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Hub Polynomial of a Graphs

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Abstract

The *hub polynomial* of a connected graph G of order n is the polynomial $H_G(x) = \sum_{i=h(G)}^{n} h_{G,i}x^i$ where $h_{G,i}$ denotes the number of hub sets of G of cardinality i and $h(G)$ is the hub number of G. In this paper, we obtain hub polynomial of helm graph, flower graph, corona of two graphs, some windmill graphs and some transformation graphs. Further we defined n-wounded spider graph and obtained its hub polynomial.

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Keywords: Hub set, hub polynomial, windmill graphs transformation graphs.

1. INTRODUCTION

In the theory of graphs, a graph polynomial is a graph invariant whose values are considered as a polynomials. Invariants of this type are studied in algebraic graph theory. In the literature, many polynomials are connected with graphs. Like domination polynomial, hosoya polynomial, clique polynomial, characteristic polynomial and Tutte polynomial. For more on polynomials one can refer [1, 2, 6, 10, 14, 15].

Hub number of a graph G was introduced by Walsh [17] in 2006, which is defined as follow. Let G be a graph with vertex set $V(G)$ and S be a subset in a graph G such that $S \subseteq V(G)$ and let $x, y \in V(G)$. An S-path between x and y is a path where all internal vertices are from S. A set $S \subseteq V(G)$ is a hub set of G if it has the property that, for any $x, y \in V(G)\backslash S$ there is an S-path in G between x and y. The minimum cardinality of a hub set is called *hub number* and is denoted by $h(G)$. More study about hub number one can refer [5, 8]. Recently, Veettil et al. [16] have introduced the concept of hub polynomial of a graph G.

Definition 1 [16] *The hub polynomial of a connected graph G of order n is the polynomial*

$$H_G(x) = \sum_{i=h(G)}^{n} h_{G,i}x^i$$

where $h_{G,i}$ denotes the number of hub sets of G of cardinality i.

Definition 2 [7] *The helm graph H_n is the graph obtained from wheel W_n with central vertex x, by attaching a pendant edge at each vertex of outer circle.*

Definition 3 [7] *The flower graph Fl_n is the graph obtained from a helm by joining each pendant vertex to the central vertex x of the helm.*

Definition 4 [9] *The corona of two graphs $G_1 \circ G_2$ is the graph obtained by taking one copy of G_1 (which has p_1 points) and p_1 copies of G_2, and then joining the ith point of G_1 to every point in the ith copy of G_2.*

Definition 5 [7] *The French windmill graph F_n^m is the graph obtained by taking $m \geq 2$ copies of $K_n, n \geq 2$ with a vertex in common.*

Definition 6 [12] *The Kulli cycle windmill graph C_{n+1}^m is the graph obtained by taking m copies of the graph $K_1 + C_n$ for $n \geq 3$ with a vertex K_1 in common.*

Definition 7 [13] *The Kulli path windmill graph P_{n+1}^m is the graph obtained by taking m copies of the graph $K_1 + P_n$ for $n \geq 3$ with a vertex K_1 in common.*

Definition 8 [7] *A wounded spider is the graph formed by subdividing at most $t - 1$ of the edges of a star $k_{1,t}$ for $t \geq 0$.*

We have defined a n-wounded spider graph, is based on the distance between the common vertex to every pendant vertices of star graph, is defined as follows.

Definition 9 *A n-wounded spider $(n \geq 2)$, is the graph obtained from star graph $k_{1,m}$ by inserting $(n - 1)$ for new vertices of degree 2 into every edge of a star graph $k_{1,m}$ for $m \geq 2$. It is denoted as $S_n(K_{1,m})$.*

2. PRELIMINARIES

Here we have considered only nontrivial, connected, simple and undirected graphs. Let G be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$. Thus $|V(G)| = n$ and $|E(G)| = m$ where, n and m are called *order* and *size* of graph G respectively. The *null graph* is a graph G [11] with nonempty vertex set and no edges. The *complement* of a graph G [9] is denoted by $\overline{G}$ whose vertex set is $V(G)$ and two vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. The *line graph* $L(G)$ of a graph G [18] is the graph with vertex set as the edge set of G and two vertices of $L(G)$ are adjacent whenever the corresponding edges in G have a vertex in common. The *subdivision graph* $S(G)$ of a graph G [9] whose vertex set is $V(G) \cup E(G)$ where two vertices are adjacent if and only if one is a vertex of G and other is an edge of G incident with it. The *jump graph* $J(G)$ of a graph G [4] is the graph with vertex set as the edge set of G and two vertices of $J(G)$ are adjacent if and only if they are nonadjacent in G. A path, a cycle, a complete graph of order n are denoted by P_n, C_n and K_n respectively. For undefined terminology and notations refer [3, 9].

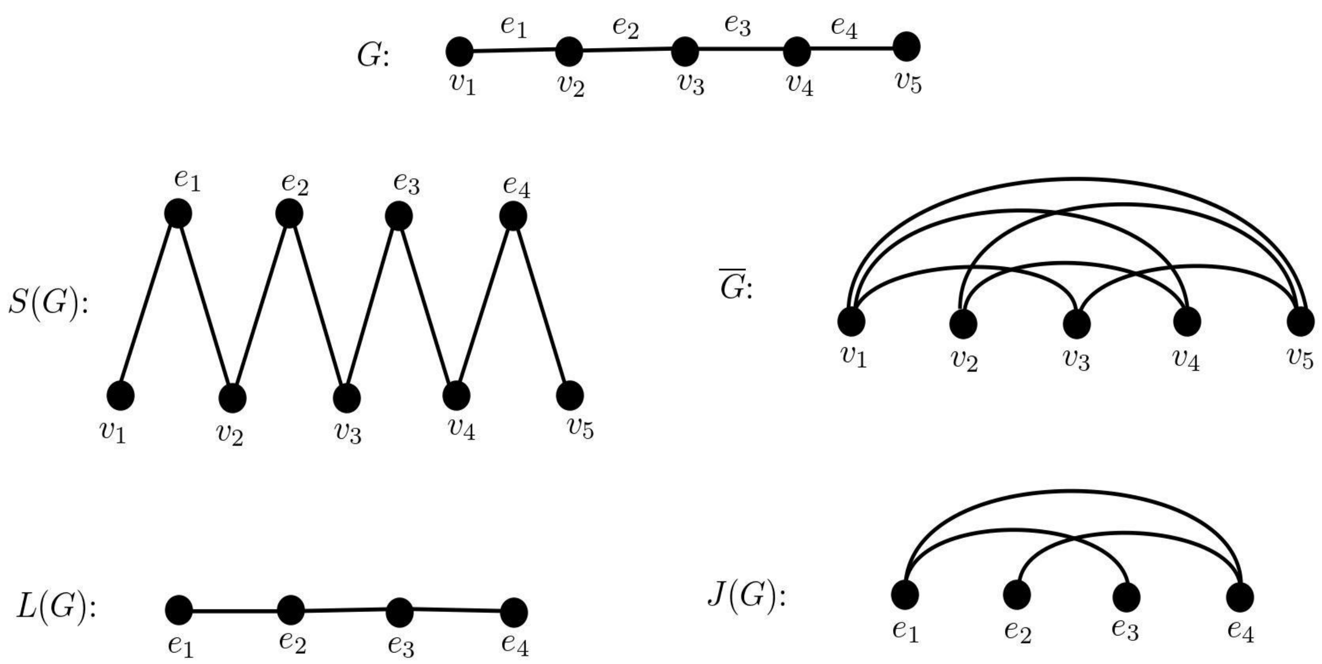

Figure 1: Graph G and its transformation graphs $S(G)$, $\overline{G}$, $L(G)$ and $J(G)$.

Proposition 2.1 [16] *The hub polynomial*

i) For path P_n is $H_{P_n}(x) = \binom{n}{2}x^{n-2} + \binom{n}{1}x^{n-1} + x^n$,

ii) For cycle C_n is $H_{C_n}(x) = \binom{n}{3}x^{n-3} + \binom{n}{2}x^{n-2} + \binom{n}{1}x^{n-1} + x^n$,

iii) For Complete graph K_n is $H_{K_n}(x) = (1+x)^n - 1$.

3. HUB POLYNOMIAL OF GRAPHS

Theorem 3.1 *The hub polynomial of the helm graph H_n, where $n \geq 4$ of order $2n-1$ is*

$$H_{H_n}(x) = x^{n-1}(1+x)^n + \sum_{i=2n-4}^{2n-2}\binom{n}{i-n+1}x^i + x^n(1+x)^{n-1} - x^{2n-1}.$$

Proof. Let $H_n = \{x, v_1, v_2, v_3, \dots, v_{n-1}, v_1', v_2', v_3', \dots, v_{n-1}'\}$ be the vertices of a helm graph. The vertices $\{v_1, v_2, v_3, \dots, v_{n-1}\}$ be a rim vertices, $\{v_1', v_2', v_3', \dots, v_{n-1}'\}$ be a pendant vertices and x be the central vertex of helm. Then $\{v_1, v_2, v_3, \dots, v_{n-1}\}$ is the only one hub set of cardinality $n-1$. Therefore $h(H_n) = n-1$.

For $n-1 \leq i \leq 2n-1$, every hub set must include all rim vertices of helm. Hence there are $\binom{n}{i-n+1}$ hub set of cardinality i. Like $\{v_1, v_2, v_3, \dots, v_{n-1}\}$,$\{x, v_1, v_2, v_3, \dots, v_{n-1}\}$,..., $\{x, v_1, v_2, v_3, \dots, v_{n-1}, v_1', v_2', v_3', \dots, v_{n-1}'\}$.

For $n \leq i \leq 2n-2$, every hub set must include pendant vertices with central vertex x. Hence there are $\binom{n-1}{i-n}$ hub set of cardinality i. Like $\{x, v_1', v_2', \dots, v_{n-1}'\}$, $\{x, v_1', v_2', \dots, v_{n-1}', v_1\}$, $\{x, v_1', v_2', \dots, v_{n-1}', v_1, v_2\}$,..., $\{x, v_1', v_2', \dots, v_{n-1}', v_1, v_2, \dots, v_{n-1}\}$.

For $2n-4 \leq i \leq 2n-2$, every hub set must include pendant vertices but not central vertex x. Hence there

are $\binom{n}{i-n+1}$ hub set of cardinality i. Like
$\{v_1', v_2', \ldots, v_{n-1}', v_1, v_2, \ldots, v_{2n-4}\}$,
$\{v_1', v_2', \ldots, v_{n-1}', v_1, v_2, \ldots, v_{2n-3}\}$,
$\{v_1', v_2', \ldots, v_{n-1}', v_1, v_2, \ldots, v_{2n-2}\}$.

$$H_{H_n}(x) = \sum_{i=n-1}^{2n-1} \binom{n}{i-n+1} x^i + \sum_{i=n}^{2n-2} \binom{n-1}{i-n} x^i + \sum_{i=2n-4}^{2n-2} \binom{n}{i-n+1} x^i$$

$$= x^{n-1}(1+x)^n + \sum_{i=2n-4}^{2n-2} \binom{n}{i-n+1} x^i + x^n(1+x)^{n-1} - x^{2n-1}.$$

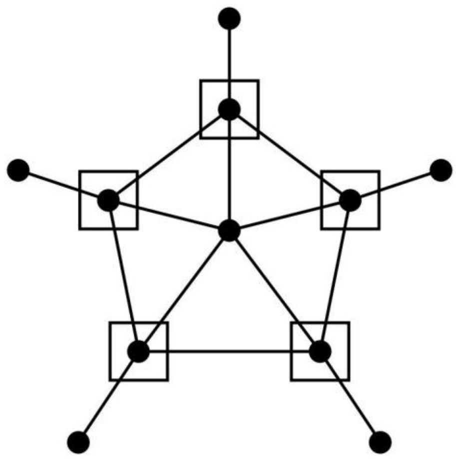

Figure 2: Choosing a minimum hub set in helm graph H_6.

Theorem 3.2 *The hub polynomial of the flower graph Fl_n, where $n \geq 4$ of order $2n-1$ is*

$$H_{Fl_n}(x) = x(1+x)^{2n} + x^{n-1}(1+x)^n - x^{2n-1}(2nx + x^2 + 1).$$

Proof. Let $Fl_n = \{x, v_1, v_2, v_3, \ldots, v_{n-1}, v_1', v_2', v_3', \ldots, v_{n-1}'\}$ be the vertices of a flower graph. The vertices $\{v_1, v_2, v_3, \ldots, v_{n-1}, v_1', v_2', v_3', \ldots, v_{n-1}'\}$ be the vertices of flower graph and x be the central vertex. Then $\{x\}$ is the only one hub set of cardinality 1. Therefore $h(Fl_n) = 1$.

For $1 \leq i \leq 2n-1$, every hub set must include central vertex x. Hence there are $\binom{2n}{i-1}$ hub set of cardinality i. Like
$\{x\}$,$\{x, v_1\}$,$\{x, v_1, v_2\}$...,
$\{x, v_1, v_2, v_3, \ldots, v_{n-1}, v_1', v_2', v_3', \ldots, v_{n-1}'\}$.

For $n-1 \leq i \leq 2n-2$, every hub set does not contain central vertex x. Hence there are $\binom{n}{i-n+1}$ hub set of cardinality i. Like
$\{v_1, v_2, \ldots, v_{n-1}\}$,$\{v_1, v_2, \ldots, v_{n-1}, v_1'\}$,
$\{v_1, v_2, \ldots, v_{n-1}, v_1', v_2'\}$,...,
$\{v_1, v_2, v_3, \ldots, v_{n-1}, v_1', v_2', v_3', \ldots, v_{n-1}'\}$.

$$H_{Fl_n}(x) = \sum_{i=1}^{2n-1} \binom{2n}{i-1} x^i + \sum_{i=n-1}^{2n-2} \binom{n}{i-n+1} x^i$$

$$= x(1+x)^{2n} + x^{n-1}(1+x)^n - x^{2n-1}(2nx + x^2 + 1).$$

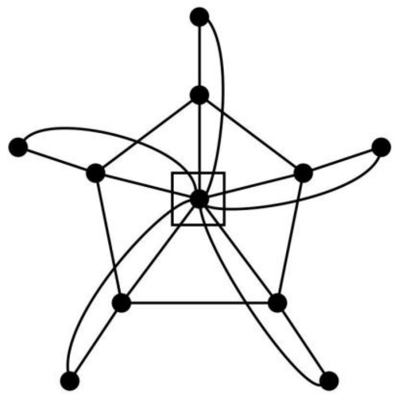

Figure 3: Choosing a minimum hub set in flower graph Fl_6.

Theorem 3.3 *Let K_2 and G be two graphs of order 2 and n respectively. Then hub polynomial of the corona of two graphs $K_2 \circ G$ of order $2n+2$ is*

$$H_{K_2 \circ G}(x) = x^2(1+x)^{2n} + 2\sum_{i=n+1}^{2n+1} \binom{n}{i-4} x^i + x^{2n}.$$

Proof. Let $K_2 \circ G = \{a, b, v_1, v_2, v_3, \ldots, v_{2n-1}, v_{2n}\}$ be the vertices of a corona of two graphs. Then $\{a, b\}$ is the only one hub set of cardinality two. Therefore $h(K_2 \circ G) = 2$.

For $3 \leq i \leq 2n+2$, every hub set must include both a and b vertices. Hence there are $\binom{2n}{i-2}$ hub set of cardinality i. Like $\{a, b, v_1\}$,$\{a, b, v_1, v_2\}$,...,
$\{a, b, v_1, v_2, v_3, \ldots, v_{2n-1}, v_{2n}\}$.

For $n+1 \le i \le 2n+1$, every hub set must include vertex a but not b. Hence there are $\binom{n}{i-4}$ hub set of cardinality i. Like $\{a, v_1, v_2, ..v_{n+1}\}$,$\{a, v_1, v_2, \dots, v_{n+2}\}$, $\{a, v_1, v_2, \dots, v_{n+3}\}$, ...,$\{a, v_1, v_2, v_3, \dots, v_{2n-1}, v_{2n}\}$.

Similarly, for $n+1 \le i \le 2n+1$, every hub set must include vertex b but not a. Hence there are $\binom{n}{i-4}$ hub set of cardinality i. Like $\{b, v_1, v_2, ..v_{n+1}\}$,$\{b, v_1, v_2, \dots, v_{n+2}\}$, $\{b, v_1, v_2, \dots, v_{n+3}\}$, ...,$\{b, v_1, v_2, v_3, \dots, v_{2n-1}, v_{2n}\}$.

For $i = 2n$, there is only one hub set which does not contain vertices a and b of cardinality i. That is, $\{v_1, v_2, v_3, \dots, v_{2n-1}, v_{2n}\}$.

$$H_{K_2 \circ G}(x) = x^2 + \sum_{i=3}^{2n+2} \binom{2n}{i-2} x^i + 2\sum_{i=n+1}^{2n+1} \binom{n}{i-4} x^i + x^{2n}$$

$$= x^2(1+x)^{2n} + 2\sum_{i=n+1}^{2n+1} \binom{n}{i-4} x^i + x^{2n}.$$

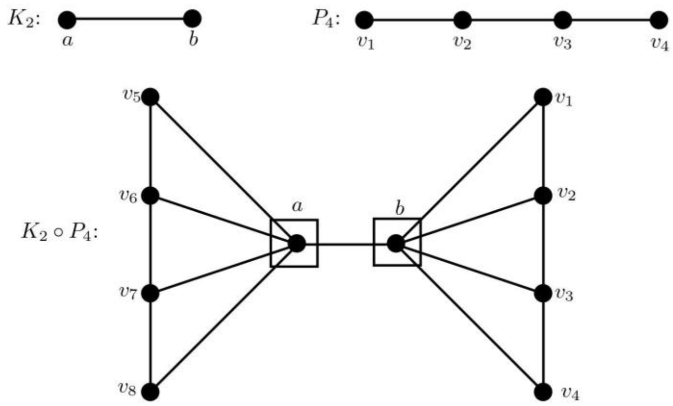

Figure 4: Graphs K_2, P_4 and choosing a minimum hub set in corona graph $K_2 \circ P_4$.

Theorem 3.4 *The hub polynomial of the French windmill graph F_4^m is*

$$H_{F_4^m}(x) = x(1+x)^{3m} + x^{3m-3}(m + 3mx^2 + x^3) + \binom{3m}{2} x^{3m-2}.$$

Proof. Let v be the central vertex of F_4^m. Then $\{v\}$ is the exactly only one hub set of cardinality 1. For $2 \le i \le 3m-4$ there are exactly $\binom{3m}{i-1}$ hub sets of cardinality i which contains the central vertex v.

For $i = 3m-3$ there are $\binom{3m}{4}$ hub sets which contains central vertex v and there are m hub sets which does not contain central vertex v, like, the set of vertices obtained by removing each complete graph of order 4, one by one.

For $i = 3m-2$ there are $\binom{3m}{3}$ hub sets which contains central vertex v and there are $\binom{3m}{2}$ hub sets which does not contain central vertex v, like, $\{1,2,3,\dots,3m-2\}$, $\{1,2,3,\dots,3m-3,3m\}$,...,$\{3,4,5\dots,3m\}$.

For $i = 3m-1$ there are $\binom{3m}{2}$ hub sets which contains central vertex v and there are $3m$ hub sets like, $\{1,2,3,\dots,3m-1\}$, $\{1,2,3,\dots,3m-2,3m\}$,...,$\{2,3,4,\dots,3m\}$ which does not contain central vertex v.

For $i = 3m$ there are $3m$ hub sets which contains central vertex v and one hub sets like, $\{1,2,3,\dots,3m\}$ which does not contain central vertex v.

For $i = 3m+1$ there is only one hub set which contains all the elements of the graph. Hence,

$$H_{F_4^m}(x) = x(1+x)^{3m} + x^{3m-3}(m + 3mx^2 + x^3) + \binom{3m}{2} x^{3m-2}.$$

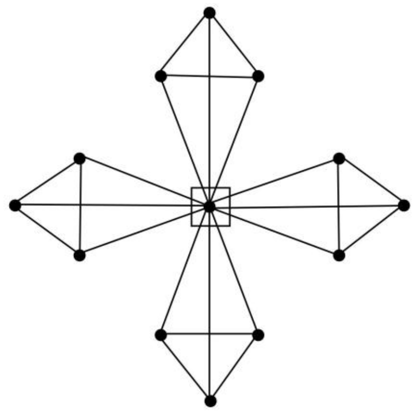

Figure 5: Choosing a minimum hub set in French windmill graph F_4^m.

Theorem 3.5 *The hub polynomial of the Kulli cycle windmill graph C_5^m is*

$$H_{C_5^m}(x) = x(1+x)^{4m} + x^{4m-2}(4m + 4mx + x^2).$$

Proof. Let v be the central vertex of C_5^m. Then $\{v\}$ is the exactly only one hub set of cardinality 1. For $2 \le i \le 4m-3$ there are exactly $\binom{4m}{i-1}$ hub sets of cardinality i which contains the central vertex v.

For $i = 4m-2$ there are $\binom{4m}{3}$ hub sets which contains central vertex v and there are $4m$ hub sets which does not contain central vertex v, like, removing any two consecutive adjacent vertices except central vertex v, $\{1,2,3,\ldots,4m-2\}$,...,$\{3,4,5\ldots,4m-1,4m\}$.

For $i = 4m-1$ there are $\binom{4m}{2}$ hub sets which contains central vertex v and there are $\binom{4m}{1}$ hub sets which does not contain central vertex v, like, $\{1,2,3,\ldots,4m-1\}$, $\{1,2,3,\ldots,4m-2,4m\}$,...,$\{2,3,4,5\ldots,4m-1,4m\}$.

For $i = 4m$ there are $\binom{4m}{1}$ hub sets which contains central vertex v and one hub sets like, $\{1,2,3,\ldots,4m\}$ which does not contain central vertex v.

For $i = 4m+1$ there is only one hub set which contains all the elements of the graph. Like $\{v,1,2,3,\ldots,4m\}$. Hence,

$$H_{C_5^m}(x) = x(1+x)^{4m} + x^{4m-2}(4m+4mx+x^2).$$

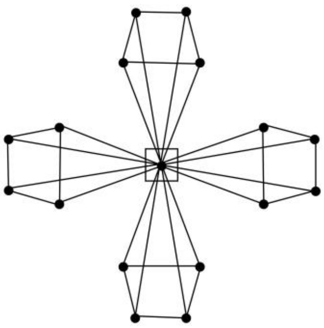

Figure 6: Choosing a minimum hub set in Kulli cycle windmill graph C_5^m.

Theorem 3.6 *The hub polynomial of the Kulli path windmill graph P_6^m is*

$$H_{P_6^m}(x) = x(1+x)^{5m} + x^{5m-2}(10m+5mx+x^2).$$

Proof. Let v be the central vertex of P_6^m. Then $\{v\}$ is the exactly only one hub set of cardinality 1.

For $2 \le i \le 5m-3$ there are exactly $\binom{5m}{i-1}$ hub sets of cardinality i which contains the central vertex v.

For $i = 5m-2$ there are $\binom{5m}{3}$ hub sets which contains central vertex v and there are $10m$ hub sets which does not contain central vertex v, like, removing any two vertices in every m copy in the graph, except the central vertex v, like $\{1,2,3,\ldots,5m-2\}$, $\{1,3,4,\ldots,5m-2\}$,...,$\{3,4,5\ldots,5m-1,5m\}$.

For $i = 5m-1$ there are $\binom{5m}{2}$ hub sets which contains central vertex v and there are $\binom{5m}{1}$ hub sets which does not contain central vertex v, like, $\{1,2,3,\ldots,5m-1\}$, $\{1,2,3,\ldots,5m-2,5m\}$,...,$\{2,3,4,5\ldots,5m-1,5m\}$.

For $i = 5m$ there are $\binom{5m}{1}$ hub sets which contains central vertex v and one hub sets like, $\{1,2,3,\ldots,5m\}$ which does not contain central vertex v. Hence,

$$H_{P_6^m}(x) = x(1+x)^{5m} + x^{5m-2}(10m+5mx+x^2).$$

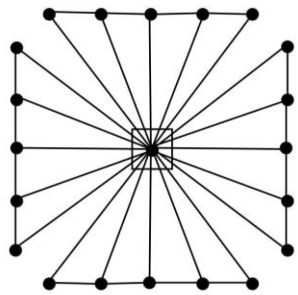

Figure 7: Choosing a minimum hub set in Kulli path windmill graph P_6^m.

Theorem 3.7 *Let $S_n(K_{1,m})$ be a n-wounded spider graph of order $m \ge 2$ and distance $n \ge 2$. Then*

$$H_{S_n(K_{1,m})}(x) = x(1+x)^{nm} - x(1+x)^{nm-m} + nx^{nm-m+1}(1 + mx^{m-2}) + x^{nm}.$$

Proof. Let $S_n(K_{1,m}) = \{x, v_1^1, v_2^1, v_3^1, \ldots, v_m^1, v_1^2, v_2^2, v_3^2, \ldots, v_m^2, \ldots, v_1^n, v_2^n, v_3^n, \ldots, v_m^n\}$ be the vertices of a n-wounded spider graph of order m and length n. Then we have $h(S_n(K_{1,m})) = (n-1)m+1$. Let x be the common vertex of $S_n(K_{1,m})$.

For $i = nm-m+1$ there are exactly n hub set which contains common vertex x of cardinality $nm-m+1$. Like,

$\{x, v_1^1, v_2^1, v_3^1, \ldots, v_m^1, v_1^2, v_2^2, v_3^2, \ldots, v_m^2, \ldots, v_1^{n-1}, v_2^{n-1}, v_3^{n-1}, \ldots, v_m^{n-1}\}$,...,$\{x, v_1^2, v_2^2, v_3^2, \ldots, v_m^2, v_1^3, v_2^3, v_3^3, \ldots, v_m^3, \ldots, v_1^n, v_2^n, v_3^n, \ldots, v_m^n\}$.

For $nm-m+2 \le i \le nm-2$ there are exactly

$\binom{nm}{i-1}$ hub sets of cardinality i which contains the common vertex x.

For $i = nm - 1$ there are $\binom{nm}{2}$ hub sets which contains common vertex x and there are nm hub sets which does not contain common vertex x. Like,

$\{v_1^1, v_2^1, v_3^1, \ldots, v_m^1, \ldots, v_1^n, v_2^n, v_3^n, \ldots, v_{(m-1)}^n\}$,..., $\{v_2^1, v_3^1, \ldots, v_m^1, \ldots, v_1^n, v_2^n, v_3^n, \ldots, v_m^n\}$.

For $i = nm$ there are $\binom{nm}{1}$ hub sets which contains common vertex x and one hub sets like, $\{v_1^1, v_2^1, v_3^1, \ldots, v_m^1, v_1^2, v_2^2, v_3^2, \ldots, v_m^2, \ldots, v_1^n, v_2^n, v_3^n, \ldots, v_m^n\}$ which does not contain common vertex x.

For $i = nm + 1$ there is only one hub sets which contains all vertices, like

$\{x, v_1^1, v_2^1, v_3^1, \ldots, v_m^1, v_1^2, v_2^2, v_3^2, \ldots, v_m^2, \ldots, v_1^n, v_2^n, v_3^n, \ldots, v_m^n\}$.

Hence,

$$H_{S_n(K_{1,m})}(x) = x(1+x)^{nm} - x(1+x)^{nm-m} + nx^{nm-m+1}(1 + mx^{m-2}) + x^{nm}.$$

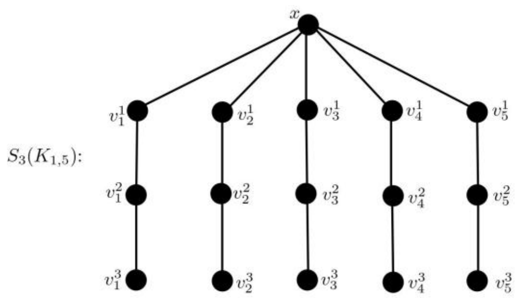

Figure 8: 3-wounded spider graph

4 HUB POLYNOMIAL OF TRANSFORMATION GRAPHS

Corollary 4.1 *Let $S(P_n)$ be a subdivision graph of path graph P_n of order $n \geq 3$ and size m. Then*

$$H_{S(P_n)}(x) = \binom{n+m}{2} x^{n+m-2} + \binom{n+m}{1} x^{n+m-1} + x^{n+m}.$$

Proof. The subdivision graph of a path $S(P_n) \cong P_{n+m}$. From the Proposition 2.1,

$$H_{S(P_n)}(x) = \binom{n+m}{2} x^{n+m-2} + \binom{n+m}{1} x^{n+m-1} + x^{n+m}.$$

Corollary 4.2 *Let $S(C_n)$ be a subdivision graph of cycle graph C_n of order $n \geq 3$ and size m. Then*

$$H_{S(C_n)}(x) = \binom{n+m}{3} x^{n+m-3} + \binom{n+m}{2} x^{n+m-2} + \binom{n+m}{1} x^{n+m-1} + x^{n+m}.$$

Proof. The subdivision graph of a cycle $S(C_n) \cong C_{n+m}$. From the Proposition 2.1,

$$H_{S(C_n)}(x) = \binom{n+m}{3} x^{n+m-3} + \binom{n+m}{2} x^{n+m-2} + \binom{n+m}{1} x^{n+m-1} + x^{n+m}.$$

Theorem 4.3 *Let $\overline{P_n}$ be a complement of a path graph P_n of order $n \geq 4$ and size m. Then*

$$H_{\overline{P_n}}(x) = (1+x)^n - (1+nx).$$

Proof. Let $\overline{P_n} = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of a complement of a path. Then we have $h(\overline{P_n}) = 2$. Also every subsets of vertex set for the complement of path $\overline{P_n}$ consisting of two elements and all its super sets form a hub set for the complement of path $\overline{P_n}$. Hence

$$H_{\overline{P_n}}(x) = \sum_{i=2}^{n} \binom{n}{i} x^i$$
$$= (1+x)^n - (1+nx).$$

Theorem 4.4 *Let $\overline{C_n}$ be a complement of cycle graph C_n of order $n \geq 5$ and size m. Then*

$$H_{\overline{C_n}}(x) = (1+x)^n - (1+nx).$$

Proof. Let $\overline{C_n} = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of a complement of a cycle. Then we have $h(\overline{C_n}) = 2$. Also every subsets of vertex set for the complement of cycle $\overline{C_n}$ consisting of two elements and all its super sets form a hub set for the complement of cycle $\overline{C_n}$. Hence

$$H_{\overline{C_n}}(x) = \sum_{i=2}^{n} \binom{n}{i} x^i$$
$$= (1+x)^n - (1+nx).$$

Corollary 4.5 *Let $\overline{G}$ be a complement of a null graph of G of order $n \geq 3$. Then*

$$H_G(x) = (1+x)^n - 1.$$

Proof. The complement of null graph $G \cong K_n$. From the Proposition 2.1,

$$H_G(x) = (1+x)^n - 1.$$

Corollary 4.6 *Let $L(P_n)$ be a line graph of path graph P_n of order $n \geq 4$ and size m. Then*

$$H_{L(P_n)}(x) = \binom{n-1}{2} x^{n-3} + \binom{n-1}{1} x^{n-2} + x^{n-1}.$$

Proof. The line graph of a path $L(P_n) \cong P_{n-1}$. From the Proposition 2.1,

$$H_{L(P_n)}(x) = \binom{n-1}{2} x^{n-3} + \binom{n-1}{1} x^{n-2} + x^{n-1}.$$

Corollary 4.7 *Let $L(C_n)$ be a line graph of cycle graph C_n of order $n \geq 4$ and size m. Then*

$$H_{L(C_n)}(x) = \binom{n}{3} x^{n-3} + \binom{n}{2} x^{n-2} + nx^{n-1} + x^n.$$

Proof. The line graph of a cycle $L(C_n) \cong C_n$. From the Proposition 2.1,

$$H_{L(C_n)}(x) = \binom{n}{3} x^{n-3} + \binom{n}{2} x^{n-2} + nx^{n-1} + x^n.$$

Corollary 4.8 *Let $J(P_n)$ be a jump graph of path graph P_n of order $n \geq 5$ and size m. Then*

$$H_{J(P_n)}(x) = (1+x)^{n-1} - (1+(n-1)x).$$

Proof. The jump graph of a path $J(P_n) \cong \overline{P_{n-1}}$. From the Theorem 4.3,

$$H_{J(P_n)}(x) = (1+x)^{n-1} - (1+(n-1)x).$$

Corollary 4.9 *Let $J(C_n)$ be a jump graph of cycle C_n of order $n \geq 5$ and size m. Then*

$$H_{J(C_n)}(x) = (1+x)^n - (1+nx).$$

Proof. The jump graph of a cycle $J(C_n) \cong \overline{C_n}$. From the Theorem 4.4,

$$H_{J(C_n)}(x) = (1+x)^n - (1+nx).$$

5 CONCLUSION

In this paper, we have obtained explicit formulae for hub polynomial of helm graph, flower graph, corona of two graphs, some windmill graphs and some transformation graphs. Further we defined n-wounded spider graph and obtained its hub polynomial. One can obtain hub polynomial of some other graphs.

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