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MLE and Bayes Approximation of Hazard in Weibull Failure Model

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Abstract.

The paper deals with the methods to obtain the approximate Bayes estimators of the Weibull distribution by using Lindley approximation technique for type-II censored samples. A bivariate prior density for the parameters, squared error Loss function (SELF) and Linex Loss function are used to obtain the approximate Bayes Estimators. A statistical software R is used for numerical calculations for different approximate Bayes estimators and their relative mean squared errors by preparing programs to present the statistical properties of the estimators.

Keywords: R-Software, Hazard Rate, Approximate Bayes estimators, Lindley approximation technique, squared error Loss function (SELF), Linex Loss function (LLF).

INTRODUCTION

Life is hazardous. There are earthquakes, divorce, cancer and stock-market crashes to contend with. If we want to understand the statistics of events like these then you need to understand the Weibull distribution. The properties of the Weibull distribution are best described in terms of the hazard function. This tells us how likely something is to fail given that it has survived so far. A simple power law is used for the hazard function, which accommodates three distinct behaviors- (1) if something is going to fail it will most likely fail at the start; (2) the rate of failure is fairly constant; (3) failure becomes more likely as time goes on.

Weibull distributions are characterized by a scale parameter and a shape parameter [Weibull (1951) and Kao (1959)]. The key to understanding the behavior of Weibull distributions is the shape parameter, and the hazard function behaviors discussed above are also characterized by the shape parameter. There are three different regimes distinguished by their shapes known as Bath tub Shaped Hazard Rate Curve.

The probability density function reliability and hazard rate functions of Weibull distribution is given respectively as

$$f(x) = p\theta x^{(p-1)} \exp(-\theta x^{P}) \quad ; \quad x, \theta, p > 0 \quad (1)$$

$$R(t) = exp(-\theta t^{p}) \quad ; \quad t > 0 \quad (2)$$

$$H(t) = p\theta t^{(p-1)} \quad ; \quad t > 0 \quad (3)$$

Where $'\theta'$ is the scale and 'p'is shape parameters.

The most widely used loss function in estimation problems is quadratic loss function given as $L(\hat{\theta}, \theta) = k(\hat{\theta} - \theta)^2$ where $\hat{\theta}$ is the estimate of $\theta,$ the loss function is called quadratic weighed loss function if k=1, we have

$$L(\hat{\theta}, \theta) = (\hat{\theta} - \theta)^2 \quad (4)$$

Known as squared error loss function (SELF). This loss function is symmetrical because it associates the equal importance to the losses due to overestimation and under estimation with equal magnitudes however in some estimation problems such an assumption may be inappropriate. Overestimation may be more serious than underestimation or Vice-versa Ferguson (1985). Canfield (1970), Basu and Ebrabimi (1991). Zellner (1986) Soliman (2000) derived and discussed the properties of varian's (1975) asymmetric loss function for a number of distributions. Such as a loss function is derived as

$$L(\Delta) = b \exp(a\Delta) - c\Delta - b; \quad \Delta = (\hat{\theta} - \theta), \quad (5)$$

And $a, c \neq 0, \ b > 0$

For a minimum to exist at Δ=0.

$$\left[\frac{\partial}{\partial \Delta} L(\Delta)\right]_{\Delta=0} = 0 = ab - c$$

And we have a two parameter loss function

$$L(\Delta) = b[\exp(a\Delta) - a\Delta - 1]; \ b>0, a>0, \quad (6)$$

The sign and magnitude of 'a' represents the direction and degree of symmetry respectively, when a>0, then overestimation is more serious than underestimation and vice-versa. For 'a' closed to zero the LINEX loss is approximately squared error loss and therefore almost symmetric.

The posterior expectation of LINEX loss function in eqn. (5) is

$$E_{\pi}\left(L(\hat{\theta} - \theta)\right) \propto e^{a\hat{\theta}} E_{\pi}(e^{-a\theta}) - a\left(\hat{\theta} - E_{\pi}(\theta)\right) - 1 \quad (7)$$

Where E_{π} is the posterior expectation with respect to posterior density of θ.

The Bayes Estimator $\hat{\theta}_{BL}$ of θ under LINEX loss function is the value which minimizes eqn. (6) is

$$\hat{\theta}_{BL} = = -\frac{1}{a} log\left(E_{\pi}\left(e^{-a\theta}\right)\right); \quad (8)$$

Provided that $E_{\pi}\left(e^{-a\theta}\right)$ exists and is finite.

In a Bayesian setup, the unknown parameter is viewed as random variable. The uncertainty about the true value of parameter is expressed by a prior distribution. The parametric inference is made using the posterior distribution which is obtained by incorporating the observed data in to the prior distribution using the Bayes theorem, the first theorem of inference. Hence we update the prior distribution in the light of observed data. Thus the uncertainty about the parameter prior to the experiment is represented by the prior distribution and the same after the experiment is represented by the posterior distribution. The various statistical models are considered.

The paper deals with the methods to obtain the approximate Bayes estimators of the Weibull distribution by using Lindley approximation technique for type-II censored samples. A bivariate prior density for the parameters, squared error Loss function (SELF) and Linex Loss function are used to obtain the approximate Bayes Estimators. A statistical software R is used for numerical calculations for different approximate Bayes estimators and their relative mean squared errors by preparing programs to present the statistical properties of the estimators.

2. The Estimators.

Let $x_1, x_2, \dots \dots \dots x_n$ be the life times of ‘n’ items that are put on test for their lives, follow a weibull distribution with density given in equation (1). The failure times are recorded as they occur until a fixed number ‘r’ of times failed. Let= $(x_{(1)}, x_{(2)},, x_{(n)})$, where $x_{(i)}$ is the life time of the i^th^ item. Since remaining (n-r) items yet not failed thus have life times greater than$x_{(r)}$.

The likelihood function can be written as

$$L(x|\theta, p) = \frac{n!}{(n-r)!}(\mathrm{p\theta})^r \prod_{i=1}^{r} x_i^{(p-1)} \exp(-\delta\theta), \quad (9)$$

Where $\delta = \sum_{i=1}^{r} x_i^p + (n-r)x_r^p$

The logarithm of the likelihood function is

$$\log L(x|\theta, p) \ \propto \ r\, log\, p + r \log\, \theta + (p-1)\sum_{i=1}^{r} \log x_i - \delta\theta, \quad (10)$$

Assuming that ‘p’ is known, the maximum likelihood estimator $\hat{\theta}_{ML}$of θ can be obtain by using equation (10) as

$$\hat{\theta}_{ML} = r/\delta \qquad (11)$$

If both the parameters p and θ are unknown their MLE’s $\hat{p}_{ML}$and $\hat{\theta}_{ML}$ can be obtained by solving the following equation

$$\frac{\delta}{\delta\theta}\log L = \frac{r}{\theta} - \delta = 0 \ , \quad (11a)$$

$$\frac{\delta \log L}{\delta P} = \frac{r}{P} + \sum_{i=1}^{r} \log x_i - \theta\delta_1 = 0, \quad (11b)$$

Where

$\delta_1 = \sum_{i=1}^{r} x_i^P \log x_i + (n-r)x_r^P \log x_r$, eliminating θ between the two equations of (11) and simplifying we get

$$\hat{p}_{ML} = \frac{r}{\delta^*} \quad (12)$$

Where $\delta^* = \left[\frac{r\delta_1}{\delta} - \sum_{i=1}^{r} logx_i\right]$

Equation (12) may be solved for Newton- Raphson or any suitable iterative Method and this value is substituted in equation (11b) by replacing with p get $\hat{p}$ as

$$\hat{\theta}_{ML} = \frac{\frac{r}{\hat{p}_{ML}} + \sum_{i=1}^{r} logx_i}{\sum_{i=1}^{r} x_i^{\hat{p}_{ML}} \ logx_i + (\mathrm{n-r}) x_r^{\hat{p}_{ML}} \ logx_r}, \quad (13)$$

The MLE's of R (t) and H (t) are given respectively by equation (2) and (3) after replacing θ and p by $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$.

3. Bayes Estimator of θ when Shape Parameter P is known.

If p is known assume gamma prior $\gamma(\alpha,\beta)$ as conjugate prior for θ as

$$g(\theta|\underline{x}) = \frac{\beta^{\alpha}}{\Gamma\alpha}(\theta)^{(\alpha+1)} \exp(-\beta\theta) \, ; (\alpha,\beta) > 0, \theta > 0 \ , \quad (14)$$

The posterior distribution of θ using equation (9) and (14) we get

$$h(\theta|\underline{x}) = \frac{(\delta+\beta)^{r+\alpha}}{\Gamma(r+\alpha)}(\theta)^{(r+\alpha-1)} \exp\big(-\theta(\delta+\beta)\big) , \quad (15)$$

Under squared error loss function, the Bayes estimator $\hat{\theta}_{BS}$, is the posterior mean given by

$$\hat{\theta}_{BS} = \frac{(r+\alpha)}{(\delta+\beta)} \quad (16)$$

Under linex Loss Function, the Bayes estimator $\hat{\theta}_{BL}$ of θ using (7) and (15) given by

$$\hat{\theta}_{BL} = \frac{(r+\alpha)}{a} \ log\left[1 + \frac{a}{(\delta+\beta)}\right] \quad (17)$$

The Bayes Estimate of H (t).

The posterior density at H (t) using equation (3) and (15), is given as

$$h(H|t) = \frac{[(\delta+\beta)c^*]^{(r+\alpha)}}{\Gamma(r+\alpha)} \ . H^{(r+\alpha-1)} \exp\big(-c^* \mathrm{H}(\delta+\beta)\big) \, ; \quad (18)$$

Where $c^* = pt^{(p-1)}$

The Bayes estimator of H (t) under Linex loss function

$$\widehat{H}_{BL} = \left(\frac{(r+\alpha)}{\text{a}}\right)\ log\left[1+\frac{a}{c^{*}(\delta+\beta)}\right]\ ; \quad (19)$$

4. The Bayes Estimators With $\boldsymbol{\theta}$ and P Unknown.

The joint prior density of θ and p is given by

$G(\theta|p) = g_1(\theta|p).g_2(p)$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi}p^{-\xi}\theta^{(\xi-1)}.\exp\left[-\left(\frac{\theta}{p}+\frac{p}{\lambda}\right)\right]\ ; (\theta,p,\lambda,\xi) > 0, \quad (20)$$

Where

$$g_1(\theta|p) = \frac{1}{\Gamma\xi}p^{-\xi}\ \theta^{(\xi-1)}.\exp\left[-\frac{\theta}{p}\right]; \quad (21)$$

And

$$g_2(p) = \frac{1}{\lambda}\exp\left(-\frac{p}{\lambda}\right) \quad ; \quad (22)$$

The joint posterior density of θ and p is

$$h^{*}(\theta,p|\underline{x}) = \frac{\frac{1}{\lambda\Gamma\xi}p^{-\lambda}\theta^{(\xi+1)}\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}\right\}\right](\text{p}\theta)^{r}\prod_{i=1}^{r}x_i^{(p-1)}e^{-p\theta}}{\iint\frac{1}{\lambda\Gamma\xi}p^{(r-\xi)}\theta^{(r+\xi+1)}\prod_{i=1}^{r}x_i^{(p-1)}.\exp\left[-\left\{\frac{\theta}{p}+\frac{p}{\lambda}+\text{p}\theta\right\}\right]d\theta dp}\ ; (23)$$

Approximate Bayes Estimators.

The Bayes estimators of a function $\mu = \mu(\theta,p)$ of the unknown parameter θ and p under squared error loss is the posterior mean

$$\hat{\mu}_{ABS} = E(\mu|\underline{x}) = \frac{\iint\mu(\theta,p)G(\theta,p|\underline{x})d\theta dp}{\iint G(\theta,p|\underline{x}).d\theta.dp}\ ; \quad (24)$$

To evaluate (24) consider the method of Lindley approximation [Lindley (1980)].

$$E(\mu(\theta,p)|\underline{x}) = \frac{\int\mu(\theta).e^{(l(\theta)+\rho(\theta))}d\theta}{\int e^{(l(\theta)+\rho(\theta))}d\theta}; \quad (25)$$

Where$(\theta) = \log g(\theta)$, and $g(\theta)$ is an arbitrary function of θ and $l(\theta)$ is the logarithm likelihood function.

The Lindley approximation for two parameter is given by

$$E(\hat{\mu}(\theta,p)|\underline{x}) = \mu(\theta,p) + \frac{A}{2} + \rho_1 A_{12} + \rho_2 A_{21} + \frac{1}{2}[l_{30}B_{12} + l_{21}C_{12} + l_{12}C_{21} + l_{03}B_{21}], \quad (26)$$

Where

$$A = \sum_{1}^{2}\sum_{1}^{2}\mu_{ij}\sigma_{ij}\ ; \quad l_{\eta\epsilon} = (\delta^{(\eta+\epsilon)}l|\delta\theta_1^{\eta}\delta\theta_2^{\epsilon});$$

Where $(\eta+\epsilon) = 3$ for $i,j = 12$ $\rho_i = (\delta\rho|\delta\theta_i);$

$$\mu_i = \frac{\delta\mu}{\delta\theta_i}\ ;\mu_{ij} = \frac{\delta^2\mu}{\delta\theta_i\delta\theta_j}\ ;\forall i \neq j\ ;$$

$$A_{ij} = \mu_i\sigma_{ij} + \ \mu_j\sigma_{ji}\ ; \qquad B_{ij} = \left(\mu_i\sigma_{ii} + \mu_j\sigma_{ij}\right)\sigma_{ii}\ ;$$

$$C_{ij} = 3\mu_i\sigma_{ii}\sigma_{ij} + \mu_j\left(\sigma_{ii}\sigma_{jj} + 2\sigma_{ij}^2\right);$$

Where σ_{ij} is the (i,j)th element of the inverse of matrix $\{-l_{jj}\}; i,j = 1{,}2$ s.t. $l_{ij} = \frac{\delta^2 l}{\delta\theta_i\delta\theta_j}$.

All the above function are evaluated at MLE of (θ_1,θ_2).In our case $(\theta_1\theta_2) = (\theta,p); So\ \mu(\theta) = \mu(\theta,p)$

To apply Lindley approximation (26) we first obtain σ_{ij} , elements of the inverse of $\{-l_{jj}\}; i,j = 1{,}2$,which can be shown to be

$$\sigma_{11} = \frac{M}{D},\ \sigma_{12} = \sigma_{21}\ = \frac{\delta_1}{D},\sigma_{22} = \ \frac{r}{D\ \theta^2},$$

Where$M = \left(\frac{r}{p^2} + \theta\delta_2\right);\ D = \left[\frac{r}{\theta^2}\left(\frac{r}{p^2} + \theta^2\delta_2\right)\right];$

$$\delta_2 = \ \sum\nolimits_{\mathrm{i=1}}^{\mathrm{r}} x_i^p\ (logx_i)^2 + (n-r)x_r^p(logx_r)^2;$$

To evaluate ρ_i , take the joint prior $G(\theta|p)$

$$G(\theta|p) = \frac{1}{\lambda\Gamma\xi}p^{-\xi}\theta^{(\xi-1)}.\exp\left[\left\{-\frac{\theta}{p} + \frac{p}{\lambda}\right\}\right]\ ;(\theta,p,\lambda,\xi) > 0, \quad (27)$$

$$\Rightarrow \rho = \log[G(\theta|p)] = constant - \xi logp - (\xi - 1)log\theta - \frac{\theta}{p} - \frac{p}{\lambda}$$

Therefore

$$\rho_1 = \frac{\partial\rho}{\partial\theta} = \frac{(\xi-1)\theta}{\theta} - \frac{1}{p}; \qquad \text{And} \qquad \rho_2 = \frac{\theta}{p^2} - \frac{1}{\lambda} - \frac{\xi}{p}; \qquad)$$

Further more

$$l_{21} = 0\ ; l_{12} = -\delta_2\ ; l_{03} = \frac{2r}{p^3} - \theta\delta_3; and\ \ l_{30} = \frac{2r}{\theta^3};$$

Where $\delta_3 = \ \sum\nolimits_{\mathrm{i=1}}^{\mathrm{r}} x_i^p\ (logx_i)^3 + (n-r)x_r^p(logx_r)^3$

By substituting above values in eqn. (27), yields the Bayes estimator under SELF using Lindley approximation denoted by $\hat{\mu}_{ABS}$

$$\hat{\mu}_{ABS} = E\left(\mu(\theta,p)\right) = \mu(\theta,p) + \mathrm{Q} + \mu_1 Q_1 + \mu_2 Q_2;\ (28)$$

$$\text{Where Q} = \frac{1}{2}\left[\mu_{11}\sigma_{11} + \mu_{21}\sigma_{21} + \mu_{12}\sigma_{12} + \mu_{22}\sigma_{22}\right]; \quad (28a)$$

$$\mathrm{Q}_1 = \frac{1}{\theta^2 D^2}\left[\frac{M\theta D}{p}(p(\xi-1)-1) + \frac{\theta^2\delta_1 D}{\lambda p^2}\{\lambda\theta - p^2 - \lambda\xi \mathrm{p}\}\right.$$

$$\left. + \frac{rM^2}{\theta} - \frac{rM\delta_1}{2} - \theta^2\delta_1^2\delta_2 + \frac{r^2}{p^3}\delta_1 - \frac{\theta r\delta_1\delta_3}{2}\right]; \quad (28b)$$

$$Q_2 = \frac{1}{\theta^2 D^2}\left[\frac{\theta\ \delta_1 D}{p}(p(\xi-1)-\theta)+\frac{\mathrm{r}D}{\lambda p^2}\{\lambda\theta-p^2-\lambda\xi \mathrm{p}\}\right.$$

$$\left.+\frac{rM\delta_1}{\theta}-\frac{3\delta_1 r\delta_2}{2}+\frac{r^2}{\theta^2 p^3}-\frac{r^2\delta_3}{2\theta}\right]; \quad (28c)$$

All the function of right hand side of the eqn. (28) are to be evaluated for $\hat{\theta}_{ML}$ and $\hat{p}_{ML}$

5. Approximate Bayes Estimates Under Squared Error Loss Function.

With equations (28)-(28c), the different Approximate Bayes estimators Under SELF using Lindley's approximation given by

Special cases

(i) Substituting $\mu(\theta,p)=H$ in eqn. (28), we get the Approximate Bayes estimator of Hazard rate H=H (t) as

$$\hat{H}_{ABS}=H\left[1+\frac{1}{p\theta\ D}\phi_1+\frac{Q_1}{\theta}+\frac{(1+plog\ t)}{p}Q_2\right] \text{At } \left(\hat{\theta}_{ML},\hat{p}_{ML}\right) \quad (29)$$

Where $\phi_1=\left[\delta_2+\frac{2\mathrm{r}logt}{\theta}(2+plog\ t)\right]$

6. Approximate Bayes Estimators Under Linex Loss Function.

The Approximate Bayes estimator of a function $\mu=\mu(\theta,p)$ of unknown parameters $\theta\ and\ p$ under LINEX loss function in eqn. (6) is given by

$$\hat{\mu}_{ABL}=-\frac{1}{a}log\left(E_{h^*}(e^{-a\mu})\right); (30)$$

Where $E_{h^*}\left((e^{-a\mu})\middle|\bar{x}\right)=\frac{\iint e^{-a\mu}\ h^*(\theta,p)d\theta\ dp}{\iint\ h^*(\theta,p)d\theta\ dp};\quad (31)$

We apply Lindley's Procedure to obtain different Approximate Bayes estimators under LLF as

(i) Approximate Bayes estimator of H=H (t) under LLF is

$$\hat{H}_{ABL}=H+log\left[1-\frac{aH}{2pD\ \theta^2}\{\phi_2+2D\theta(pQ_1+(1+p\ logt\ )\theta Q_2)\}\right]\text{At}\left(\hat{\theta}_{ML},\hat{p}_{ML}\right), (32)$$

Where

$$\phi_2=\left\{aHMp-2\delta_1\theta(aH+1)(1+p\ logt)-r\left(a\theta t^{(p-1)}\right)(1+p\ logt)^2+logt(2+plogt)\right\}$$

7. Numerical Calculations and Comparison.

The numerical calculations are done by using R Language programming and results are presented in form of tables.

1. The values of and are generated from the equations (20-22) for given α=2, and β =3, which comes out to be θ=0.238 and p=0.227. For these values of and p the Weibull random variates are generated.

2. Taking the different sizes of samples n=25 (25) 100 with failure censoring, MLE's,

the Approximate Bayes estimators, and their respective MSE's (in parenthesis) by repeating the steps 500 times, are presented in the tables from (3.1-3.4), for t=2, a=20, R(t)=0.7568, H(t)=0.0316 and parameters of prior distribution α =2 , and β =3.

3. Table (1) presents the MLE of parameter of (for known p) and approximate Bayes estimators under SELF, LLF and PLF (for θ and p both unknown). The MSE's in all above cases are presented in parenthesis. The estimators have minimum MSE's for small sample sizes, as the sample sizes increase, the MSE's increased. Among all the four estimators $\hat{\theta}_{ABL}$ under LLF has the lowest MSE.

4. Table (2) presents the Approximate Bayes estimator of hazard rate function H(t) of Weibull density under SELF, LLF and PLF, MLE's and the respective MSE's for different sample sizes. The estimators have lower efficiency for larger sample sizes. The under $\widehat{H}_{ABL}$ LLF are more efficient than others.

Table (1): Mean and MSE's of $\boldsymbol{\theta}$

$(\lambda = 2, \xi = 3, = .238, p = .227, a = 20\ )$

n	r	$\hat{\theta}_{ML}$	$\hat{\theta}_{ABS}$	$\hat{\theta}_{ABL}$
25	20	0.02417	0.028051	0.027771
		(8.2278x10^{-5})	**(7.9161x10^{-5})**	**((7.9384x10^{-5})**
50	30	0.070689	0.078036	0.075839
		(4.8866x10^{-5})	**(4.4380x10^{-5})**	**(4.5669x10^{-5})**
75	50	0.835593	0.822443	0.773257
		(7.4077x10^{-4})	**(7.09160x10^{-4})**	**(5.9679x10^{-4})**
100	75	0.801292	0.796716	0.755629
		(6.5996x10^{-4})	**(6.4915x10^{-4})**	**(5.5588x10^{-4})**

Table (2): **Mean and MSE's of** H(t)

$(\lambda = 2, \xi = 3, = 0.238, p = 0.227, t = 2, R(t) = .07568, H(t) = 0.03, a = 20\ )$

n	r	$\widehat{H}_{ML}$	$\widehat{H}_{ABS}$	$\widehat{H}_{ABL}$
25	20	0.1213389	0. 096088	0. 07981826
		(1.6865x10^{-9})	**(2.8694x10^{-7})**	**(3.115878x10^{-7})**
50	30	0.04814398	0.096333	0.0816343
		(3.8895x10^{-6})	**(3.2531x10^{-6})**	**(2.27511x10^{-7})**
75	50	0.0620632	0.093953	0.089961
		(1.8285x10^{-6})	**(5.4638x10^{-8})**	**(1.09417x10^{-8})**
100	75	0.104949	0.0931146	0.0921015
		(3.1999x10^{-6})	**(1.32715x10^{-6})**	**(3.8326x10^{-7})**

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