Improved Lumped Model for Transient Heat Conduction in a Hollow Sphere for different Heat Conduction Problems

Saroj Kumar Pradhan
Student, Department of Mechanical Engineering,
Shibani Institute of Technical Education, Bhubaneshwar, Khorda, Odisha, India.

Soumya Narayan Behuria
M.tech Scholar, Department of Mechanical Engineering,
Centre for Advanced Post Graduate Studies, BPUT, Rourkela, Odisha, India.

Shuprasad Mohanty, Subhashree Suchismita
Assistant Professor, Department of Mechanical Engineering,
Shibani Institute of Technical Education, Bhubaneshwar, Khorda, Odisha, India.

Abstract
Improved lumped models are developed for the transient heat conduction of a hollow sphere by using polynomial approximation method for two different heat conduction problems. In first problem the inner and the outer walls of the hollow sphere is being subjected to different convective heat transfer coefficient without internal heat generation, where as in the second problem the internal heat generation is taken into consideration. Here, first the governing equations and the boundary conditions are non-dimensionalised. Then, the non-dimensional equation is solved by employing polynomial approximation method. The transient temperature is obtained for different heat conduction problems. The results obtained are plotted in form of graph and analyzed.

Keywords: Lumped model, Improved lumped model, non-dimensional, polynomial approximation method.

Introduction
The Classical lumped model restricts the problem with Biot number less than 0.1. As in many engineering application as the analysis of thermo hydraulic nuclear reactor, boiling water reactor involve higher Biot number so classical lumped model is not valid[10]. To counteract this limitation improved lumped models are developed by using different analytical methods as perturbation method[3,4], two point Hermite method[1,2,5], polynomial approximation method[6,10].

In one of the earliest work on this lumped model analysis, Clarissa R. Regis et al.[1] used hermite approximation method on a nuclear fuel rod to develop a lumped model. The lumped model approach was used for determining the average temperature and heat flux in radial direction. Jian Su[2] developed an improved lumped model for unsteady cooling of a long slab by two point Hermite integral method which worked well with higher values of Biot numbers. H. Sadat[3] applied perturbation method to develop a lumped model of unsteady one-dimensional heat conduction problem. And the center, surface and average temperature for different geometries was also obtained for different Biot numbers. Shijun Liao et al.[5] solved a nonlinear model of combined convective and radiative cooling of a spherical body using homotopy analysis method. This series solution agreed well with the numerical solution.

P Keshavraj and M Taheri[6] developed an improved lumped model by employing Polynomial Approximation method and it was found that the improved model was able to calculate average temperature for more higher Biot numbers than that was obtained by finite difference method. An improved lumped model was developed by Devanshu Prasad[7] by utilizing polynomial approximation method with a number of approximate temperature profiles on slab and cylinder under different conditions. On the basis of the analysis a modified Biot number was obtained. Noorul Haque and Amitesh Paul[8] studied the improved lumped parameter in transient heat conduction. Zheng Tan et al.[9] developed an improved model by using two point hermite approximations for integrals for combined convective and radiative cooling wall. Jian su et al.[11] developed improved lumped models for transient combined convective and radiative cooling of multilayer spherical media. Two point Hermite approximation methods is used to obtain the average temperature and heat flux in each layer. The plain trapezoidal rule was employed in all layers, except for the innermost layer where the second-order two sided corrected trapezoidal rule is used to obtain average temperatures. S.Mohanty and B.Dalai [12] developed an improved lumped model for transient heat conditions in a sphere with different boundary conditions and different assumed temperature profiles using polynomial approximation method. A modified lumped model for transient heat conduction in a hollow sphere where the boundary condition for the inner side and outer side of the hollow sphere were fixed heat flux and convective heat transfer with the ambient respectively, was developed S.Mohanty and B.Dalai[13] by polynomial approximation method.

Contribution
In this paper, an improved lumped model is developed for a hollow sphere by using polynomial approximation method for two different heat conduction problems. In first case the inner side and outer side of the hollow sphere is exposed to different convective heat transfer coefficients without taking internal heat generation into consideration, whereas in second case internal heat generation is taken into consideration.
Table 1: Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h_1$</td>
<td>Heat transfer coefficient at the inner side</td>
<td></td>
</tr>
<tr>
<td>$h_2$</td>
<td>Heat transfer coefficient at the inner side</td>
<td></td>
</tr>
<tr>
<td>$T$</td>
<td>Object temperature</td>
<td></td>
</tr>
<tr>
<td>$T_\infty$</td>
<td>Ambient temperature</td>
<td></td>
</tr>
<tr>
<td>$T_0$</td>
<td>Initial temperature</td>
<td></td>
</tr>
<tr>
<td>$k$</td>
<td>Thermal conductivity</td>
<td></td>
</tr>
<tr>
<td>$Q$</td>
<td>Internal heat generation</td>
<td></td>
</tr>
<tr>
<td>$r_1$</td>
<td>Inner radius of hollow sphere</td>
<td></td>
</tr>
<tr>
<td>$r_2$</td>
<td>Outer radius of hollow sphere</td>
<td></td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Constant</td>
<td></td>
</tr>
<tr>
<td>$R$</td>
<td>Length coordinates</td>
<td></td>
</tr>
</tbody>
</table>

Mathematical Formulation

Considering a hollow sphere of inner and outer radii $r_1$ and $r_2$ been subjected to convective heat transfer $h_1$ and $h_2$ respectively as shown in figure 1.

![Figure 1: A hollow sphere with inner and outer radius $r_1$ and $r_2$ respectively without heat generation.](image)

![Figure 2: A hollow sphere with inner and outer radius $r_1$ and $r_2$ respectively with heat generation 'q'.](image)

Analysis of temperature variation with time for a hollow sphere

The governing one dimensional heat conduction equation for a solid sphere of radius $r$ without internal heat generation with temperature $T(x,t)$ and thermal diffusivity $\alpha$ is given by:

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial T}{\partial r} \right)$$  \hspace{1cm} (1)

The required boundary conditions for the hollow sphere are:

$$k \frac{\partial T}{\partial r} = h_1(T - T_\infty) \text{ at } r = r_1$$  \hspace{1cm} (2)

$$-k \frac{\partial T}{\partial r} = h_2(T - T_\infty) \text{ at } r = r_2$$  \hspace{1cm} (3)

Dimensionless parameters are:

$$\theta = \frac{T - T_\infty}{T_0 - T_\infty}, Bi = \frac{h_i r_i}{k}, \epsilon = \frac{at}{R^2}, x_i = \frac{r_i}{r_2}$$  \hspace{1cm} (4)

Where $\theta$ is the dimensionless temperature, $B$ the Biot number, $\epsilon$ the dimensionless time, $x_i$ the dimensionless length, and $i$ indicates for 1 and 2.

The dimensionless governing equation and boundary conditions are:

$$\frac{\partial \theta}{\partial \tau} = \frac{1}{x^2} \frac{\partial}{\partial x} \left(x^2 \frac{\partial \theta}{\partial x} \right)$$  \hspace{1cm} (5)

$$\frac{\partial \theta}{\partial x} = B_1 \theta \text{ at } x = x_1 = \epsilon$$  \hspace{1cm} (6)

Where $\epsilon = \frac{r_1}{r_2}$

$$\frac{\partial \theta}{\partial x} = -B_2 \theta \text{ at } x = x_2 = 1$$  \hspace{1cm} (7)

For the solution of Eq.(5) by polynomial approximation method, let the assumed temperature profile be:

$$\theta = a_0 + a_1 x + a_2 x^2$$  \hspace{1cm} (8)

On differentiating the above equation

$$\frac{\partial \theta}{\partial x} = a_1 + 2a_2 x$$  \hspace{1cm} (9)

On applying first and second boundary conditions, values of $a_1$ and $a_2$ are obtained as:

$$a_1 = \frac{-B_2 \theta \epsilon - B_1 \theta}{\epsilon - 1}$$  \hspace{1cm} (10)

$$a_2 = \frac{B_1 \theta + B_2 \theta}{2(\epsilon - 1)}$$  \hspace{1cm} (11)

Utilizing the equation (8), (9), (10) and (11) in equation (7), it can be written as:

$$a_0 = \theta \left(1 + \frac{B_2(2\epsilon - 1)}{2(\epsilon - 1)} + \frac{B_1}{2(\epsilon - 1)} \right)$$  \hspace{1cm} (12)

The average temperature ($\bar{\theta}$) of the sphere is given by

$$\bar{\theta} = \frac{3}{x_2^2 - x_1^2} \int_{x_1}^{x_2} x^2 \theta dx$$  \hspace{1cm} (13)

$$\bar{\theta} = \frac{3}{1 - \epsilon^3} \int_{\epsilon}^{1} x^2(a_0 + a_1 x + a_2 x^2) dx$$  \hspace{1cm} (14)

$$\bar{\theta} = \theta [1 + B_1(L_2 - L_3 + L_5) - B_2(L_1 - L_4 + L_3)]$$  \hspace{1cm} (15)
Where \( L_1 = \frac{2\varepsilon - 1}{2(\varepsilon - 1)} \), \( L_2 = \frac{1}{2(\varepsilon - 1)} \), \( L_3 = \frac{3(1 - \varepsilon^2)}{4(\varepsilon - 1)(1 - \varepsilon^3)} \), \( L_4 = \frac{3(1 - \varepsilon^2)}{4(\varepsilon - 1)(1 - \varepsilon^3)} \), \( L_5 = \frac{3(1 - \varepsilon^2)}{10(\varepsilon - 1)(1 - \varepsilon^3)} \).

On differentiating Eq. (15),
\[
\frac{\partial \overline{\theta}}{\partial \tau} \quad\frac{\partial \theta}{\partial x} = \theta[1 + B(L_1 - L_4 + L_5) - Q(L_2 - L_3 + L_5)]
\]
Integrating equation (5) with respect to \( x \)
\[
\int_{\varepsilon}^{1} x^2 \frac{\partial \theta}{\partial x} \, dx = \int_{\varepsilon}^{1} \frac{1}{\varepsilon} \frac{\partial}{\partial x} \left( x^2 \frac{\partial \theta}{\partial x} \right) \, dx
\]
On solving further, it is obtained as
\[
\frac{\partial \overline{\theta}}{\partial \tau} = B_1 \theta(M_3 - M_4) + B_2 \theta(M_2 - M_3)
\]
Where \( M_1 = \frac{3(1 - \varepsilon^2)}{3(1 - \varepsilon^3)}, M_2 = \frac{3(1 - \varepsilon^2)}{3(1 - \varepsilon^3)}, M_3 = \frac{3}{\varepsilon - 1} \) and

From Eq. (16), it can be written as
\[
\frac{\partial \theta}{\partial \tau} = \frac{-\theta[B_1(M_1 - M_3) + B_2(M_2 - M_3)]}{1 + B_1(L_2 - L_3 + L_5) + B_2(L_1 - L_4 + L_5)}
\]
Integrating the above equation, it is obtained:
\[
\theta = e^{-\varepsilon X}
\]
Where \( X = \frac{[B_1(M_1 - M_3) + B_2(M_2 - M_3)]}{1 + B_1(L_2 - L_3 + L_5) + B_2(L_1 - L_4 + L_5)} \).

Equation (20) represents the non-dimensional temperature distribution along the radius of the hollow sphere with variation of non-dimensional time \( \tau \) without internal heat generation.

Similarly, lumped analysis was carried out for a hollow sphere with inner and outer radius been subjected to \( h_1 \) and \( h_2 \) respectively and taking internal heat generation into consideration as shown in figure 2. The non dimensional temperature for the same was found to be
\[
\theta = \frac{e^{-\varepsilon X} + Y}{X}
\]
Where,
\[
X = \frac{[B_1(M_1 - M_3) + B_2(M_2 - M_3)]}{1 + B_1(L_2 - L_3 + L_5) + B_2(L_1 - L_4 + L_5)}
\]
\[
Y = \frac{G}{1 + B_1(L_2 - L_3 + L_5) + B_2(L_1 - L_4 + L_5)}
\]

**Discussion**
The non dimensional temperature \( \theta \) is calculated with respect to \( \tau \) using equation (20) at different \( B_1 \) and \( B_2 \) with internal radius and external radius 0.5 and 1 respectively without taking heat generation into consideration, and is plotted as shown in figure 3. It is observed that with increase in non dimensional time, the non dimensional temperature decreases. It is also observed that on keeping a biot number constant and increasing the other, the non dimensional temperature decreases at a particular instant of time.

Figure 3: Dimensionless temperature Vs Dimensionless time for a hollow sphere at different \( B_1 \) and \( B_2 \) without heat generation.

Figure 4: Dimensionless temperature Vs Dimensionless time for a hollow sphere at different \( B_1 \) and \( B_2 \) with heat generation, here \( G=1 \).

Figure 5: Dimensionless temperature Vs Dimensionless time for a hollow sphere with different values of \( G \) at \( B_1=1 \) and \( B_2 =5 \).

Figure 4 shows the variation of dimensionless temperature with respect to dimensionless time with heat generation. From the figure it is found that the dimensionless temperature
The present approach can be further utilized to determine the temperature variation with respect to time for a region. The temperature drop decreases with respect to time up to certain instant of time. The method is used to find the temperature distribution with respect to time, whereas for the lower values of G the non-dimensional temperature changes gradually with time.

Two particular cases can be identified. If \( r_1 = B_1 = 0 \) in equation (20) and equation (21), the problem reduces to the modified lumped model of a solid sphere subjected to natural convection cooling and that of solid sphere with heat generation respectively. This case has been discussed by S.Mohanty and B.Dalai [11]. Figure 6 shows the variation of dimensionless temperature to dimensionless time for a solid sphere without heat generation and it is found that the result obtained matches very well with that of S.Mohanty and B.Dalai [11].

Figure 6: Comparison of result of Dimensionless Temperature Vs Dimensionless time for a solid sphere obtained from [11] and the result obtained by considering \( r_1=B_1=0 \) in equation (20).

**Conclusion**

An Improved lumped model is developed for a hollow sphere which works well for higher values of Biot number. A temperature versus time relationship for a hollow sphere for two boundary conditions are developed using polynomial approximation method. In first case both the inner and outer side of the hollow sphere is supplied with different heat transfer coefficient \( h_1 \) and \( h_2 \) respectively without internal heat generation, where as in the second case internal heat generation is taken into account. Polynomial approximation method is used to find the temperature distribution with effect of time, Biot number and heat source parameter. The temperature drop decreases with respect to time up to certain region. The present approach can be further utilized to determine the temperature variation with respect to time for a multi layer hollow sphere.

**References**


