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Complete Controllability of Automobile Suspension System

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Abstract
The objective of this paper is to study the controllability of automobile suspension. The suspension system dynamics was captured using a mathematical model. The state-space representation was then subjected to controllability test using MATLAB commands. The result of the test shows that the suspension system was completely controllable and a control function was then found to bring the initially displaced suspension system to equilibrium position after some given time.

Keywords: Complete controllability, Automobile Suspension and Linear system.

Introduction

Controllability is one of the fundamental concepts in the mathematical control theory and widely used in many fields of science and technology. Controllability of linear and non-linear systems in finite dimensional space are represented by ordinary differential equation. The controllability property plays an important role in nearly all control problems, such as stabilization of unstable systems by feedback, or optimal control. The concept of controllability implies the ability to move a system around in its entire configuration space using some control effort. The exact definition varies slightly within the framework or the type of models applied.The primary responsibility of the control system engineers is to design and implement controller.

The concept of controllability for finite dimensional deterministic linear control systems was introduced by Kalman [9]. The basic concepts of control theory in finite dimensional space have been discussed in [5], [7] and [10]. Anurag Shukla et al. studied the approximate and complete controllability of semi-linear delay control systems using fixed point theory in [2]. For stochastic systems Klamka and Locha [6] has given some remarks on stochastic controllability. The work of [6] has been extended by Anurag Shukla et al. in [2].

Karnopp and Heess [4] examine some fundamental aspects of controllable suspensions and discussed the kind of devices which can be used in suspensions. O.I. Ignatius and Oviawe C.I. [8] focused on the controllability and observability of an active suspension system used in automobiles. The system transfer function model was determined by using the road disturbance as input and the car response as output.

However in best of our knowledge there is no result on simultaneous study of controllable and uncontrollable systems for automobile suspension system. So it is interesting to see for which control the system will be completely controllable. The present paper is dedicated to study of complete controllability of automobile suspension system.

Preliminaries

Let us consider a time varying controlled system S_1 which is defined as

$$\bar{x}(t) = A(t)x(t) + B(t)u(t) \qquad (2.1)$$

subject to initial condition $x(t_0) = x_0$. Then its solution will be given as

$$x(t) = \phi(t,t_0)[x_0 + \int_{to}^{t} \phi(t_0,\tau)B(t)u(\tau)d\tau] \qquad (2.2)$$

where $\phi(t,t_0)$ is the state transition matrix. Here we can generalize the state transition matrix for this systems by writing

$$\phi(t,t_0) = X(t)X^{-1}(t_0) \qquad (2.3)$$

Where X(t) is the fundamental matrix of matrix A(t).

- For the uncontrolled case the matrix B(t) = 0, so the solution then will be,

$$x(t) = \phi(t,t_0)x_0 \qquad (2.4)$$

- For time invariant controlled systems, the solution is given by

$$x(t) = \exp(At)\,[x_0 + \int_0^t \exp\,(A\tau)Bu(\tau)d\tau] \qquad (2.5)$$

Controllability

The concept of controllability implies the ability to move a system around in its entire configuration space using some control effort.

For the system S_1 defined in 2.1, where A is $n \times n$ matrix, B is a $n \times m$ matrix, is said to be "completely controllable if for any t_0, any initial state $x(t_0) = x_0$ and any given final state x_f there exists a finite time $t_f > t_0$, and a control $u(t)$, such that $x(t_f) = x_f$."

For the time invariant systems, if initial time t_0 in the controllability definition is set equal to zero, then a general algebraic criterion can be derived. Let us take a constant system

$$\overline{x} = Ax + Bu \qquad (2.6)$$

is completely controllable if and only if the $n \times nm$ controllability matrix

$$C = [B, AB, A^2B, \ldots\ldots A^{n-1}B] \qquad (2.7)$$

has rank n.

Theorem 2.1: "*The system S_1 is completely controllable if and only if the n × n symmetric controllability matrix*

$$M(t_0, t_f) = \int_{t_0}^{t_f} \phi(t_0,\tau)B(\tau)B^T(\tau)\phi^T(t_0,\tau)d\tau \qquad (2.8)$$

where ϕ is the state transition matrix of system S_1. The system is said to be controllable if this matrix M is positive definite and non-singular. In this case the control $u(t)$ is given by

$$u(t) = -B^T\phi^T(t_0,t_f)M^{-1}(t_0,t_f)[x_0 - \phi(t_0,t_f)x_f] \qquad (2.9)$$

which is defined on $t_0 \le t \le t_f$, transfers $x(t_0) = x_0$ to $x(t_f) = x_f$".

Problem Formulation

A platform is supported by springs and dampers as shown, it being assumed that the forces they produce act at the end points P and Q, and that x_1 and x_2 are the displacement of these points from equilibrium

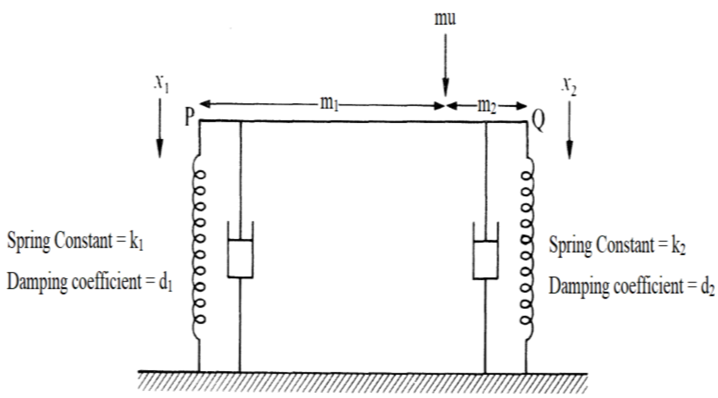

Fig. 1

This could be thought of as a simple representation of an automobile suspension system. The forces exerted by the dampers are proportional to velocity and the springs obey Hooke's law. Assuming that the mass of the platform can be neglected so that the spring motions can be regarded as independent. If a control force mu (t) is applied one quarter the way along from one end, then the system equation

1. Without Control:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k_1/d_1 & 0 \\ 0 & -k_2/d_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad (3.1)$$

2. With Control:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -k_1/d_1 & 0 \\ 0 & -k_2/d_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} m_2/d_1 \\ m_1/d_2 \end{bmatrix}u(t) \qquad (3.2)$$

Verify that the system is completely controllable. If the ends of the platform are each given an initial displacement of x_0 units, find a control function u(t) which returns the system to equilibrium at $t = t_f$.

Results

For a real life problem, considering:
$k_1 = 1, k_2 = 2, d_1 = 1, d_2 = 1, m = 4,\ m_1 = 3, m_2 = 1,$
$x_0 = 10, t_0 = 0$ and $t_f = 1$.

1. **Without control:** Solution of system (3.1) is given as

$$x_1 = 10e^{-t} \qquad (4.1.1)$$

$$x_2 = 10e^{-2t} \qquad (4.1.2)$$

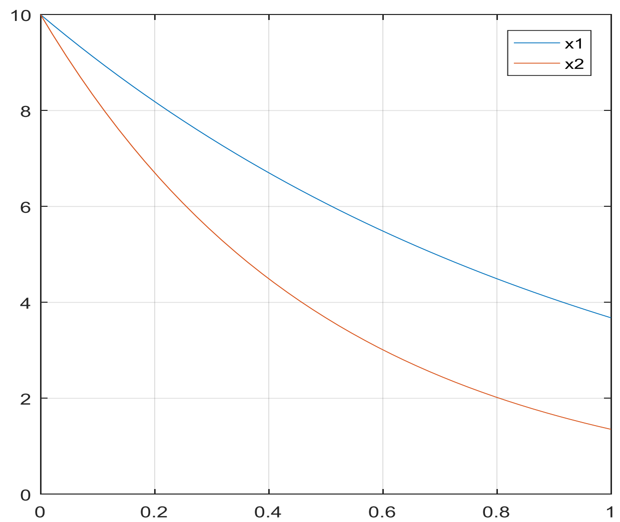

Fig. 2

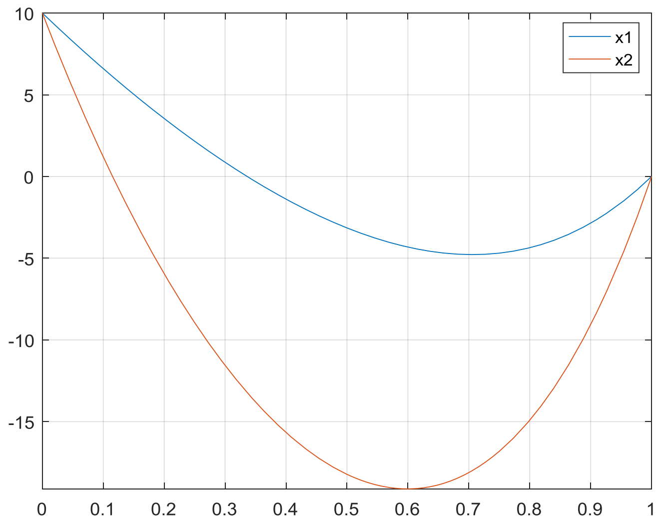

Fig. 3

2. With Control

For the system (3.2)

- Controllability matrix,

$$C = \begin{bmatrix} 1 & -1 \\ 3 & -6 \end{bmatrix}$$

- Rank of controllability matrix = 2
- Control function,

$$u(t) = \frac{100e^{t} - 120e^{2t} + 100e^{t+1} + 100e^{t+2} + 180e^{t+3} - 120e^{2t+1} - 240e^{2t+2}}{e - 8e^2 + 8e^3 - e^4 - e^5 + 1} \quad (4.2.1)$$

- System displacement,

$$x_1(t) = -e^{t}\left(\frac{50e^2 + 10e^3 + 10e^4 + 10e^5}{e - 8e^2 + 8e^3 - e^4 - e^5 + 1} - \frac{e^{2t}(50e + 50e^2 + 90e^3 + 50) - e^{3t}(40e + 80e^2 + 40)}{e - 8e^2 + 8e^3 - e^4 - e^5 + 1}\right) \quad (4.2.2)$$

$$x_2(t) = -e^{2t}\left(\frac{100e^3 + 10e^4 + 10e^5}{e - 8e^2 + 8e^3 - e^4 - e^5 + 1} - \frac{e^{3t}(100e + 100e^2 + 180e^3 + 100) - e^{4t}(90e + 180e^2 + 90)}{e - 8e^2 + 8e^3 - e^4 - e^5 + 1}\right) \quad (4.2.3)$$

Result Analysis

In figure 2 we can see that without using control our state space is not achieving equilibrium state for any finite time t. But for controlled system state space is achieving equilibrium (Figure 3) for the obtained control function u(t) (defined in 2.9) given in 4.2.1. So for this control u(t) our control system is completely controllable.

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Supplementary Data

Matlab programs for finding the displacement without and with control applied to the state space model.

1. Without control:

```
a = [-1 0 ; 0 -2];
b = [0 ; 0];
c = [0 0 ; 0 0];
d= [0 ; 0];
X0 = [10 ; 10];
t = sym('t', 'real');
phi = expm(a*t)
x = phi*X0
fplot( x, [0,1]);
grid on
legend('x1','x2')
```

2. With control:

```
a=[ -1 0; 0 -2];
b=[1;3];
c=[0 0];
d=0;
G= ss(a,b,c,d)          % state space model of given problem%
C= ctrb(a,b)
rank(C)
X0= [10;10];             %initial conditions%
t=sym('t','real');
y=rank(ctrb(a,b))        %calculating rank of the controllability matrix%
phi = expm(-a*t);
v= phi*b*b'*phi';
U= int(v,t,0,1)           %controllability grammian%
z= -1*b'*phi'*inv(U)*X0 ; %control function calculation%
u=simplify(z)
syms x1(t) x2(t);
Y = [x1;x2];
odes= diff(Y)== a*Y + b*u
C= Y(0)==[10;10];
[x1sol(t), x2sol(t)]=dsolve(odes,C)
fplot(x1sol, [0,1])
hold on
fplot(x2sol,[0,1])
grid on
legend('x1','x2','Location','best')
```

About the Authors

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Arvind Kumar is pursuing B.Tech in Electrical Engineering from Rajkiya Engineering College, Kannauj. At present, he is in his final year. His area of interest is controllability, observability and stability of dynamical control systems.

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