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Algebraic Properties of Convolution of Arithmetic Functions on the Partial Sub-Basic Sequence of Square-Free Odd Integers

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Abstract

In this paper, we introduce the notion of partial sub-basic sequence on the sub set of square-free odd integers and using convolution definition of arithmetic functions from the set of square free positive integers to real numbers and obtain some basic algebraic properties of convolution. We also define partial multiplicative functions with respect to partial basic sequences and obtain their properties. These results are extended the results given in Sridevi [7] relating to the arithmetic functions, thus this paper is a sequel to Sridevi [7].

Keywords: Arithmetic functions, Square-free integers, Square-free odd integers, Convolution, Partial sub-basic sequence, Partial multiplicative functions.

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1. INTRODUCTION

A real or complex valued function defined of the set of all natural numbers or the set of all positive integers is called an arithmetical function. Their various properties were investigated by several authors and they represent an important research topic [1, 2, 3, 4]. Properties of $\mathcal{B}$-multiplicative and quasi $\mathcal{B}$-multiplicative functions are studied in [5, 6].

Sridevi [7] considered the set Å of square-free integers and defined partial multiplication with respect to partial operator $*$ on Å × Å. In [7] a partial basic sequence of Å × Å satisfying three properties is introduced. In this paper, we introduce partial sub-basic sequence and convolution operator $\circ$ on partial basic sequence. Using this convolution we establish some algebraic results of arithmetic functions. We also introduce the notion of partial multiplicative functions and study their properties. This paper is sequel to Sridevi [7].

2. PRELIMINARIES

Let $\mathbb{Z}^+$ be the set of all positive integers.

Define Å $= \{n \in \mathbb{Z}^+ | n \text{ is square} - \text{free}\}$ (i.e., p is a prime, $p|n \implies p^2 \dagger n$). Clearly $1 \in$ Å.

Sridevi [7] introduced the notion of partial binary operator on Å and partial basic sequence on Å as follows.

Let $*$ be the partial binary operator defined on Å as follows.

For $m, n \in$ Å, $m * n = mn$ whenever $(m, n) = 1$.

We observe that $m * n$ is not always defined but it is defined only when $(m, n) = 1$. (i.e., gcd of $m, n = 1$).

Let F be the sub set of Å × Å such that $(m, n) \in$ F if $(m, n) = 1$.

Thus F $= \{(m, n) | m, n \in$ Å$, (m, n) = 1\}$.

We observe that $m * n$ is defined if $(m, n) \in$ F.

F has the following properties:

(i) $(a, b) \in$ F $\Leftrightarrow (b, a) \in$ F

(ii) Suppose $a, b, c \in$ Å. Then $(a, bc) \in$ F $\Leftrightarrow (a, b) \in$ F, $(a, c) \in$ F and $(b, c) = 1$.

(iii) $(1, a) \in$ F $for\ all\ a \in$ Å.

F is called partial basic sequence on Å.

3. RESULTS

Let $\dot{B} = \{m \in \mathbb{Z}^+ | m \text{ is square} - \text{free odd integer}\}$

Then clearly Ḃ is a proper subset of Å.

Write $F_{\dot{B}} = \{(m,n) | m, n \in \dot{B}, (m,n) = 1\}$.

Now we show that $F_{\dot{B}}$ has the following properties:

(i) $(a,b) \in F_{\dot{B}} \Leftrightarrow (b,a) \in F_{\dot{B}}$

(ii) Suppose $a,b,c \in \dot{B}$. Then $(a,bc) \in F_{\dot{B}} \Leftrightarrow (a,b) \in F_{\dot{B}}, (a,c) \in F_{\dot{B}}$ and $(b,c) = 1$

(iii) $(1,a) \in F_{\dot{B}}$ for all $a \in \dot{B}$.

Since $(a,b) \in F_{\dot{B}} \Longrightarrow a, b \in \dot{B}$ and $(a,b) = 1$

$\Longrightarrow b, a \in \dot{B}$ and $(b,a) = 1 \Longrightarrow (b,a) \in F_{\dot{B}}$.

Thus $F_{\dot{B}}$ satisfies property (i).

Clearly $1 \in \dot{B}$ and hence $(1,a) \in F_{\dot{B}}$ for all $a \in \dot{B}$. (since $(1,a) = 1$).

Thus $F_{\dot{B}}$ satisfies property (iii).

Now we show that $F_{\dot{B}}$ has property (ii).

Suppose $(a,bc) \in F_{\dot{B}}$. Then $a \in \dot{B}$ and $b \in \dot{B}$, so that $'a'$ is a square-free odd integer and bc is a square-free odd integer.

Hence b *and* c are square-free odd integers and $(b,c) = 1$.

Further, $(a,bc) = 1 \Longrightarrow (a,b) = 1$ and $(a,c) = 1$.

Hence $(a,b) \in F_{\dot{B}}$ *and* $(a,c) \in F_{\dot{B}}$.

Thus $(a,bc) \in F_{\dot{B}} \Leftrightarrow (a,b) \in F_{\dot{B}}, (a,c) \in F_{\dot{B}}$ *and* $(b,c) = 1$.

Conversely, suppose $(a,b) \in F_{\dot{B}}, (a,c) \in F_{\dot{B}}$ and $(b,c) = 1$.

This shows that a, b, c are square-free odd integers, bc is a square-free odd integer since $(b,c) = 1$.

Hence $(a,bc) \in F_{\dot{B}}$.

Hence $F_{\dot{B}}$ has property (ii).

Thus $F_{\dot{B}}$ has properties (i), (ii) and (iii).

$F_{\dot{B}}$ is called a partial sub basic sequence of Å.

We also say that $F_{\dot{B}}$ is a partial basic sequence of Ḃ.

We use the following two Lemmas without any specific mention in the further developments.

3.1 Lemma: For all $n \in$ Å $, d \mid n \Longrightarrow \left(d, \frac{n}{d}\right) = 1$.

3.2 Lemma: If m, n are positive integers and $(m,n) = 1$ then $d \mid mn \Longrightarrow \exists$ unique pair (δ_1, δ_2) such that $\delta_1 \mid m$, $\delta_2 \mid n$ and $\delta_1 \delta_2 = d$.

Now we define the convolution operator $\circ$ for arithmetic functions defined on Ḃ.

Suppose $f : \dot{B} \longrightarrow \mathbb{R}$, $g : \dot{B} \longrightarrow \mathbb{R}$. Define $f \circ g : \dot{B} \longrightarrow \mathbb{R}$ by

$$(f \circ g)(n) = \sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right), \quad \text{for all } n \in \dot{B}$$

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3.3 Definition: Define $I_{\dot{B}} : \dot{B} \longrightarrow \mathbb{R}$ by

$$I_{\dot{B}} = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Write $\mathfrak{B} = \{f \mid f : \dot{B} \longrightarrow \mathbb{R}\}$.

Now we obtain the following properties on $(\mathfrak{B}, \circ)$:

3.4 Property : $f, g \in \mathfrak{B}$ implies $f \circ g \in \mathfrak{B}$.

3.5 Property : $f, g \in \mathfrak{B} \implies f \circ g = g \circ f$.

Proof : For $n \in \dot{B}$,

$$(f \circ g)(n) = \textstyle\sum_{d \mid n} f(d)\, g\left(\frac{n}{d}\right)$$

Now

$$(g \circ f)(n) = \sum_{d \mid n} g(d) f\left(\frac{n}{d}\right)$$

$$= \sum_{d \mid n} f\left(\frac{n}{d}\right) g(d)$$

$$= \sum_{\delta \mid n} f(\delta) g\left(\frac{n}{\delta}\right) \quad \text{where } \delta = \frac{n}{d}$$

$$= (f \circ g)(n)$$

So that $(g \circ f)(n) = (f \circ g)(n)$ for all $n \in \dot{B}$.

Hence $f \circ g = g \circ f$.

3.6 Property : $f \in \mathfrak{B} \implies f \circ I_{\dot{B}} = f$.

Proof :

$$(f \circ I_{\dot{B}})(1) = \sum_{d \mid 1} f(d)\, I_{\dot{B}}\left(\frac{1}{d}\right) = f(1) I_{\dot{B}}(1) = f(1)$$

$$(f \circ I_{\dot{B}})(3) = \sum_{d \mid 3} f(d)\, I_{\dot{B}}\left(\frac{3}{d}\right) = f(1) I_{\dot{B}}(3) + f(3) I_{\dot{B}}(1) = f(3)$$

Assume that $(f \circ I_{\dot{B}})(m) = (I_{\dot{B}} \circ f)(m)$ for $m \in \dot{B}$ and $m < n \in \dot{B}$.

Then

$$(f \circ I_{\dot{B}})(\mathrm{n}) = \sum_{d \mid \mathrm{n}} f(d)\, I_{\dot{B}}\left(\frac{n}{d}\right)$$

$$= f(n) I_{\dot{B}}(1) + \sum_{\substack{d \mid \mathrm{n} \\ 1 \le d < n}} f(d)\, I_{\dot{B}}\left(\tfrac{n}{d}\right)$$

$$= f(n).1 + 0 = f(n)$$

Therefore $f \circ I_{\dot{B}} = f$.

3.7 Note: $I_{\dot{B}}$ is called the identity element of $\mathfrak{B}$.

3.8 Property : For all $f, g, h \in \mathfrak{B}$, we have $f \circ (g \circ h) = (f \circ g) \circ h$.

Proof : For $n \in \dot{B}$,

$$\big((f \circ g) \circ h\big)(n) = \sum_{d|n} (f \circ g)(d)\, h\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} \left(\sum_{\delta|d} f(\delta) g\left(\frac{d}{\delta}\right) \right) h\left(\frac{n}{d}\right)$$

$$= \sum_{d|n} h\left(\frac{n}{d}\right) \sum_{\delta|d} f(\delta) g\left(\frac{d}{\delta}\right)$$

$$= \sum_{n=\eta\lambda\delta} h(\eta) f(\delta) g(\lambda) = \big(f \circ (g \circ h)\big)(n)$$

Therefore $f \circ (g \circ h) = (f \circ g) \circ h$.

3.9 Property : Suppose $f \in \mathfrak{B}\ and\ f(1) \neq 0$. Then there exists unique $g \in \mathfrak{B}$ such that $f \circ g = I_{\dot{B}}$.

Proof : Define

$$g(1) = \frac{1}{f(1)} \qquad (\text{since } f(1) \neq 0)$$

So that $(f \circ g)(1) = f(1)g(1) = I_{\dot{B}}(1)$.
Define

$$g(3) = -\frac{f(3)g(1)}{f(1)} = -\frac{f(3)}{(f(1))^2}$$

So that $(f \circ g)(3) = f(1)g(3) + f(3)g(1) = 0 = I_{\dot{B}}(3)$

If $n > 1$, for $n \in \dot{B}$ we define g inductively by

$$g(n) = -\frac{1}{f(1)} \left(f(n)g(1) + \sum_{\substack{d|n \\ 1<d<n}} f(d)\, g\left(\frac{n}{d}\right) \right)$$

So that

$$(f \circ g)(n) = \textstyle\sum_{d|n} f(d)\, g\left(\frac{n}{d}\right)$$

$$= f(1)g(n) + f(n)g(1) + \sum_{\substack{d|n \\ 1<d<n}} f(d)\, g\left(\frac{n}{d}\right) = 0 = I_{\dot{B}}(n)$$

Therefore,

$$f(1)g(n) = -f(n)g(1) - \sum_{\substack{d|n \\ 1<d<n}} f(d)\, g\left(\frac{n}{d}\right)$$

Thus $(f \circ g)(n) = I_{\dot{B}}(n) \quad \forall\, n \in \dot{B}$.

Hence $f \circ g = I_{\dot{B}}$.

It can be shown that g is unique. g is denoted by f^{-1}.

Hence $\mathfrak{B}' = \{f \in \mathfrak{B}, f(1) \neq 0\}$ is a commutative group.

4. MAIN RESULTS

4.1 Definition: Suppose $f \in \mathfrak{B}$, we say that f is a partially multiplicative function if f is not identically zero and $f(m * n) = f(m)f(n)$ whenever $(m, n) \in F_{\dot{B}}$.

4.2 Theorem: If f is a partially multiplicative function then $f(1) = 1$.

Proof : Suppose f is partially multiplicative. Then

$f(1) = f(1.1) = f(1)f(1)$ since $(1, 1) \in F_{\dot{B}}$.

Therefore $f(1) = 0$ or 1.

Suppose $f(1) = 0$. Then for any $n \in \dot{B}$,

We have $f(n) = f(1.n) = f(1)f(n)$ since $(1, n) \in F_{\dot{B}}$.

Therefore, $f(n) = 0 \ \forall n \in \dot{B}$

i.e., f is identically zero on $\dot{B}$, a contradiction.

Therefore $f(1) = 1$.

4.3 Theorem: If f, g are partially multiplicative then $f \circ g$ is also partially multiplicative.

Proof : Clearly $(f \circ g)(1) = f(1)g(1) = I_{\dot{B}}(1) = 1$.

Suppose $m, n \in \dot{B}$ and $(m, n) = 1$ so that $mn \in \dot{B}$.

By Lemma 2, there exists a unique pair (δ_1, δ_2) such that $d \mid mn \Leftrightarrow d = \delta_1\delta_2$, $\delta_1 \mid m$, $\delta_2 \mid n$.

(infact $\delta_1 = (d, m), \delta_2 = (d, n)$ since $(m, n) = 1$).

Now

$$(f \circ g)(m * n) = (f \circ g)(mn) = \sum_{d \mid mn} f(d)\, g\left(\frac{mn}{d}\right)$$

$$= \sum_{\delta_1\delta_2 \mid mn} f(\delta_1\delta_2) g\left(\frac{mn}{\delta_1\delta_2}\right) \quad \text{where } \delta_1 = (d, m), \delta_2 = (d, n).$$

$$= \sum_{\delta_1\delta_2 \mid mn} f(\delta_1)f(\delta_2)\, g\left(\frac{m}{\delta_1}\right) g\left(\frac{n}{\delta_2}\right)$$

$$= \sum_{\delta_1 \mid m,\ \delta_2 \mid n} f(\delta_1)f(\delta_2) g\left(\frac{m}{\delta_1}\right) g\left(\frac{n}{\delta_2}\right)$$

$$= \sum_{\delta_1 \mid m,\ \delta_2 \mid n} f(\delta_1) g\left(\frac{m}{\delta_1}\right) f(\delta_2) g\left(\frac{n}{\delta_2}\right)$$

$$= \left(\sum_{\delta_1 \mid m} f(\delta_1) g\left(\frac{m}{\delta_1}\right)\right)\left(\sum_{\delta_2 \mid n} f(\delta_2) g\left(\frac{n}{\delta_2}\right)\right)$$

$$= (f \circ g)(m)(f \circ g)(n)$$

Therefore $(f \circ g)(m * n) = (f \circ g)(m)(f \circ g)(n)$.

Therefore $f \circ g$ is partially multiplicative.

4.4 Theorem: If f is partially multiplicative then f^{-1} is also partially multiplicative.

Proof : If $m = 1$ then

$f^{-1}(mn) = f^{-1}(1.n) = f^{-1}(n) = 1.f^{-1}(n) = f^{-1}(1)f^{-1}(n) = f^{-1}(m)f^{-1}(n)$.

Therefore $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$.

If $m = 1,\ n = 3$ then $f^{-1}(mn) = f^{-1}(1.3) = f^{-1}(3) = f^{-1}(1)f^{-1}(3)$.

Therefore $f^{-1}(mn) = f^{-1}(m)f^{-1}(n)$.

Suppose this result is true for $(m', n') \in F_{\dot{B}},\ m'n' < mn\ and\ (m, n) \in F_{\dot{B}}$.

Then $0 = (f \circ f^{-1})(m * n)$

$$= \sum_{d | mn} f(d)\, f^{-1}\left(\frac{mn}{d}\right)$$

$$= \sum_{\delta_1\delta_2 | mn} f(\delta_1\delta_2)\, f^{-1}\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= \sum_{\delta_1 | m,\ \delta_2 | n} f(\delta_1\delta_2)\, f^{-1}\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= f(1\cdot 1)f^{-1}(mn) + f(mn)f^{-1}(1\cdot 1) + \sum_{\substack{\delta_1 | m,\ \delta_2 | n \\ 1<\delta_1\delta_2<mn}} f(\delta_1\delta_2)f^{-1}\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= f(1)f^{-1}(mn) + f(mn)f^{-1}(1) + \sum_{\substack{\delta_1 | m,\ \delta_2 | n \\ 1<\delta_1\delta_2<mn}} f(\delta_1\delta_2)f^{-1}\left(\frac{m}{\delta_1}\cdot\frac{n}{\delta_2}\right)$$

$$= f(mn) + \sum_{\substack{\delta_1 | m,\ \delta_2 | n \\ 1<\delta_1\delta_2<mn}} f(\delta_1)\, f(\delta_2)\, f^{-1}\left(\frac{m}{\delta_1}\right)f^{-1}\left(\frac{n}{\delta_2}\right) + f^{-1}(mn)$$

$$= f(m)f(n) + \sum_{\substack{\delta_1 | m,\ \delta_2 | n \\ 1<\delta_1\delta_2<mn}} f(\delta_1)\, f(\delta_2)\, f^{-1}\left(\frac{m}{\delta_1}\right)f^{-1}\left(\frac{n}{\delta_2}\right)$$

$$+\left(f^{-1}(mn) - f^{-1}(m)f^{-1}(n) + f^{-1}(m)f^{-1}(n)\right)$$

$$= \sum_{\substack{\delta_1 | m,\ \delta_2 | n \\ 1\le\delta_1\delta_2\le mn}} f(\delta_1)\, f(\delta_2)\, f^{-1}\left(\frac{m}{\delta_1}\right)f^{-1}\left(\frac{n}{\delta_2}\right) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= \left(\sum_{\delta_1 | m} f(\delta_1)f^{-1}\left(\frac{m}{\delta_1}\right)\right)\left(\sum_{\delta_2 | n} f(\delta_2)f^{-1}\left(\frac{n}{\delta_2}\right)\right) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= (f \circ f^{-1})(m)\cdot(f \circ f^{-1})(n) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= I_{\dot{B}}(m)\cdot I_{\dot{B}}(n) + f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

$$= f^{-1}(mn) - f^{-1}(m)f^{-1}(n)$$

Therefore $f^{-1}(m * n) = f^{-1}(m)f^{-1}(n)$.

Therefore, f is partially multiplicative implies f^{-1} is also a partially multiplicative.

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4.5 Theorem: If f and $f \circ g$ are partially multiplicative then g is partially multiplicative.

Proof : Suppose $(m,n) \in F_{\dot{B}}$. Then

$g(m * n) = g(mn)$

$$= \left(f^{-1} \circ (f \circ g)\right)(mn)$$

$$= \sum_{d|mn} f^{-1}(d)(f \circ g)\left(\frac{mn}{d}\right)$$

$$= \sum_{\delta_1|m,\ \delta_2|n} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{mn}{\delta_1\delta_2}\right)$$

$$= f^{-1}(mn)(f \circ g)(1) + \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{mn}{\delta_1\delta_2}\right) + f^{-1}(1)(f \circ g)(mn)$$

$$= f^{-1}(mn) + (f \circ g)(mn) + \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{m}{\delta_1}\right)(f \circ g)\left(\frac{n}{\delta_2}\right)$$

$$= f^{-1}(mn) + (f \circ g)(mn) + f^{-1}(1)(f \circ g)(m)(f \circ g)(n) - f^{-1}(1)(f \circ g)(m)(f \circ g)(n)$$

$$+ f^{-1}(mn)(f \circ g)(1)(f \circ g)(1) - f^{-1}(mn)(f \circ g)(1)(f \circ g)(1)$$

$$+ \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1\delta_2)(f \circ g)\left(\frac{m}{\delta_1}\right)(f \circ g)\left(\frac{n}{\delta_2}\right)$$

$$= f^{-1}(mn) + (f \circ g)(mn) - (f \circ g)(m)(f \circ g)(n) - f^{-1}(mn)$$

$$+ \sum_{\substack{\delta_1|m,\ \delta_2|n \\ 1<\delta_1\delta_2<mn}} f^{-1}(\delta_1)f^{-1}(\delta_2)(f \circ g)\left(\frac{m}{\delta_1}\right)(f \circ g)\left(\frac{n}{\delta_2}\right)$$

$$= \left(\sum_{\delta_1|m} f(\delta_1)(f \circ g)\left(\frac{m}{\delta_1}\right)\right)\left(\sum_{\delta_2|n} f(\delta_2)(f \circ g)\left(\frac{n}{\delta_2}\right)\right)$$

$$= \left(f^{-1} \circ (f \circ g)\right)(m)\left(f^{-1} \circ (f \circ g)\right)(n)$$

$$= g(m)g(n)$$

Therefore $g(m * n) = g(m)g(n)$.

Therefore, if f and $f \circ g$ are partially multiplicative then g is partially multiplicative.

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