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# On the L(2,1) Labeling Number of Some Trees

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***Abstract***
An $L(2,1)$ labeling (or) distance two labeling of a graph $G$ is a function $f$ from the vertex set $V(G)$ to the set of all non-negative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y)=2$.The $L(2, 1)$ labeling number $\lambda(G)$ of $G$ is the smallest number $k$ such that $G$ has an $L(2, 1)$ labeling with max{$f(v)$, $v$belongs to $V(G)$} = $k$. In this paper we determine the $L(2, 1)$ labeling number $\lambda(G)$ for some trees like B(r,s,t-1), B(s,t) and <$K_{1,t}$:2>.

**Keywords**: *L*(2, 1) labeling, L(2,1) labeling number, tree, bistar graph.
**AMS Subject Classification:** *O5C78.*

## 1. INTRODUCTION

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication networks and data base management and models for constraint programming over finite domain. The concept of graph labeling was introduced by Rosa in 1967[11]. Hence in the intervening years various labeling of graphs such as graceful labeling, harmonious labeling, magic labeling, antimagic labeling, bimagic labeling, prime labeling, cordial labeling, total cordial labeling, *k*-graceful labeling and odd graceful labeling etc., have been studied in over 2100 papers. Hale [7] introduced the graph theory model of assignment of channels in 1980. A channel assignment problem was designed in such a way that the vertices of distance two are considered to be close and vertices which are adjacent, are considered to be very close which paved way for distance two labeling of graphs. Labeling with a condition of distance two was introduced by J.R. Griggs and R.K. Yeh [6] who proved that every graph with maximum degree $k$ has an $L(2,1)$-labeling with span at most $k^2+2k$ and proved the conjecture for 2-regular graphs. G.J. Chang and D. Kuo [3] improved this upper bound to $k^2+k$. Chang et al. [4] generalized this to obtain $k^2+(d-1)k$ as an upper bound on the minimum span of an $L(d,1)$-labeling. R. Ponrajet al. [10] determined the difference cordial labeling of triangular and quadrilateral snake graphs. S.S. Sandhya et al. [13] determined the Heronian mean labeling of triple triangular and triple quadrilateral snake graphs.

## 2. PRELIMINARIES

In this section we give the basic notations relevant to this paper. In this paper, the graphs considered are all finite, undirected and simple. $V(G)$ and $E(G)$ denote the vertex set and the edge set of $G$.

**Definition 2.1.**
An L(2,1) labeling (or) distance two labeling of a graph G is a function f from the vertex set V(G) to the set of all non-negative integers such that $|f(x) - f(y)| \geq 2$ if $d(x,y)=1$ and $|f(x) - f(y)| \geq 1$ if $d(x,y) =2$. The L(2, 1) labeling number $\lambda(G)$ of G is the smallest number k such that G has an L(2, 1) labeling with max{f(v), $v \in V(G)$} = k.

**Definition 2.2.**

The graph $B(r,s,t-1)$ is obtained by attaching the stars $K_{1,r}$and$K_{1,s}$ with the pendent vertices of a path on $t$ vertices. The graph B (7, 6, 7) is shown in figure 2.8.

**Definition 2.3.**

The bistar graph $B(s,t)$ is obtained by attaching the stars $K_{1,s}$ and$K_{1,t}$ by an edge. The bistar graph $B(6,5)$ is shown in figure .

**Definition 2.4.**

The tree <$K_{1,t}$:2>is obtained from the bistar $B_{t,t}$ by subdividing the edge joining the two stars. The tree <$K_{1,7}$:2>is shown in figure 2.10.

## 3. MAIN RESULTS

**Theorem 3.1.**

The L(2,1) labeling number λ(G) of bistar graph $B_{s,t}$ is max(s,t)+3, $s \geq 3, t \geq 3$

**Proof.**

Let the vertices of a bistar graph $B_{s,t}$ be $V(G) = V_1 \cup V_2 \cup V_3$

$V_1 = \{c_k / 1 \leq k \leq s\}$

$V_2 = \{d_m / 1 \leq m \leq t\}$

$V_3 = \{z_j / 1 \leq j \leq 2\}$

Where $c_k$ are the vertices on the cycle on one side of the bistar (the head part) and $d_m$ are the vertices on the other end of the bistar (the tail part) and $z_j$ be the vertices on the path joining the head and the tail part of the bistar graph with degree of $z_1$ is $s+1$ and degree of $z_2$ is $t+1$.

Define a mapping ψ: V (G) → N ∪ {0} by

$$\psi(c_k) = k-1; \quad 1 \leq k \leq s$$

$$\psi(d_m) = m-1; \quad 1 \leq m \leq t$$

$$\psi(z_1) = \begin{cases} s+1, s \geq t \\ t+3, s < t \end{cases}$$

$$\psi(z_2) = \begin{cases} t+1, s < t \\ s+3, s \geq t \end{cases}$$

**Case (i):**

Let a,b be any two vertices in $V_1$. Clearly a,b are non - adjacent.

Hence d(a,b) ≥ 2. Let x = $c_k$ and b= $c_{k+1}$, $1 \leq k \leq s$ then ψ(a) = k-1, ψ(b) = k. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |k-1-k| ≥ 3.

**Case (ii):**

Let a,b be any two vertices in $V_2$. Clearly a,b are non - adjacent.

Hence d(a,b) ≥ 2. Let a = $c_m$ and b= $c_{m+1}$. Then ψ(a) = m−1, ψ(b) = m. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |m−1−m| ≥ 3.

**Case (iii):**

Let a,b be any two vertices in $V_3$.

Clearly both the vertices are adjacent. Hence d(a,b) = 1.

Let a = $z_1$ and b= $z_2$, then ψ(a) = s+1, ψ(b) = s+3. $s \geq t$

Therefore d(a,b) + |ψ(a) −ψ(b)| = 1 +|s+1−(s+3)| = 1 + |−2 | ≥ 3.

**Case (iv):**

Let a,b be any two vertices in $V_1$ and $V_2$ respectively.

Clearly for all the vertices a and b, d(a,b) ≥ 3.

**Case (v):**

Let a,b be any two vertices in $V_1$ and $V_3$ respectively.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_1$ and $V_3$ respectively

Hence d(a,b) =1. Let a = $c_k$ and b= $z_1$ then ψ(a) = k−1, 1 ≤ k ≤ s, ψ(b) = s+1, $s \geq t$ and $s \geq 3, t \geq 3$. Therefore d(a,b) + |ψ(a) −ψ(b)| =1+|k−1−(s+1)| ≥ 3.

**Subcase (ii):**

Let a,b be any two non -adjacent vertices in $V_1$ and $V_3$ respectively

Hence d(a,b) ≥ 2. Let a = $c_k$ and b= $z_2$ .Then ψ(a) = k−1, 1 ≤ k ≤ s, ψ(b) = s+3, $s \geq t$ and $s \geq 3, t \geq 3$. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |k−1−(s+3)| ≥ 3.

**Case (vi):**

Let a,b be any two vertices in $V_2$ and $V_3$ respectively.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_2$ and $V_3$ respectively

Hence d(a,b) =1. Let a = $z_2$ and b= $d_m$. Then ψ(a) =s+3, $s \geq t$, $1 \leq k \leq s$, where $s \geq 3, t \geq 3$, ψ(b) = m−1, $1 \leq m \leq t$. Therefore d(a,b) + |ψ(a) −ψ(b)| =1+|s+3−(m−1)|≥ 3.

**Subcase (ii):**

Let a,b be any two non-adjacent vertices in $V_2$ and $V_3$ respectively

Hence d(a,b) ≥ 2. Let a = $z_1$ and b= $d_m$ . Then ψ(a) = s+1, 1 ≤ m ≤ t, ψ(b) = m-1, $s \geq t$ and $s \geq 3, t \geq 3$. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |s+1−(m−1)| ≥ 3.

Similarly for all the other possibilities of a and b, d(a,b) + |ψ(a) −ψ(b)| ≥ 3. Therefore the L(2,1) labeling number λ(G) of a bistar graph $B_{s,t}$ is max(s,t)+3, $s \geq 3, t \geq 3$.

**Example 2.5.2.**

L(2,1) labeling of a bistar B(6,5) is shown in Figure 3.1.

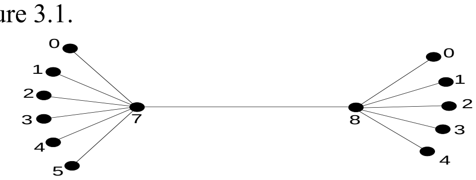


Figure 3.1. L(2,1) labeling of a bistar B(6,5)

**Theorem 3.2.**

The L(2,1) labeling number λ(G) of the tree <$K_{1,t}$: 2>is t+5, $t \geq 3$

**Proof.**

Let the vertices of the tree <$K_{1,t}$: 2> be

$V(G) = V_1 \cup V_2 \cup V_3$

$V_1$ = {$c_k$ / 1 ≤k≤t}

$V_2$ = {$d_r$ / 1 ≤r≤t}

$V_3$ = {$e_s$ / 1 ≤s≤ 3}

where $c_k$ are the vertices on the cycle on one side of the bistar (the head part) and $d_r$are the vertices on the other end of the bistar (the tail part) and $e_s$are the vertices on the path joining the head and the tail part of the bistar graph such that the degree of $e_1$and $e_2$ is (*t*+ 1).

Define a mappingψ: V (G) → N ∪ {0}by

$\psi(c_k) = k - 1; \quad 1 \leq k \leq t$

$\psi(d_r) = r - 1; \quad 1 \leq r \leq t$

$\psi(e_s) = \{t + 2s - 1, 1 \leq s \leq 3\}$

**Case (i):**

Let a,b be any two vertices in $V_1$.
Clearly a,b are non - adjacent. Hence d(a,b) ≥ 2. Let a = $c_k$ and b = $c_{k+1}$, $1 \leq k \leq t$ ,$t \geq 3$ thenψ(a) = k-1, ψ(b) = k. Therefore d(a, b) + |ψ(a) –ψ(b)| ≥ 2 + |k-1-k| ≥ 3.

**Case (ii):**

Let a,b be any twovertices in $V_2$. Clearly a,b are non - adjacent.
Hence d(a,b) ≥ 2. Let a = $d_r$ and b = $d_{r+1}$. Thenψ(a) = r-1, $1 \leq r \leq t$ ,$t \geq 3$ ψ(b) = r. Therefore d(a,b) + |ψ(a) –ψ(b)| ≥ 2 + |r-1-r| ≥ 3.

**Case (iii):**

Let a,b be any two vertices in $V_3$.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_3$.
Hence d(a,b) = 1. Let a = $e_1$ and b= $e_2$, thenψ(a) = t+1, ψ(b) = t+3 ,$t \geq 3$
Therefore d(a,b) + |ψ(a) –ψ(b)| = 1 +|t+1-(t+3)| = 1 + |-2 | ≥ 3.

**Subcase (ii):**

Let a,b be any two non -adjacent vertices in $V_3$.
Hence d(x, y) ≥ 2. Let a = $e_1$ and b= $e_3$, thenψ(a) = t+1, ψ(b) = t+5 ,$t \geq 3$
Therefore d(a,b) + |ψ(a) –ψ(b)| ≥ 2 +|t+1-(t+5)| ≥ 2 + |-4 | ≥ 3.

**Case (iv):**

Let a,b be any two vertices in $V_1$ and $V_2$ respectively.
Clearly for all the vertices a and b, d(a,b)≥ 3.

**Case (v):**

Let a,b be any two vertices in $V_1$ and $V_3$ respectively.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_1$ and $V_3$ respectively
Hence d(a,b) =1. Let a = $c_k$ and y= $e_1$ thenψ(a) = k–1, $1 \leq k \leq t$ ,ψ(b) = t+1 ,$t \geq 3$ Therefore d(a,b) + |ψ(a) –ψ(b)| =1+|k–1–(t+1)| ≥ 3.

**Subcase (ii):**

Let a,bbe any two non -adjacent vertices in $V_1$ and $V_3$ respectivelyHence d(a,b) ≥ 2. Let a = $c_k$ and b= $e_2$ . Then ψ(a) = k–1,$1 \leq k \leq t$ , ψ(b) = t+3 ,$t \geq 3$. Therefore d(a,b) + |ψ(a) –ψ(b)| ≥ 2 + |k–1–(t+3)| ≥ 3.

**Case (vi):**

Let a,bbe any two vertices in $V_2$ and $V_3$ respectively.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_2$ and $V_3$ respectively
Hence d(a,b) = 1. Let a = $e_3$ and b= $d_r$. Thenψ(a) =t+5, $t \geq 3$ ,ψ(b) = r–1, $1 \leq r \leq t$.Therefore d(a.b) + |ψ(a) –ψ(b)| =1+|t+5–(r–1)| ≥ 3.

**Subcase (ii):**

Let a,bbe any twonon-adjacentvertices in $V_2$ and $V_3$ respectively
Hence d(a,b) ≥ 2. Let a = $e_1$ and b= $d_r$. Then ψ(a) = t+1,$1 \leq r \leq t$ ,ψ(b) = r–1, $t \geq 3$ . Therefore d(a,b) + |ψ(a) –ψ(b)| ≥ 2 + |t+1–(r–1)| ≥ 3.

Similarly for all the other possibilities of aandb, d(a,b) + |ψ(a) –ψ(b)| ≥ 3.Therefore theL(2,1) labeling number λ(G) of tree <$K_{1,t}$: 2>ist+5 ,$t \geq 3$.

**Example 3.2.**

TheL (2, 1) labeling tree <$K_{1,7}$: 2>is shown in figure 2.19.

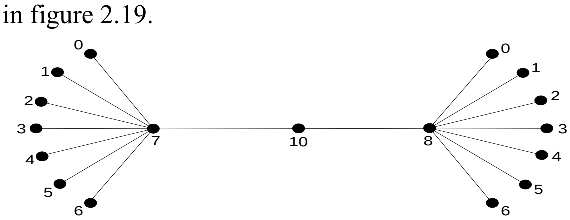


Figure 3.2.L(2,1) labeling of a tree

<$K_{1,7}$: 2>

**Theorem 3.3.**

The L (2,1) labeling number λ(G) of the graph $B(t, s, r-1)$ is max(t,s) + r, $t,s,r \geq 3$.

**Proof.**

Let the vertices of the graph B(t, s, r-1) be

$V(G) = V_1 \cup V_2 \cup V_3$

$V_1 = \{c_k / 1 \leq k \leq t\}$

$V_2 = \{d_m / 1 \leq m \leq s\}$

$V_3 = \{e_n / 1 \leq n \leq r\}$

where $c_k$ are the vertices on one side of the graph $B(t, s, r-1)$ (the head part) and $d_m$ are the vertices on the other end of the graph $B(t, s, r-1)$ ( the tail part) and $e_n$ be the vertices on the path joining the head and the tail part of the graph $B(t, s, r-1)$.

Define a mappingψ: V (G) → N ∪ {0} by

$\psi(c_k) = k - 1; \quad 1 \leq k \leq t$

$\psi(d_m) = m - 1; \quad 1 \leq m \leq s$

$\psi(e_n) = \max(t,s) + 2n - 1; \quad 1 \leq n \leq \left\lceil \frac{r}{2} \right\rceil$

$\psi(e_{\left\lceil \frac{r}{2} \right\rceil + n}) = \max(t,s) + 2n; \quad 1 \leq n \leq \left\lfloor \frac{r}{2} \right\rfloor$

**Case (i):**

Let a,b be any two vertices in $V_1$.

Clearly a,b are non - adjacent. Hence d(a,b) ≥ 2. Let a = $c_k$ and b= $c_{k+1}$, $1 \leq k \leq t$ thenψ(a) = k−1, ψ(b) = k. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |k−1−k| ≥ 3.

**Case (ii):**

Let a,b be any twovertices in $V_2$. Clearly a,b are non - adjacent.

Hence d(a,b) ≥ 2. Let a = $d_m$ and b = $d_{m+1}$, $1 \leq m \leq s$ . Thenψ(a) = k−1, ψ(b) = k. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |k−1−k| ≥ 3.

**Case (iii):**

Let a,b be any two vertices in $V_3$.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_3$ . Hence d(a,b) =1.

Let a = $e_n$ and b= $e_{n+1}$ ,1≤n≤r . Thenψ(a) =max(t,s) +2n−1, ψ(b) = max(t,s) +2n+1.

Therefore d(a,b) + |ψ(a) −ψ(b)| =1+|max(t,s) +2n−1−(max(t,s) +2n+1)| ≥ 3.

**Subcase (ii):**

Let a,b be any two non -adjacent vertices in $V_3$. Hence d(a,b) ≥ 2.

Let a = $e_n$ and b= $e_{\left\lceil \frac{r}{2} \right\rceil + 1}$ .Then ψ(a) = max(t,s) +2n−1,ψ(y) = max(t,s) +2n. Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |max(t,s) +2n−1−(max(t,s) +2n)| ≥ 3.

**Case (iv):**

Let a,b be any two vertices in $V_1$ and $V_2$ respectively.

Clearly for all the vertices a and b, d(a,b) ≥ 3.

**Case (v):**

Let a,b be any two vertices in $V_1$ and $V_3$ respectively.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_1$ and $V_3$ respectively

Hence d(a,b) =1. Let a = $c_k$ and b= $e_1$ thenψ(a) = k−1, 1 ≤ k ≤ t, ψ(b) = max(t,s)+1.

Therefore d(a,b) + |ψ(a)−ψ(b)| =1+|k−1−(max(t,s)+1)| ≥ 3.

**Subcase (ii):**

Let a,b be any two non -adjacent vertices in $V_1$ and $V_3$ respectively

Hence d(a,b) ≥ 2. Let a = $c_k$ and b= $e_2$ . Then ψ(a) = k−1, 1 ≤ k ≤ t,ψ(b) = max(t,s)+3.

Therefore d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |k−1−(max(t, s) +3)| ≥ 3.

**Case (vi):**

Let a,b be any two vertices in $V_2$ and $V_3$ respectively.

**Subcase (i):**

Let a,b be any two adjacent vertices in $V_2$ and $V_3$ respectively

Hence d(a,b) =1. Let a = $e_r$and b = $d_m$. Thenψ(a) =max(t,s) +r, ψ(b) = m−1, 1 ≤ m ≤ s. Therefore d(a,b) + |ψ(a) −ψ(b)| =1+|max(t, s) +r− (m−1)| ≥ 3.

**Subcase (ii):**

Let a,b be any two non -adjacent vertices in $V_2$ and $V_3$ respectively

Hence d(a,b) ≥ 2. Let a =$d_m$and b =$e_{r-1}$. Then ψ(a) = m-1, $1 \leq m \leq s$ ,ψ(b)=max(t,s)+2( $\left\lfloor \frac{r}{2} \right\rfloor$ −1),Therefore

d(a,b) + |ψ(a) −ψ(b)| ≥ 2 + |max(t, s) +2( $\left\lfloor \frac{r}{2} \right\rfloor$ −1)−(m−1)| ≥ 3.

Similarly for all the other possibilities of a andb, d(a,b) + |ψ(a) −ψ(b)| ≥ 3. Hence theL(2,1) labeling number of the graph $B(t, s, r-1)$isλ(G) = max(t, s) + r, where $t,s,r \geq 3$.

**Example 3.3.**

L(2,1) labeling of B(7,6,7) is shown in Figure 3.3.

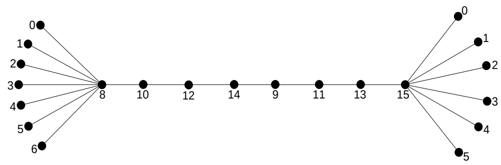


Figure 3.3. L (2,1) labeling of B (7,6,7)

## 4. CONCLUSION

In this paper the *L* (2,1) labeling number for triple quadrilateral snake and alternate triple quadrilateral snake graphsaredetermined.

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