convex set containing \( P_{k+1} \), the result follows from the Proposition 4.1.

**5. A View on Intuitionistic Fuzzy Digital Cut Sets**

**Definition 5.1**

Let \( A_{\omega} \) be an intuitionistic fuzzy subset of \( E \), \( A'_{\omega} \) be the IFD subset of the lattice points \( \Sigma \) and \( \sigma, \omega \in [0,1] \) such that \( \sigma + \omega \leq 1 \), then intuitionistic fuzzy digital upper cut sets (or **IFD upper cut sets**) are defined and denoted as follows,

i. \( A'_{\omega}[\sigma, \omega]^+ = \left\{ P : \mu_{A_{\omega}}(P) \geq \sigma \text{ and } \nu_{A_{\omega}}(P) \leq \omega \right\} \)
Definition 5.2

Let $A_\omega$ be an intuitionistic fuzzy subset of $E$, $A'_\omega$ be the IFD subset of the lattice points $\Sigma$ and $\sigma, \omega \in [0,1]$ such that $\sigma + \omega \leq 1$, then intuitionistic fuzzy digital lower cut sets (or IFD lower cut sets) are defined and denoted as follows,

- $A'_\omega[\sigma, \omega]^+ = \{ P : \mu_{A_\omega}(P) \geq \sigma \text{ and } \nu_{A_\omega}(P) \leq \omega \}$
- $A'_\omega(\sigma, \omega)^+ = \{ P : \mu_{A_\omega}(P) > \sigma \text{ and } \nu_{A_\omega}(P) < \omega \}$
- $A'_\omega(\sigma, \omega)^- = \{ P : \mu_{A_\omega}(P) \leq \sigma \text{ and } \nu_{A_\omega}(P) \geq \omega \}$
- $A'_\omega(\sigma, \omega)^- = \{ P : \mu_{A_\omega}(P) < \sigma \text{ and } \nu_{A_\omega}(P) > \omega \}$
- $A'_\omega[\sigma, \omega]^+ = \{ P : \mu_{A_\omega}(P) \geq \sigma \text{ and } \nu_{A_\omega}(P) < \omega \}$

Proposition 5.1

Let $A_\omega$ be an intuitionistic fuzzy subset of $E$, $A'_\omega$ be the IFD subset of the lattice points $\Sigma$, then the following statements are equivalent.

- $A'_\omega$ is an IFD convex set.
- $A'_\omega[\sigma, \omega]^-$ is an IFD convex set.
- $A'_\omega(\sigma, \omega)^-$ is an IFD convex set.
- $A'_\omega(\sigma, \omega)^-$ is an IFD convex set.
- $A'_\omega(\sigma, \omega)^-$ is an IFD convex set.

Proof:

(i) $\Rightarrow$ (ii)

Assume that (i) is true. Since $A'_\omega$ is an IFD convex set, $A_\omega$ is IFD regular and convex. Then for all $P, Q$ in $A'_\omega[\sigma, \omega]^-$ and for all $P_i, 0 \leq i \leq n$ on the line segment, $P = P_0, P_1, \ldots, P_n = Q$ from $P$ to $Q$, we have...
\[ \mu_{A_+}((1 - \alpha)P + \alpha Q) \geq \min(\mu_{A_+}(P), \mu_{A_+}(Q)) \geq \sigma ; \]

\[ \nu_{A_+}((1 - \alpha)P + \alpha Q) \geq \min(\nu_{A_+}(P), \nu_{A_+}(Q)) \leq \omega. \]

\[ \Rightarrow \mu_{A_+}(P_i) \geq \sigma \text{ and } \nu_{A_+}(P_i) \leq \omega \]

\[ \Rightarrow P_i \in A_+[[\sigma, \omega]] \] and so \( A_+[[\sigma, \omega]] \) is convex. Hence \( A'_+[[\sigma, \omega]] \) is an IFD convex set and also IFD regular.

In the similar manner, the other parts of the Proposition can also be proved.

**Proposition 5.2**

Let \( A_+ \) be an intuitionistic fuzzy subset of \( E \), \( A'_+ \) be the IFD subset of the lattice points \( \Sigma \), then the following statements are equivalent.

i. \( A'_+ \) is an IFD concave set.

ii. \( A'_+[[\sigma, \omega]] \) is an IFD convex set.

iii. \( A'_+[[\sigma, \omega]] \) is an IFD convex set.

iv. \( A'_+[[\sigma, \omega]] \) is an IFD convex set.

v. \( A'_+[[\sigma, \omega]] \) is an IFD convex set.

**Proof:**

The proof is similar to the proof of Proposition 5.1.

**References**


