Certain Dual Series Equations Involving Generalized Laguerre Polynomials

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Abstract

In this paper an exact solution is obtained for the dual series equations involving generalized Laguerre polynomials by Noble’s modified multiplying factor technique

**Key words:** Integral equation, Series equation, Laguerre polynomial

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1 Introduction

We consider the following dual series equations

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{s} a_{ij} A_{nj} \frac{L_{n+\beta}^\alpha[(x+b)^h]}{I(\alpha + ni + p + 1)} = f_i(x), \quad 0 < x < a
\]

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{s} b_{ij} A_{nj} \frac{L_{n+\beta}^\sigma[(x+b)^h]}{I(\alpha + ni + p + \beta)} = g_i(x), \quad a < x < \infty
\]

Where \(\alpha + \beta + 1 > \beta > 1 - m\), \(\sigma + 1 > \alpha + \beta > 0\), \(m\) is a positive integer, and \(0 < h < \infty\), \(0 \leq b < \infty\) and \(h\) and \(b\) are finite constants. \(L_{n+\beta}^\sigma[(x+b)^h]\) is a Laguerre polynomial, \(p\) is a non-negative integer. \(A_{nj}\) are unknown coefficients to be determine, and \(f_i(x)\) and \(g_i(x)\) are prescribed functions for \(i = 1, 2, \ldots, s\).

Srivastava [8] has solved following dual series equations involving Laguerre polynomials:
\[ \sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(x)}{\Gamma(\alpha + n + 1)} = f(x), 0 < x < a \]  \hspace{1cm} (1.3)  

\[ \sum_{n=0}^{\infty} \frac{A_n L_n^{(\alpha)}(x)}{\Gamma(\alpha + n + \beta)} = g(x), a < x < \infty \]  \hspace{1cm} (1.4)  

The dual series equations (1.3) and (1.4) are a special case of the dual series equations (1.1) and (1.2) when \( p = 0, b = 0 \), \( h = 1 \), \( a_i = b_j = 1 \), \( A_{ij} \) is replaced by \( A_{ij} \) and \( n_i \) is replaced by \( n \) for \( j = 1, 2, \ldots, s \) and \( i = 1, 2, \ldots, s \) and \( s = 0 \). Lowndes and Srivastava [4] have solved the triple series equations involving Laguerre polynomials. References for the solutions of dual and triple series equations involving Laguerre polynomials are given in Lowndes and Srivastava [4]. Connected to this work, references and solutions for dual series equations are given by Sneddon [7].  

The dual series equations (1.1) and (1.2) are new in the literature and have importance due to the closed-form solution. The analysis is purely formal and no justification had been given for the change of order of integrations and summation.  

2 Some Useful Results  
In this section, we will discuss some results which are useful in solving simultaneous dual series equations (1.1) and (1.2). The orthogonality relation for Laguerre polynomials is given by Erdelyi [2] from which we have  
\[ \int_{0}^{\infty} x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \frac{\Gamma(\alpha + n + 1)}{\Gamma(n + 1)} \delta_{nm}, \alpha > -1, \]  \hspace{1cm} (2.1)  

Where \( \delta_{nm} \) is the kronecker delta.  

We can easily find, with the help of integrals Erdelyi [2], that  
\[ \int_{0}^{\xi} x^\alpha (\xi - x)^{\beta + m - 2} L_n^{(\alpha)}(x) dx = \frac{\Gamma(\alpha + n + 1) \Gamma(\beta + m - 1)}{\Gamma(\alpha + \beta + m + n)} \xi^{\alpha + \beta + m - 1} L_n^{(\alpha + \beta + m - 1)}(\xi) \]  \hspace{1cm} (2.2)  

Where \( \alpha > -1, \beta + m > 1 \).  
\[ \int_{\xi}^{\infty} e^{-x} (x - \xi)^{\sigma - \alpha - \beta} L_n^{(\sigma)}(x) dx = \Gamma(\sigma - \alpha - \beta + 1)e^{-\xi} L_n^{(\alpha + \beta - 1)}(\xi) \]  \hspace{1cm} (2.3)  

Where \( \sigma + 1 > \alpha + \beta > 0 \).
From Erdelyi [1], we find that

\[
\frac{d^m}{dx^m} \left[ x^\alpha m L_n^{(\alpha m)}(x) \right] = \frac{\Gamma(\alpha + m + n + 1)}{\Gamma(\alpha + n + 1)} x^\alpha L_n^\alpha(x)
\]

(2.4)

3 Solution of Dual Series Equations

We assume that

\[
x + b = X \bar{\pi}, f \left( \frac{1}{X \bar{\pi} - b} \right) = f_{1i}(X), g \left( \frac{1}{X \bar{\pi} - b} \right) = g_{1i}(X),
\]

(3.1)

\[ b^h = c, \ (a + b)^h = d \]

then the simultaneous dual series equations (1.1) and (1.2) can be written in the following form:

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{S} a_{ij} \frac{A_{nj} L_{ni+p}^{\alpha}(X)}{\Gamma(\alpha + ni + p + 1)} = f_{1i}(X), c < X < d,
\]

(3.2)

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{S} b_{ij} \frac{A_{nj} L_{ni+p}^{\sigma}(X)}{\Gamma(\alpha + \beta + ni + p)} = g_{1i}(X), d < X < \infty,
\]

(3.3)

We assume that

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{S} a_{ij} \frac{A_{nj} L_{ni+p}^{\alpha}(X)}{\Gamma(\alpha + ni + p + 1)} = f_{2i}(X), 0 < X < c,
\]

(3.4)

Combining the series equations (3.4) and (3.2), we can write the simultaneous dual series equations (3.4) and (3.2) in the form

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{S} a_{ij} \frac{A_{nj} L_{ni+p}^{\alpha}(X)}{\Gamma(\alpha + ni + p + 1)} = F_i(X), 0 < X < d,
\]

(3.5)

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{S} b_{ij} \frac{A_{nj} L_{ni+p}^{\sigma}(X)}{\Gamma(\alpha + \beta + ni + p)} = g_{1i}(X), d < X < \infty,
\]

(3.6)

Where

\[ F_i(X) = \begin{cases} 
  f_{2i}(X), & 0 < X < c \\
  f_{1i}(X), & c < X < d 
\end{cases} \]

(3.7)

Multiplying (3.5) by \( X^{\alpha} (\xi - X)^{\beta m - 2} \), where \( m \) is a positive integer, integrating with respect to \( X \) over \((0, \xi)\) and interchanging the order of integrations, we find on using (2.2) that
\[ \sum_{n=0}^{\infty} \sum_{j=1}^{S} a_{ij} A_{n+1}^{(\alpha+\beta+m-1)}(\xi) \frac{\Gamma(\alpha+\beta+m+ni+p)}{\Gamma(\beta+m-ni+p)} = \frac{\xi^{-\alpha-\beta-m+1}}{\Gamma(\beta+m-1)} \int_{0}^{\xi} X^{\alpha} (\xi - X)^{\beta+m-2} F_i(X) dX, 0 < \xi < d \]

(3.8)

Where \( \alpha > -1, \beta + m > 1 \).

If we now multiply (3.8) by \( \xi^{\alpha+\beta+m-1} \) differentiate both sides \( m \) times with respect to \( \xi \), and use formula (2.4), we find that

\[ \sum_{n=0}^{\infty} \sum_{j=1}^{S} b_{ij} A_{nj} L_{ni+p}^{(\alpha+\beta-1)}(\xi) \frac{\Gamma(\alpha+\beta+n+ni+p)}{\Gamma(\beta+n-1)} \]

\[ = \sum_{j=1}^{S} c_{ij} \frac{\xi^{-\alpha-\beta-1}}{\Gamma(\beta+m-ni+p)} \frac{d^m}{d^m} \int_{0}^{\xi} X^{\alpha} (\xi - X)^{\beta+m-2} F_i(X) dX, 0 < \xi < d \]

(3.9)

where \( c_{ij} \) are the element of matrix \([b_{ij}]^{-1}\) and \( \alpha > -1, \beta + m > 1, 0 < \xi < d, i = 1, 2, 3, \ldots, s \).

Multiplying (3.6) by \( e^{-X} (X - \xi)^{\sigma-\alpha-\beta} \), integrating with respect to \( X \) over by \((\xi, \infty)\), and interchanging the order of integrations, we find by using formula (2.3) that

\[ \sum_{n=0}^{\infty} \sum_{j=1}^{S} b_{ij} A_{nj} L_{ni+p}^{(\alpha+\beta-1)}(\xi) \frac{\Gamma(\alpha+\beta+n+ni+p)}{\Gamma(\beta+n-1)} \]

\[ = \frac{e^{\xi}}{\Gamma(\sigma-\alpha-\beta+1)} \int_{\xi}^{\infty} e^{-X} (X - \xi)^{\sigma-\alpha-\beta} g_{1i}(X) dX, d < \xi < \infty \]

(3.10)

Where \( \sigma + 1 > \alpha + \beta > 0 \) and \( i = 1, 2, 3, \ldots, s \).

The left-hand sides of (3.9) and (3.10) are now identical. Making use of the orthogonality relation (2.1), we find from (3.9) (3.10) that

\[ A_{nj} = \sum_{i=1}^{S} d_{ij} \left[ \sum_{j=1}^{S} c_{ij} \frac{\Gamma(ni+p+1)}{\Gamma(\beta+m-1)} \int_{0}^{d} e^{-\xi} L_{ni+p}^{(\alpha+\beta-1)}(\xi) F_i(\xi) d\xi \right. \]

\[ + \left. \frac{\Gamma(ni+p+1)}{\Gamma(\sigma-\alpha-\beta+1)} \int_{d}^{\infty} \xi^{\alpha+\beta-1} L_{ni+p}^{(\alpha+\beta-1)}(\xi) G_i(\xi) d\xi \right] \]

(3.11)

Where \( n = 0, 1, 2, \ldots \) and where \( d_{ij} \) are the elements of the matrix \([b_{ij}]^{-1}\) and
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\[ F_{1i}(\xi) = \frac{d^m}{d\xi^m} \int_0^{\xi} X^\alpha (\xi - X)^{\beta + m - 2} F_i(X)\,dX \]  

(3.12)

\[ G_i(\xi) = \int_{\xi}^{\infty} e^{-X}(X - \xi)^{\sigma - \alpha - \beta} g_{1i}(X)\,dX \]  

(3.13)

Provided that \( \alpha + \beta + 1 > 1 - m \) and \( \sigma + 1 > \alpha + \beta > 0 \), \( m \) being a positive integer with the help of (3.7), (3.12) can be written in the form

\[ F_{1i}(\xi) = \frac{d^m}{d\xi^m} \left[ \int_0^{c} X^\alpha (\xi - X)^{\beta + m - 2} f_{2i}(X)\,dX \right. \\
+ \left. \int_{c}^{\xi} x^\alpha (\xi - x)^{\beta + m - 2} f_{1i}(x)\,dx \right], \quad c < \xi \]  

(3.14)

References
