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On detection number of graphs

Sanma G.R[1] and T. Nicholas[2]

[1.] *Research scholar, Research Department of Mathematics, St. Jude's College, Thoothoor – 629176, Kanyakumari District, Tamil Nadu, India.*

[2] *Associate Professor and Head, Research Department of Mathematics, St. Jude's College, Thoothoor – 629176, Kanyakumari District, Tamil Nadu, India*

Abstract

Let G be a connected graph of order $n \geq 3$ and let k-labeling c: $E(G) \rightarrow \{1, 2, 3, \ldots, k\}$ of the edges of G, (where adjacent edges may be colored the same). For each vertex v of G, the color code of v with respect to c is the k-tuple $c(v) = (a_1, a_2, \ldots, a_k)$ where a_i is the number of edges incident with v that are colored i ($1 \leq i \leq k$). The k-labeling c is detectable if every two adjacent vertices of G have distinct codes. The minimum positive integer k for which G has a detectable k-labeling is the detection number det(G) of G. In this paper we obtain the detection number of some known graphs such as P_n x P_m, circular halin graph of level two, wheel, crown graph etc.

Keywords: detection number, Pn x Pm, circular halin graphs of level two.

MSC AMS classification 2010: 05C15, 05C70.

1. INTRODUCTION

Let G be a finite, connected, undirected and simple graph with order $n \geq 3$. Let c: $E(G) \rightarrow \{1, 2, \ldots, k\}$ be a labeling of the edges of G, where k is a positive integer. The color code of a vertex v of G is the ordered k-tuple $c(v) = (a_1, a_2, \ldots, a_k)$ where a_i is the number of edges incident with v that are labeled i for $i \in \{1, 2, \ldots, k\}$. The labeling c is called a detectable coloring of G if any pair of adjacent vertices of G have distinct color codes. The detection number or detectable chromatic number of G, denoted det(G) is the minimum positive integer k for which G has detectable k-coloring.

In order to distinguish the vertices of a connected graph and to distinguish adjacent vertices of a graph, with the minimum number of colors, the concept of detection number was introduced by Karonski et al.[6] (2004). To differentiate the vertices of a connected graph[4], the concept of detective coloring was originated. In [1] (G. Chartrand, H. Escuadro, F. Okamoto and P.Zhang) the detective number of stars, double stars, cycles, paths, complete graphs and complete bipartite graphs are determined. Also in [3] (H. Escuardo, Ping Zhang) they establish a formulae for the detection number of path in terms of its order. Furthermore it is shown that for integers $n \geq 3$ and $k \geq 2$, there exists a unicyclic graph G of order n having det(G) = k if and only if $d_u(n) \leq k \leq D_u(n)$ where $d_u(n)$ and $D_u(n)$ are the minimum and maximum detection number among all unicyclic graphs. In [5] (Frederic Havet, Nagarajan Paramaguru, Rathinasamy Sampathkumar) showed that it is NP-complete to decide if the detection number of a cubic graph is 2. They also show that the detection number of every bipartite graph of minimum degree at least 3 is atmost 2. Also they gave some sufficient conditions for a cubic graph to have detection number 3. In this paper we find the detection number of P_n x P_m, , helm graph, gear graph, wheel, circular halin graphs of level two[9], fan, friendship graph.

We make use of the following result which is already proved in [3] [7].

Observation 1.1: [3] Every connected graph of order 3 or more has detection number at least 2.

2. MAIN RESULTS.

Theorem 2.1. Let G = P_n x P_m , $m \geq 2$, then det(G) = 2.

Proof. We first prove that det(G) $\leq$ 2.

Consider the graph P_n x P_m where $m \geq 2$.

Let E_i = { e_{i1}, e_{i2}, e_{i3}, . . . , $e_{i(m-1)}$ } the edges in the i^{th} row and H_j = { h_{1j}, h_{2j}, h_{3j}, , , , ,$h_{(n-1)j}$} be the vertical edges of the j^{th} column where i = 1, 2, 3, ... , n and j = 1, 2, 3, . . . , m.

Let c: E(G) → {1, 2, 3, . . . , k} be a labeling of the edges of G where k is a positive integer then

$$c(e) = \begin{cases} 1 \text{ for } E_i, H_i \text{where i is odd,} \\ 2 \text{ for } E_i, H_i \text{where i is even.} \end{cases}$$

In P_n x P_m the corner vertices have degree two, remaining boundary vertices have degree three and the internal vertices all have degree four each.

Let P_{ij} be any internal vertex. Then, c(P_{ij}) has one of the codes (4, 0) or (2, 2) or (0, 4).

If $c(P_{ij}) = (4, 0)$ then the edges incident to the adjacent vertices receive any of the labels 221, 1122 resulting in the codes (1, 2), (2, 2) which is not equal to $c(P_{ij})$.

If $c(P_{ij}) = (0, 4)$ then the edges incident to the adjacent vertices receive any of the labels 112, 1122 resulting in the codes (2, 0), (2, 2) which is not equal to $c(P_{ij})$.

If $c(P_{ij}) = (2, 2)$ then the edges incident to the adjacent vertices receive any of the labels 111, 222, 1111, 2222 resulting in the codes (3, 0), (0, 3), (4, 0), (0, 4) which is not equal to $c(P_{ij})$.

For the boundary vertices (except the corner vertices) $c(P_{ij})$ accepts one of the codes (2, 1), (3, 0), (1, 2), (0, 3).

If $c(P_{ij}) = (2, 1)$ then the edges incident to the adjacent vertices receive any of the labels 11, 111, 2222 resulting in the codes (2, 0), (3, 0), (0, 4) which is not equal to $c(P_{ij})$.

If $c(P_{ij}) = (3, 0)$ then the edges incident to the adjacent vertices receive any of the labels 12, 112, 1122 resulting in the codes (1, 1), (2, 1), (2, 2) which is not equal to $c(P_{ij})$.

If $c(P_{ij}) = (1, 2)$ then the edges incident to the adjacent vertices receive any of the labels 22, 222, 1111 resulting in the codes (0, 2), (0, 3), (4, 0) which is not equal to $c(P_{ij})$. If $c(P_{ij}) = (0, 3)$ then the edges incident to the adjacent vertices receive any of the labels 12, 122, 1122 resulting in the codes (1, 1), (1, 2), (2, 2) which is not equal to $c(P_{ij})$.

In the corner vertex, if $c(P_{11}) = (2, 0)$, then the edges incident to the adjacent vertices receive the label 112 only giving the code (2, 1) which is not equal to $c(P_{11})$.

For the remaining corner vertices $c(P_{ij}) = (1,1)$ or (2, 0) or (0, 2) depending on i and j are even or odd.

Case 1. n is even.

Here, $c(P_{n1}) = (1, 1)$ then the edges incident to the adjacent vertices receive any of the labels 111, 222 resulting in the codes (3, 0), (0, 3) which is not equal to $c(P_{n1})$.

Subcase 1.1. m is odd.

The corner vertex has the code $c(P_{1m}) = (2, 0)$, then the edges incident to the adjacent vertices receive the labels 112 only resulting in the code (2, 1) which is not equal to $c(P_{1m})$.

The corner vertex has the code $c(P_{nm}) = (1, 1)$, then the edges incident to the adjacent vertices receive the labels 111, 222 resulting in the codes (3, 0), (0,3) which is not equal to $c(P_{nm})$.

Example.

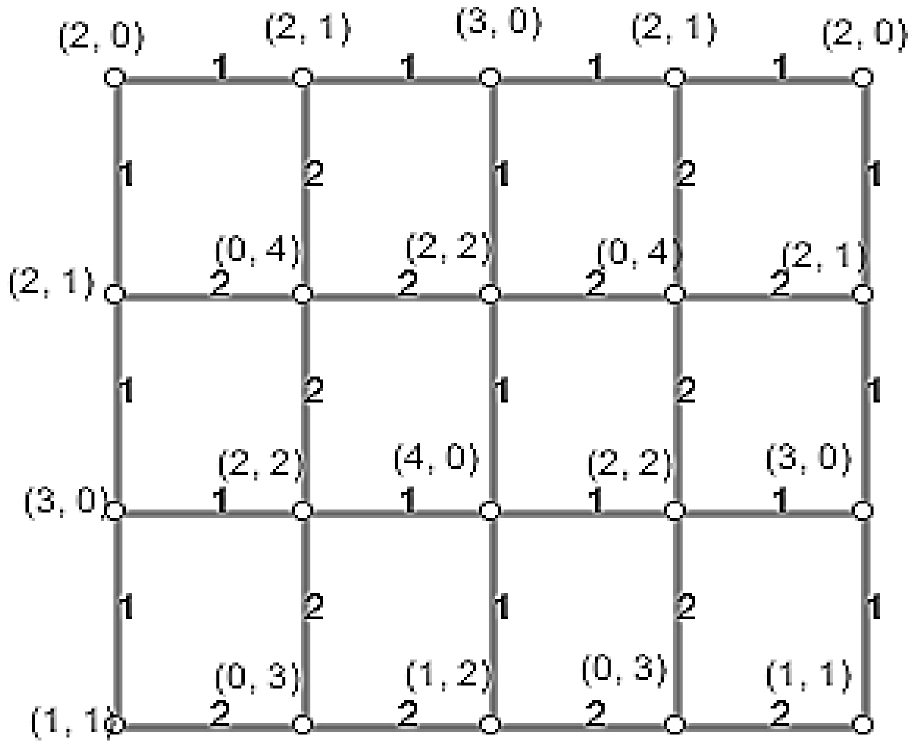

Fig 1. Detective labelings in P_4 x P_5

Subcase 1.2. m is even.

The corner vertex has the code $c(P_{1m}) = (1, 1)$, then the edges incident to the adjacent vertices receive the labels 111, 222 resulting in the codes (3, 0), (0, 3) which is not equal to $c(P_{1m})$.

The corner vertex has the code $c(P_{nm}) = (0, 2)$, then the edges incident to the adjacent vertices receive the label 122 resulting in the code (1, 2) which is not equal to $c(P_{nm})$.

Case 2. n is odd.

Here, $c(P_{n1}) = (2, 0)$ then the edges incident to the adjacent vertices receive the label 112 resulting in the code (2, 1) which is not equal to $c(P_{n1})$.

Subcase 2.1. m is odd.

The corner vertices has the code $c(P_{1m}) = c(P_{nm}) = (2, 0)$, then the edges incident to the adjacent vertices receive the label 112 only resulting in the code (2, 1) which is not equal to $c(P_{1m})$ or $c(P_{nm})$.

Example.

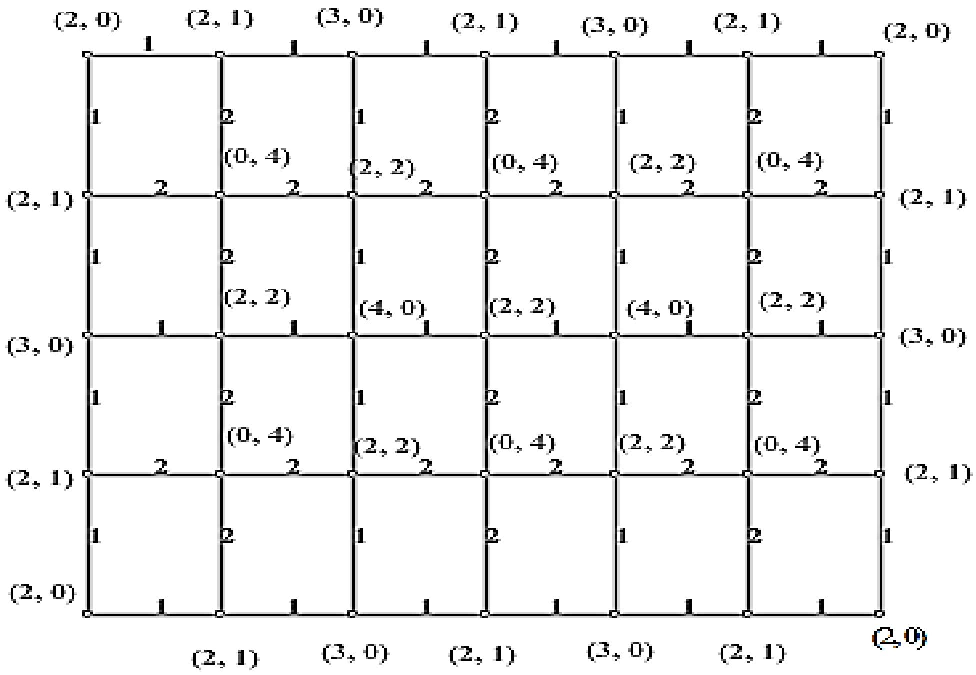

Fig 2. Detective labeling in $P_5 \times P_7$

Subcase 2.2. m is even.

The corner vertex has the code $c(P_{1m}) = c(P_{nm}) = (1, 1)$, then the edges incident to the adjacent vertices receive the labels 111, 222 resulting in the codes (3, 0), (0, 3) which is not equal to $c(P_{1m})$ or $c(P_{nm})$.

Hence all the adjacent vertices have distinct codes by the above labeling. Therefore $det(G) \le 2$.

By Observation 1.1 and $det(G) \le 2$ we get the result that $det(G) = 2$. ■

Theorem 2.2

For $G = H_1(2, D)$, $D > 2$ then $det(G) = 2$.

Proof. To prove $det(G) = 2$ first we have to show that $det(G) \le 2$.

Let x be the central vertex at level ℓ_0. Let $v_1, v_2, v_3, \ldots, v_D$ be the vertices in level ℓ_1. Let $w_{i1}, w_{i2}, w_{i3}, \ldots, w_{i(D-1)}$ be the vertices from v_i to level ℓ_2. Let the edges from x to v_i, $i = 1, 2, 3, \ldots, D$ be e_i. let the edges from v_i to w_{ij} for all i and $j = 1, 2, 3, \ldots, (D-1)$ be f_{ij}. Let c_k be the edges in the level ℓ_2 where $k = 1, 2, 3, \ldots, D(D-1)$.

Case 1. D is odd.

Let the labeling $c: E(G) \rightarrow \{1, 2, 3, \ldots, k\}$ where k is a positive integer such that

$$c(e) = \begin{cases} 1 & \text{for } e_i \text{ and } f_{ij} \text{ for all i and } j = 1, 3, 5, \dots, (D-2), \\ 2 & \text{for } c_k \text{ and } f_{ij} \text{ for all k, i and } j = 2, 4, \dots, (D-1). \end{cases}$$

Here all the inner vertices have degree D and outer vertices have degree 3. The central vertex x has all the D adjacent edges labeled as 1.Therefore c(x) = (D, 0).

If c(x) = (D, 0), then the edges incident to the adjacent vertices receive the labels 112 or 11122 or 1111222 . . . . resulting the codes (2,1) or (3, 2) or (4, 3) . . . . which is different from c(x).

If $c(v_i) = \left(\left\lceil \frac{D}{2} \right\rceil, \left\lfloor \frac{D}{2} \right\rfloor\right)$, then the edges incident to the adjacent vertices receive the labels 122, 222, 111 or 11111 or 1111111 . . . . resulting the codes (1, 2), (0, 3), (3, 0) or (5, 0) or (7, 0). . . . which is not equal to $c(v_i)$.

If $c(c_i) = (1, 2)$, $i = 1, 3, 5, \dots (D^2 - D - 1)$ then the edges incident to the adjacent vertices receive the labels 222, 112 or 11122 or 1111222 . . . . resulting the codes (0, 3), (2, 1) or (3, 2) or (4, 3). . . .which is not equal to $c(c_i)$ for $i = 1, 3, 5, \dots (D^2 - D - 1)$.

If $c(c_i) = (0, 3)$, $i = 2, 4, 6, \dots, D(D-1)$ then the edges incident to the adjacent vertices receive the labels 122, 112 or 11122 or 1111222 . . . . resulting the codes (1, 2), (2, 1) or (3, 2) or (4, 3). . . .which is not equal to $c(c_i)$ for $i = 2, 4, 6, \dots, D(D-1)$.

$$\text{Therefore } c(v) = \begin{cases} (D, 0) & \text{for } x, \\ \left(\left\lceil \frac{D}{2} \right\rceil, \left\lfloor \frac{D}{2} \right\rfloor\right) & \text{for } v_i \text{ for all i}, \\ (1, 2) & \text{for } c_i,\ i = 1, 3, 5, \dots (D^2 - D - 1), \\ (0, 3) & \text{for } c_i,\ i = 2, 4, 6, \dots, D(D-1). \end{cases}$$

That is , adjacent vertices have distinct codes and therefore det(G) ≤ 2.

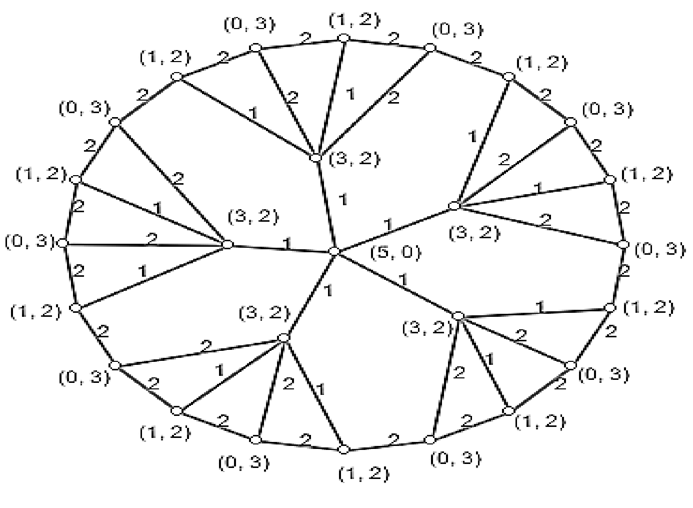

Fig 3. Detective labeling in $H_1(2, 5)$

Case 2. D is even

Let the labeling c: E(G) → {1, 2, 3, . . . , k} where k is a positive integar such that

$$c(e) = \begin{cases} 1 & \text{for all } e_i;\ f_{ij}\ \text{i. j are both odd and i, j are both even,} \\ 2 & \text{for all } c_k \text{ and } f_{ij} \text{ for odd i, even j and viceversa,} \end{cases}$$

Here all the inner vertices have degree D and outer vertices have degree 3. The central vertex x has all the D adjacent edges labelled as 1.Therefore c(x) = (D, 0).

If c(x) = (D, 0), then the edges incident to the adjacent vertices receive the labels 1112 or 1122 or 111122 or 111222 . . . .depending on degree D resulting the codes (3,1) or (2, 2) or (4, 2) or (3, 3). . . . which is different from c(x).

If $c(v_i) = \left(\left\lceil\frac{D}{2}\right\rceil + 1, \left\lfloor\frac{D}{2}\right\rfloor - 1\right)$ $i = 1, 3, 5, \ldots, (D-1)$, then the edges incident to the adjacent vertices receive the labels 122, 222, 1111 or 111111 or 11111111 . . . . resulting the codes (1, 2), (0, 3), (4, 0) or (6, 0) or (8, 0). . . . which is not equal to $c(v_i)$ $i = 1, 3, 5, \ldots, (D-1)$.

If $c(v_i) = \left(\left\lceil\frac{D}{2}\right\rceil, \left\lfloor\frac{D}{2}\right\rfloor\right)$ $i = 2, 4, 6, \ldots, D$, then the edges incident to the adjacent vertices receive the labels 122, 222, 1111 or 111111 or 11111111 . . . . resulting the codes (1, 2), (0, 3), (4, 0) or (6, 0) or (8, 0). . . . which is not equal to $c(v_i)$ $i = 2, 4, 6, \ldots, D$.

If $c(c_i)$ = (1, 2), $i = 1, 3, 5, \ldots (D^2 - D - 1)$ then the edges incident to the adjacent vertices receive the labels 222, 1112 or 1122 or 111122 or 111222 . . . . resulting the codes (0, 3), (3, 1) or (2, 2) or (4, 2) or (3, 3) . . . .which is not equal to $c(c_i)$ for $i = 1, 3, 5, \ldots (D^2 - D - 1)$.

If $c(c_i)$ = (0, 3), $i = 2, 4, 6, \ldots, D(D-1)$ then the edges incident to the adjacent vertices receive the labels 122, 1112 or 1122 or 111122 or 111222 . . . . resulting the codes (1, 2), (3, 1) or (2, 2) or (4, 2) or (3,3) . . . . which is not equal to $c(c_i)$ for $i = 2, 4, 6, \ldots, D(D-1)$.

$$\text{Therefore } c(v) = \begin{cases} (D, 0) & \text{for } x, \\ \left(\left\lceil\frac{D}{2}\right\rceil + 1, \left\lfloor\frac{D}{2}\right\rfloor - 1\right) & \text{for } v_i,\ i = 1, 3, 5, \ldots, (D-1) \\ \left(\frac{D}{2}, \frac{D}{2}\right) & \text{for } v_i, i = 2, 4, 6, \ldots, D, \\ (1, 2) & \text{for } c_i,\ i = 1, 3, 5, \ldots (D^2 - D - 1), \\ (0, 3) & \text{for } c_i,\ i = 2, 4, 6, \ldots, D(D-1). \end{cases}$$

That is , adjacent vertices have distinct codes and therefore det(G) ≤ 2.

Therefore by Observation 1.1, det(G) = 2. ■

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