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Complex-valued set-indexed Brownian motion

Arthur Yosef

Tel Aviv-Yaffo Academic College, Israel.
E-mail: yusupoa@yahoo.com

Abstract

In this study, we explore the complex-valued (planar) Brownian motion in the set-indexed framework. We present the characterization of complex-value set-indexed Brownian motion (cvsiBM) by flows. In addition, we extend some selected aspects to the cvisBM for the following issues: stationary increments, Markov property, martingale, P. Levy martingale characterization, path independent variation, Holder continuity and differentiability, stopping lines, hitting time, reflection principle, etc.

Keywords: Brownian motion, set-indexed, complex-valued, flow.

MSC: 60G15, 60G17, 60G40, 60G10

1. INTRODUCTION

In this article, we study complex-valued Brownian motion when the set index $\mathbf{A}$ is a compact set collection on a topological space (T,τ). This complex-valued Brownian motion is called complex-valued set-indexed Brownian motion (cvsiBM). Complex-valued Brownian motion ([19],[18], [2], [20], [23], [22], [7] and [3]) is a centered Gaussian process $\mathcal{B}=\{\mathcal{B}_t: t\geq 0\}$ where

$$\mathcal{B}_t = X_t + iY_t$$

and X,Y are independent real-valued Brownian motions. It is well known that the complex-valued (or, $\mathbb{C}$-valued) Brownian motion and planar (or, $\mathbb{R}^2$-valued) Brownian motion are equivalent ($\mathbb{C}\cong\mathbb{R}^2$). We extend the notion to set-indexed framework: the complex-valued set-indexed process $\mathcal{B}=\{\mathcal{B}_A: A\in\mathbf{A}\}$ where

$$\mathcal{B}_A = X_A + iY_A$$

is a complex-valued set-indexed Brownian motion (cvsiBM) with variance $\sigma=\sigma_X+i\sigma_Y$, if X and Y are independent set-indexed Brownian motions with variances σ_X,σ_Y, respectively.

The frame of set-indexed Brownian motion ([17], [10], [11], [13], [9] and [5]) is not only a new step of generalization of classical Brownian motion, but it proved a new look on Brownian motion. In recent years, there have been many new results involving the dynamical properties of random processes indexed by a class sets. Set indexed processes have many potential areas of applications. For example: environment (increased occurrence of polluted wells in a rural area could indicate a geographic region that has been subjected to industrial waste), astronomy (a cluster of black holes could be a result of an unobservable phenomenon affecting a region in space), quality control (an increased rate of breakdowns in a certain type of equipment might follow the failure of one or more components), population health (unusually frequent outbreaks of a disease such as leukemia near a nuclear power plant could signal a possible air or ground contamination of the region) and the like.

In the past, there has been much interest in properties of planar Brownian motion: geometric properties of sample path ([2], [18]), asymptotic distributions ([19], [20]), multiple points and intersection problems ([7], [22], [23]), etc.

The purpose of this article is to extend numerous results from a real-valued Brownian motion to a complex-valued set-indexed Brownian motion. In particular, we present the characterization of cvsiBM and we extend some selected aspects to the cvsiBM for the following issues: Markov property, stationary increments and self-similarity, strong law of large numbers, law of iterated logarithms, unboundedness, bounded stopping lines, P. Levy martingale characterization, martingale and path independent variation, Holder continuity and differentiability, reflection principle, hitting time, etc.

2. INDEXED COLLECTION AND SET-INDEXED PROCESS

Definition 2.1. Let T denote a non-void σ-compact connected topological space. In set-indexed works (see [17], [10], [13]), processes will be indexed by a nonempty class $\mathbf{A}$ of compact connected subsets of T is called an indexed collection if it satisfies the following:

- $\emptyset\in\mathbf{A}$. In addition, there is an increasing sequence (B_n) of sets in $\mathbf{A}$ such that $T=\bigcup_{n=1}^{\infty}B_n^{\circ}$.
- $\mathbf{A}$ is closed under arbitrary intersections and if $A,B\in\mathbf{A}$ are nonempty, then $A\cap B$ is nonempty. If (A_i) is an increasing sequence in $\mathbf{A}$ and if there exists n such that $A_i\subseteq B_n$ for every i, then $\overline{\bigcup_i A_i}\in\mathbf{A}$.
- $\sigma(\mathbf{A})=\mathbf{B}$ where $\mathbf{B}$ is the collection of Borel sets of T.
- There exists an increasing sequence of finite sub-classes $\mathbf{A}_n=\{A_1^n,\dots,A_{k_n}^n\}\subseteq\mathbf{A}$ closed under intersection with $\emptyset, B_n\in\mathbf{A}_n(u)$, ($\mathbf{A}_n(u)$ is the class of union of sets in $\mathbf{A}_n$), and a sequence of functions $g_n:\mathbf{A}\to\mathbf{A}_n(u)\cup T$ such that:
    a. g_n preserves arbitrary intersections and finite unions.
    b. For each $A\in\mathbf{A}$, $A\subseteq g_n(A)^{\circ}$ and $A=$

$\cap_n g_n(A), g_n(A) \subseteq g_m(A)$ if $n \geq m$.

c. $g_n(A) \cap \hat{A} \in \mathbf{A}$ if $\hat{A}, A \in \mathbf{A}$ and $g_n(A) \cap \hat{A} \in \mathbf{A}_n$ if $A \in \mathbf{A}$ and $\hat{A} \in \mathbf{A}_n$.

d. $g_n(\emptyset) = \emptyset$ for all n.

(Note: $\overline{(\cdot)}$ and $(\cdot)^\circ$ denote respectively the closure and the interior of a set).

We will require other classes of sets generated by **A**. The first is $\mathbf{A}(\mathrm{u})$, which is the class of finite unions of sets in **A**. We remark that $\mathbf{A}(\mathrm{u})$ it itself a lattice with the partial order induced by set inclusion. Let **C** consists of all the subsets of $T$of the form

$$C = A \backslash B, A \in \mathbf{A}, B \in \mathbf{A}(\mathrm{u}).$$

In addition, let A^{ss} be any finite sub-semilattice of **A** closed under intersection. For $A \in A^{ss}$, define the left neighbourhood of A in A^{ss} to be a set $C_A = A \backslash \cup_{B \in A^{ss}, B \subset A} B$. We note that $\cup_{A \in A^{ss}} A = \cup_{A \in A^{ss}} C_A$ and that the latter union is disjoint. The sets in A^{ss} can always be numbered in the following way: $A_0 = \emptyset'$, ($\emptyset' = \cap_{A \in \mathrm{A}, A \neq \emptyset} A$, note that $\emptyset' \neq \emptyset$) and given $A_0, \ldots, A_{i-1}$, choose A_i to be any set in A^{ss} such that $A \subset A_i$ implies that $A = A_j$, some $j = 1, \ldots, i-1$. Any such numbering $A^{ss} = \{A_0, \ldots, A_k\}$ will be called "consistent with the strong past" (i.e., if C_i is the left-neighbourhood of A_i in A^{ss}, then $C_i = \cup_{j=0}^{i} A_j \backslash \cup_{j=0}^{i-1} A_j$ and $C_i \cap A_j = \emptyset$, for all $j = 0, \ldots, i-1$, $i = 1, \ldots, k$).

Any **A**-indexed function which has a (finitely) additive extension to **C** will be called additive (and is easily seen to be additive on $\mathbf{C}(\mathrm{u})$ as well, where $\mathbf{C}(\mathrm{u})$ is a class of finite unions of sets in **C**). For stochastic processes, we do not necessarily require that each sample path be additive, but additivity will be imposed in an almost sure sense: A set-indexed stochastic process $X = \{X_A : A \in \mathbf{A}\}$ is additive if it is has an (almost sure) additive extension to **C**: $X_\emptyset = 0$ and if $C, C_1, C_2 \in \mathbf{C}$ with $C = C_1 \cup C_2$ and $C_1 \cap C_2 = \emptyset$ then almost surely $X_C = X_{C_1} + X_{C_2}$. In particular, if $C \in \mathbf{C}$ and$C = A \backslash \cup_{i=1}^{n} A_i$, $A, A_1, \ldots, A_n \in \mathbf{A}$ then almost surely

$$X_C = X_A - \sum_{i=1}^{n} X_{A \cap A_i} + \sum_{i<j} X_{A \cap A_i \cap A_j} - \ldots + (-1)^n X_{A \cap \cap_{i=1}^{n} A_i}.$$

We shall always assume that our stochastic processes are additive. We note that a process with an (almost sure) additive extension to **C** also has an (almost sure) additive extension to $\mathbf{C}(\mathrm{u})$. The class $\mathbf{C}_0$ is defined as a sub-class of **C** of elements $A \backslash B$ where $A, B \in \mathbf{A}$. Since $\emptyset \in \mathbf{A}$, we have the inclusion $\mathbf{A} \subset \mathbf{C}_0 \subset \mathbf{C}$. From any set-indexed process $X = \{X_A : A \in \mathbf{A}\}$, we define the increment process $\Delta X = \{\Delta X_C : C \in \mathbf{C}\}$ by$\Delta X_C = X_{A_0} - \Delta X_{A_0 \cup \cap B_i}$ for all $C = A_0 \backslash \cap_{i=1}^{\infty} B_i$,where $\Delta X_{A_0 \cup \cap B_i}$is given by the formula $\Delta X_{A_0 \cup \cap B_i} = \sum_{i=1}^{n} \sum_{j_1 < \ldots < j_i} (-1)^{i-1} X_{A_0 \cap B_{j_1} \cap \ldots \cap B_{j_i}}$. When $C = A \backslash B \in \mathbf{C}_0$, the expression of ΔX is reduced to $\Delta X_C = \Delta X_{A \backslash B} = X_A - X_{A \cap B}$

3. ELEMENTARY PROPERTIES

Definition 3.1.

a. A positive measure μ on $(T, \mathbf{B})$ is called strictly monotone on **A** if: $\mu(\emptyset') = 0$ and $\mu(A) < \mu(B)$ for all $A \subset B, A, B \in \mathbf{A}$ where $\emptyset' = \cap_{A \in \mathrm{A}, A \neq \emptyset} A$. The collection of these measures is denoted by $M(\mathbf{A})$.

b. Let μ be a positive, strictly monotone on **A** and continuous measure in **A**. If $A \in \mathbf{A}$ and $\varepsilon > 0$ then define $D_A^\varepsilon = \{B \in \mathbf{A} : A \subseteq B, \mu(B \backslash A) = \varepsilon\}$. Denote by A^ε an element in D_A^ε and assume that $D_A^\varepsilon \neq \emptyset$.

Hereafter, we assume that the space T has a positive and continuous measure μ such that for all $A \in \mathbf{A}$ there exists a A^ε, $\mu(A^\varepsilon \backslash A) = \mu(A^\varepsilon) - \mu(A) = \varepsilon$ for all $\varepsilon > 0$.

The classical examples are: (a) $T = \mathfrak{R}_+^d$ and $\mathbf{A} = \mathbf{A}(\mathfrak{R}_+^d) = \{[0, x] : x \in \mathfrak{R}_+^d\}$ when μ is Lebesgue measure. (b) $T = \mathfrak{R}_+^d$ and $\mathbf{A} = \mathbf{A}(Ls)$ when μ is Lebesgue measure.

Notes:

a. For $B \in D_A^\varepsilon$, $\mu(B) > \varepsilon$ there exists a set $A \in \mathbf{A}$ such that $\mu(B \backslash A) = \varepsilon$, we denote A by $B^{-\varepsilon}$. In other words, for all $A \in \mathbf{A}$ ($\mu(A) > \varepsilon$) there exists a sets A^ε and $A^{-\varepsilon}$ such that $(A^\varepsilon \backslash A) = \mu(A \backslash A^{-\varepsilon}) = \varepsilon$.

b. We have several elements in D_A^ε (i.e., $A^{-\varepsilon}$ is not uniquely for $A \in \mathbf{A}$). Therefore, we must consider it in our paper.

Definition 3.2. Let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed stochastic process.

a. X is said to have ε-stationary increments if $X_{A_1^\varepsilon} - X_{A_1} \overset{d}{=} X_{A_2^\varepsilon} - X_{A_2} \overset{d}{=} \ldots \overset{d}{=} X_{A_n^\varepsilon} - X_{A_n}$ for all $\{A_k\}_{k=1}^{n} \in \mathbf{A}$, for all $\varepsilon > 0$ and for all $A_k^\varepsilon \in D_{A_k}^\varepsilon$. The notation $\overset{d}{=}$ mean identical distribution.

b. Let $\mu \in M(\mathbf{A})$. X is said to have $\mathbf{C}_0$-stationary increments if $\forall i$, $\mu(A_i \backslash B_i) = \mu(D_i)$ then $\Delta X_{A_i \backslash B_i} \overset{d}{=} \Delta X_{D_i}$ for all $\{A_i \backslash B_i\}_{i=1}^{n} \in \mathbf{C}_0$, $\{D_i\}_{i=1}^{n} \in \mathbf{A}$.

Definition 3.3.

a. Let $\sigma \in M(\mathbf{A})$. We say that the set-indexed process X is a set-indexed Brownian motion (siBM) with variance σ if X can be extended to a finitely additive process on $\mathbf{C}(\mathrm{u})$ and if for disjoint sets $C_1, \ldots, C_n \in \mathbf{C}$, $X_{C_1}, \ldots, X_{C_n}$are independent mean-zero Gaussian random variables with variances $\sigma_{C_1}, \ldots, \sigma_{C_n}$, respectively. (For any $\sigma \in M(\mathbf{A})$, there exists a set-indexed Brownian motion with variance σ [13]).

b. Let $\sigma_X, \sigma_Y \in M(\mathbf{A})$. We say that the complex-valued set-indexed process $\mathcal{B} = X + iY$ is a complex-valued set-indexed Brownian motion (cvsiBM) with variance $\sigma = \sigma_X + \sigma_Y$, if X and Y are independent set-indexed Brownian motions with variances σ_X, σ_Y, respectively.

Note: The following statements are equivalent:

- $\mathcal{B} = X + iY$ is a $\mathbb{C}$-valued (complex-valued) set-indexed Brownian motion with variance $\sigma = \sigma_X + \sigma_Y$
- (X, Y) is a $\mathbb{R}^2$-valued set-indexed Brownian motion with variance $\sigma = (\sigma_X, \sigma_Y)$

- X and Y are independent set-indexed Brownian motions with variances σ_X, σ_Y, respectively.

Definition 3.4. Let $\mu \in M(\mathbf{A})$ and $(G,\cdot)$ is a group. A group action $*$ of $(G,\cdot)$ on $\mathbf{A}$ is defined by: $g*(A\cup B) = g*A \cup g*B$, $g*(A\backslash B) = g*A \backslash g*B$ for all $A, B \in \mathbf{A}, g \in G$ and there exists $\eta: G \to \Re_+$ such that $\mu(g*A) = \eta(g)\mu(A)$ for all $A \in \mathbf{A}, g \in G$.

(The definition and more details about group action on $\mathbf{A}$ appears in [26]).

Theorem 3.1. The cvsiBM $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ with variance $\sigma = \sigma_X + \sigma_Y, \sigma_X, \sigma_Y \in M(\mathbf{A})$ satisfies the following properties:

a. $\mathrm{Cov}(\mathcal{B}_A, \mathcal{B}_B) = \sigma_X(A \cap B) + \sigma_Y(A\cap B)$ where $\overline{\mathcal{B}} = X - iY$, for all $A, B \in \mathbf{A}$

b. $\mathcal{B}$ has a ε-stationary increments. (i.e., $\mathcal{B}_{A_1^\varepsilon} - \mathcal{B}_{A_1} \stackrel{d}{=} \ldots \stackrel{d}{=} \mathcal{B}_{A_n^\varepsilon} - \mathcal{B}_{A_n}$ for all $\{A_k\}_{k=1}^n \in \mathbf{A}$, for all $\varepsilon > 0$ and for all $A_k^\varepsilon \in D_{A_k}^\varepsilon$).

c. $\mathcal{B}$ has a $\mathbf{C}_0$-stationary increments. (i.e., $\Delta\mathcal{B}_{A_i\backslash B_i} \stackrel{d}{=} \Delta\mathcal{B}_{D_i}$ where $\sigma(A_i\backslash B_i) = \sigma(D_i)$ for all $\{A_i\backslash B_i\}_{i=1}^n \in \mathbf{C}_0$, $\{D_i\}_{i=1}^n \in \mathbf{A}$).

d. $\mathcal{B}$ is a Markov process.

e. (Markov property)

   1. Let $B \in \mathbf{A}$ then $\widehat{\mathcal{B}} = \{\widehat{\mathcal{B}}_A : A \in \mathbf{A}\}$ is a cvsiBM where $\widehat{\mathcal{B}} = \mathcal{B}_{B\cup A} - \mathcal{B}_{B\backslash A}$

   2. Let $g \in G^\uparrow$ then $\widehat{\mathcal{B}} = \{\widehat{\mathcal{B}}_A : A \in \mathbf{A}\}$ is a cvsiBM where $(G,\cdot)$ is a group, $*$ is a group action of $(G,\cdot)$ on $\mathbf{A}$, $\widehat{\mathcal{B}}_A = \mathcal{B}_{g*A} - \mathcal{B}_{g*A\backslash A}$ and $G^\uparrow = \{g \in G : \forall A, B \in \mathbf{A}, A \subseteq B \implies g*A \subseteq g*B\}$.

Proof.

The processes X, Y are independent and mean-zero Gaussian, then

a. $\mathrm{Cov}(\mathcal{B}_A, \mathcal{B}_B) = \mathrm{E}(\mathcal{B}_A\overline{\mathcal{B}}_B) = \mathrm{E}(X_AX_B) + \mathrm{E}(Y_AY_B) = \sigma_X(A\cap B) + \sigma_Y(A \cap B)$.

b. Let $A_k, A_j \in \mathbf{A}$. $\mathcal{B}_{A_k^\varepsilon} - \mathcal{B}_{A_k}$ and $\mathcal{B}_{A_j^\varepsilon} - \mathcal{B}_{A_j}$ are Gaussian and centered.

$$
\begin{aligned}
Var\left(\mathcal{B}_{A_k^\varepsilon} - \mathcal{B}_{A_k}\right) &= E\left[(\mathcal{B}_{A_k^\varepsilon} - \mathcal{B}_{A_k})(\overline{\mathcal{B}_{A_k^\varepsilon} - \mathcal{B}_{A_k}})\right] \\
&= E\left[(X_{A_k^\varepsilon} - X_{A_k})^2 + (Y_{A_k^\varepsilon} - Y_{A_k})^2\right] \\
&= [\sigma_X(A_k^\varepsilon) - 2\sigma_X(A_k^\varepsilon \cap A_k) + \sigma_X(A_k)] \\
&\quad + [\sigma_Y(A_k^\varepsilon) - 2\sigma_Y(A_k^\varepsilon \cap A_k) + \sigma_Y(A_k)] \\
&= 2\varepsilon = Var\left(\mathcal{B}_{A_j^\varepsilon} - \mathcal{B}_{A_j}\right).
\end{aligned}
$$

Therefore, the two mean-zero Gaussian processes $\mathcal{B}_{A_k^\varepsilon} - \mathcal{B}_{A_k}$ and $\mathcal{B}_{A_j^\varepsilon} - \mathcal{B}_{A_j}$ have the same law, for all $A_k, A_j \in \mathbf{A}$.

c. Let $\{A_i\backslash B_i\}_{i=1}^n \in \mathbf{C}_0, \{D_i\}_{i=1}^n \in \mathbf{A}$ such that $\sigma(A_i\backslash B_i) = \sigma(D_i)$ for all i. $\Delta\mathcal{B}_{A_k\backslash B_k}$ and $\Delta\mathcal{B}_{D_k}$ are Gaussian and centered. Therefore, one only has to prove that they have the same variance function.

$$
\begin{aligned}
Var(\Delta\mathcal{B}_{A_k\backslash B_k}) &= E\left[(\mathcal{B}_{A_k} - \mathcal{B}_{A_k\cap B_k})\overline{(\mathcal{B}_{A_k} - \mathcal{B}_{A_k\cap B_k})}\right] \\
&= E\big[(X_{A_k} - X_{A_k\cap B_k})^2 \\
&\quad + (Y_{A_k} - Y_{A_k\cap B_k})^2\big]
\end{aligned}
$$

$$
= [\sigma_X(A_k) - 2\sigma_X(A_k\cap B_k) + \sigma_X(A_k \cap B_k)] + [\sigma_Y(A_k) - 2\sigma_Y(A_k\cap B_k) + \sigma_Y(A_k\cap B_k)
$$

$= \sigma_X(A_k\backslash B_k) + \sigma_Y(A_k\backslash B_k) = \sigma_X(D_k) + \sigma_Y(D_k) = Var(\Delta\mathcal{B}_{D_k})$. Then, $\Delta\mathcal{B}_{A_k\backslash B_k} \stackrel{d}{=} \Delta\mathcal{B}_{D_k}$.

d. X and Y are independent set-indexed Brownian motions with variances σ_X, σ_Y, respectively then X and Y are Markov processes. Thus, $\mathcal{B}$ is a Markov process.

e. Easy to see that $\hat{X} = Re(\widehat{\mathcal{B}}), \hat{Y} = Im(\widehat{\mathcal{B}})$ can be extended to a finitely additive processes on $\mathbf{C}(\mathrm{u})$ and for disjoint sets $C_1, \ldots, C_n \in \mathbf{C}$, $\hat{X}_{C_1}, \ldots, \hat{X}_{C_n}$ and $\hat{Y}_{C_1}, \ldots, \hat{Y}_{C_n}$ are independent mean-zero Gaussian random variables. Enough to prove that $Var(\hat{X}_A) = \sigma_X(A), Var(\hat{Y}_A) = \sigma_Y(A)$ for all $A \in \mathbf{A}$.

   1. If $\hat{X} = X_{B\cup A} - X_{B\backslash A}, \hat{Y}_A = Y_{B\cup A} - Y_{B\backslash A}$ then

   $Var(\hat{X}_A) = Var(X_{B\cup A} - X_{B\backslash A}) = Var(X_{B\cup A}) + Var(X_{B\backslash A}) - 2E(X_{B\cup A}X_{B\backslash A}) =$

   $= \sigma_X(B\cup A) + \sigma_X(B\backslash A) - \sigma_X((B\cup A)\cap(B\backslash A)) =$

   $$
   \begin{aligned}
   &= \sigma_X(B\cup A) + \sigma_X(B\backslash A) - 2\sigma_X(B\backslash A) \\
   &= \sigma_X(B\cup A) - \sigma_X(B\backslash A) \\
   &= \sigma_X(A)
   \end{aligned}
   $$

   Similarly, it can be shown that $Var(\hat{Y}_A) = \sigma_Y(A)$

   2. If $\hat{X}_A = X_{g*A} - X_{g*A\backslash A}$, $\hat{Y}_A = Y_{g*A} - Y_{g*A\backslash A}$ then

   $Var(\hat{X}_A) = Var(X_{g*A} - X_{g*A\backslash A}) = Var(X_{g*A}) + Var(X_{g*A\backslash A}) - 2E(X_{g*A}X_{g*A\backslash A}) =$

   $= \sigma_X(g*A) + \sigma_X(g*A\backslash A) - \sigma_X((g*A)\cap(g*A\backslash A)) =$

   $$
   \begin{aligned}
   &= \sigma_X(g*A) + \sigma_X(g*A\backslash A) - 2\sigma_X(g*A\backslash A) \\
   &= \sigma_X(g*A) - \sigma_X(g*A\backslash A) \\
   &= \sigma_X(A)
   \end{aligned}
   $$

   Similarly, it can be shown that $Var(\hat{Y}_A) = \sigma_Y(A)$. □

The notation of flow is the key to reduce the proof of many theorems. It was introduced in [4], extensively studied in [13] and used by several authors [5], [9].

Definition 3.5. A increasing function $f:[a,b]\rightarrow \mathbf{A}(\mathrm{u})$ is called a flow, if

- f is a continuous flow
- $f(s)=\overline{\cup_{u<s} f(u)}=\cap_{v>s} f(v)$ for all $s\in(a,b)$
- $f(a)=\cap_{v>a} f(v), f(b)=\overline{\cup_{u<b} f(u)}$.

Given a set indexed stochastic process X and the flow $f:[a,b]\rightarrow \mathbf{A}(\mathrm{u})$, we wish to define a process X^f indexed by $[a,b]$ as follows: $X_{f(s)}=X_s^f$ for all $s\in[a,b]$.

Lemma 3.1: Let $A^{ss}=\{\emptyset'=A_0,\dots,A_k\}$ be any finite sub-semilattice of $\mathbf{A}$ equipped with a numbering consistent with the strong past where $\emptyset'=\cap_{A\in\mathrm{A},A\neq\emptyset} A$. Then there exists a flow $f:[0,k]\rightarrow \mathbf{A}(\mathrm{u})$ such that the following are satisfied:

1. $f(0)=\emptyset', f(k)=\cup_{j=0}^{k} A_j$
2. Each left-neighborhood C generated by A^{ss} is of the form $C=f(i)/f(i-1)$ for all and if $C=f(t)/fs)$ then $C\in\mathbf{C}(\mathrm{u})$, $F_{f(s)}\in \mathrm{G}_C^*$ where $\mathrm{G}_C^*=\bigvee_{A\in\mathrm{A(u)},A\cap C} F_A$.

The proof appears in [13].

Theorem 3.2 (The characterization of siBM motion by flows): Let $X=\{X_A: A\in\mathbf{A}\}$ be a square-integrable set-indexed stochastic process. Let $\sigma\in M(\mathbf{A})$ then X is a set-indexed Brownian motion with variance σ if and only if the process X^f is a time-change Brownian motion, for all continuous flows $f:[a,b]\rightarrow\mathbf{A}(\mathrm{u})$. (The process X^f is called a time-change Brownian motion if there exists $\theta:[0,\infty)\rightarrow[a,b]$ such that $X^{f\circ\theta}$ is a Brownian motion, for some a continuous flow $f:[a,b]\rightarrow\mathbf{A}(\mathrm{u})$)

You can see the proof in [17].

Based on Theorem 3.2, we derive:

Theorem 3.3. (The characterization of cvsiBM by flows) $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A: A\in\mathbf{A}\}$ be a square-integrable set-indexed complex stochastic process. $\mathcal{B}$ is a cvsiBM with variance $\sigma=\sigma_X+\sigma_Y$ if and only if $\mathcal{B}^f=X^f+iY^f$ is a complex-valued time-change Brownian motion with variance $\sigma^f=\sigma_X^f+\sigma_X^f$, where $\sigma^f(t)=\sigma(f(t))$ (i.e., there exist $\theta_X,\theta_Y:[0,\infty)\rightarrow[a,b]$ such that $\mathcal{B}^{f,\theta_X,\theta_Y}=X^{f\circ\theta_X}+iY^{f\circ\theta_Y}$ is a complex-valued Brownian motion, for all continuous flows $f:[a,b]\rightarrow\mathbf{A}(\mathrm{u})$).

Definition 3.6. Let $\{A_i\}_{i=1}^{\infty}$ be an increasing sequence in $\mathbf{A}(\mathrm{u})$. We write $A_i\uparrow T$ if $A_i\neq T$ for all i and if $\overline{\cup_\iota A_\iota}=T$.

According to Theorem 3.3 and [26], we conclude:

Lemma 3.2. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A: A\in\mathbf{A}\}$ be a cvsiBM with variance $\sigma=\sigma_X+\sigma_Y,\sigma_X,\sigma_Y\in M(\mathbf{A})$.

1. If $\{A_i\}_{i=1}^{k}$ be an increasing sequence in A(u) then there exists a continuous flow $f:[0,k]\rightarrow \mathrm{A(u)}$, $f(0)=\emptyset'$ and $f(i)=A_i$ for all $1\le i\le k$, such that $\mathcal{B}^f$ is a complex-valued time-change Brownian motion.
2. If $A_i\uparrow T$ then there exists a continuous flow $f:[0,\infty)\rightarrow\mathbf{A}(\mathrm{u})$, $f(0)=\emptyset'$ and $f(i)=A_i$ for all $1\le i$, such that $\mathcal{B}^f$ is a complex-valued time-change Brownian motion.

Corollary 3.1. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A: A\in\mathbf{A}\}$ be a cvsiBM with $\sigma=\sigma_X+\sigma_Y,\sigma_X,\sigma_Y\in M(\mathbf{A})$. If $\sigma_X(A)=\sigma_Y(A)=\mu(A)$ for all $A\in\mathbf{A}$ then

a. (Strong law of large numbers) $\lim\limits_{A\uparrow T}\frac{\mathcal{B}_A}{\mu(A)}=0$, almost surely for all $A_i\uparrow T$.

b. (Law of iterated logarithms) $\underline{\lim}_{A\uparrow T}\frac{\mathcal{B}_A}{\sqrt{2\mu(A)\ln\ln(\mu(A))}}=-1-i$ and $\overline{\lim}_{A\uparrow T}\frac{\mathcal{B}_A}{\sqrt{2\mu(A)\ln\ln(\mu(A))}}=1+i$ almost surely and for all $A_i\uparrow T$.

Proof.

Let $A_i\uparrow T$. $\mathcal{B}$ is a cvsiBM then based on Lemma 3.2, there exist continuous functions $\theta_X,\theta_Y:[0,\infty)\rightarrow[a,b]$ and $0<t_i$ such that $\mathcal{B}^{f,\theta_X,\theta_Y}=X^{f\circ\theta_X}+iY^{f\circ\theta_Y}$ is a complex-valued Brownian motion and $A_i=f(i)=f(\theta_X(t_i))=f(\theta_Y(t_i))$, $\sigma_X(A_i)=\sigma_Y(A_i)=\mu(A_i)=t_i$.

a. We recall that, if $W=\{W_t: t\ge 0\}$ is a classical Brownian motion then $\lim\limits_{t\to\infty}\frac{W_t}{t}=0$. Then,

$$\lim_{A\uparrow T}\frac{\mathcal{B}_A}{\mu(A)}=\lim_{t_i\to\infty}\frac{X_{t_i}^{f\circ\theta_X}}{t_i}+i\lim_{t\to\infty}\frac{Y_{t_i}^{f\circ\theta_Y}}{t_i}=0+i0=0$$

b. We recall that, if $W=\{W_t: t\ge 0\}$ is a classical Brownian motion then $\underline{\lim}_{t\to\infty}\frac{W_t}{\sqrt{2t\ln\ln(t)}}=-1$ and $\overline{\lim}_{t\to\infty}\frac{W_t}{\sqrt{2t\ln\ln(t)}}=1$. Then, $\underline{\lim}_{A\uparrow T}\frac{\mathcal{B}_A}{\sqrt{2\mu(A)\ln\ln(\mu(A))}}=\underline{\lim}_{t_i\to\infty}\frac{X_{t_i}^{f\circ\theta_X}}{\sqrt{2t_i\ln\ln(t_i)}}+i\,\underline{\lim}_{t_i\to\infty}\frac{Y_{t_i}^{f\circ\theta_Y}}{\sqrt{2t_i\ln\ln(t_i)}}=-1-i$ and $\overline{\lim}_{A\uparrow T}\frac{\mathcal{B}_A}{\sqrt{2\mu(A)\ln\ln(\mu(A))}}=\overline{\lim}_{t_i\to\infty}\frac{X_{t_i}^{f\circ\theta_X}}{\sqrt{2t_i\ln\ln(t_i)}}+i\,\overline{\lim}_{t_i\to\infty}\frac{Y_{t_i}^{f\circ\theta_Y}}{\sqrt{2t_i\ln\ln(t_i)}}=1+i$. □

4. MARTINGALES

Let (Ω,F,P) be a complete probability space equipped with an $\mathbf{A}$ indexed filtration $\{F_A: A\in\mathbf{A}\}$ which satisfies the following conditions: $\forall A\in\mathbf{A}$, we have $F_A\subseteq F$ and F_A contains the P-null sets, $\forall A,B\in\mathbf{A}, F_A\subseteq F_B$, if $A\subseteq B$, $F_{\cap A_i}=\cap F_{A_i}$, for any decreasing sequence (A_i) in $\mathbf{A}$.

Let $X=\{X_A: A\in\mathbf{A}\}$ be an integrable additive set-indexed stochastic process and adapted with respect to filtration $F=\{F_A: A\in\mathbf{A}\}$.

- X is said to be a martingale if for any $A, B \in \mathbf{A}$ such that $A \subseteq B$, we have $E[X_B|F_A] = X_A$.
- X is said to be a strong martingale if for any $C \in \mathbf{C}$, we have $E[X_C|\mathrm{G}_C^*] = 0$

For a study of the different kinds of martingales see [16], [27].

Definition 4.2. Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ be an integrable additive complex-valued set-indexed stochastic process and adapted with respect to filtration $F = \{F_A : A \in \mathbf{A}\}$. $\mathcal{B}$ is said to be a martingale (strong martingale) if X, Y are martingales (strong martingales).

From the well-known P. Levy martingale characterization of the Brownian motion (see [6], [21]), we derive:

Theorem 4.1. Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ be a square-integrable (L^2) complex-valued set-indexed martingale with $B_{\emptyset'} = 0$ which is inner- and outer-continuous. Let $\sigma = \sigma_X + \sigma_Y, \sigma_X, \sigma_Y \in M(\mathbf{A})$ then $\mathcal{B} = X + iY$ is a cvsiBM with variance $\sigma = \sigma_X + \sigma_Y$ if and only if $\langle X^f \rangle$ and $\langle Y^f \rangle$ are independent and deterministic, for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$. (Note: Under some hypothesis, we can define $\langle X \rangle$ to be compensator associated with the sub-martingale X^2. The definition and more details about $\langle X \rangle$ and inner- and outer-continuous can be found in [13]).

Proof.

(if) Suppose that $\mathcal{B} = X + iY$ is a cvsiBM. Based on Theorem 3.2 we conclude that the processes X^f and Y^f are independent and time-change Brownian motions, for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$. Then from P. Levy characterization we get that $\langle X^f \rangle$ and $\langle Y^f \rangle$ are independent and deterministic for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$.

(only if) Suppose that $\langle X^f \rangle$ and $\langle Y^f \rangle$ are independent and deterministic for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$. Since $\mathcal{B}$ is inner- and outer-continuous (i.e., X, Y are inner- and outer-continuous), X^f and Y^f continuous (see [13]). $\mathcal{B}$ is a set-indexed martingale (i.e., X, Y are martingales) then $\langle X^f \rangle$ and $\langle Y^f \rangle$ are martingales. But if the processes X^f and Y^f are martingales and $\langle X^f \rangle$ and $\langle Y^f \rangle$ are independent and deterministic then from P. Levy characterization we conclude that X^f and Y^f are time-changed Brownian motions, for all continuous flows f. Thus, based on Theorem 3.3 we derive, $\mathcal{B} = X + iY$ is a cvsiBM. □

Theorem 4.2. Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ be a square-integrable complex-valued set-indexed martingale with $\mathcal{B}_{\emptyset'} = 0$ which is inner- and outer-continuous. Let $\sigma = \sigma_X + \sigma_Y, \sigma_X, \sigma_Y \in M(\mathbf{A})$ then $\mathcal{B} = X + iY$ is cvsiBM with variance $\sigma = \sigma_X + \sigma_Y$ if and only if for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$,

a. The process $\mathcal{B}^f$ has independent increment and also has time-changed stationary increments ($\mathcal{B}^f$ has time-changed stationary increments if there exist continuous functions $\theta_X, \theta_Y : [0, \infty) \to [a, b]$ such that $\mathcal{B}^{f, \theta_X, \theta_Y} = X^{f \circ \theta_X} + iY^{f \circ \theta_Y}$ has stationary increments).

b. The processes X^f and Y^f are independent.

Proof.

(if) It is clear.

(only if) Suppose that X^f has independent increment and also has time-changed stationary increments, for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$. Since $\mathcal{B} = X + iY$ is inner- and outer-continuous (i.e., X, Y are inner- and outer-continuous), X^f continuous (see [13]). The process X^f has independent increment and there exists a continuous function $\theta_X : [0, \infty) \to [a, b]$ such that $X^{f \circ \theta_X}$ has stationary increments then X^f is a time-change Brownian motion for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$. Thus, based on Theorem 3.2 we conclude that X is a set-indexed Brownian motion with variance σ_X. Similarly, it can be shown that Y is a set-indexed Brownian motion with variance σ_Y. □

Definition 4.2 : Let $X = \{X_A : A \in \mathbf{A}\}$ be a square-integrable set-indexed martingale. X is said to have path independent variation on $\mathbf{A}(\mathrm{u})$ (or in shortly, path independent variation ($p.i.v.$)) if for all continuous flows $f_1, f_2 : [a, b] \to \mathbf{A}(\mathrm{u})$ with $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$, then $\langle X^{f_1} \rangle(b) = \langle X^{f_2} \rangle(b)$.

Remark: This definition of $p.i.v.$ was introduced by Cairoli and Walsh in the plane [4].

Theorem 4.3. Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ be a cvsiBM with variance $\sigma = \sigma_X + \sigma_Y, \sigma_X, \sigma_Y \in M(\mathbf{A})$ then $\mathcal{B}$ has $p.i.v.$ and $\langle \mathcal{B} \rangle(A) = \sigma(A)$ for all $A \in \mathbf{A}$.

Proof.

$\mathcal{B}$ is a cvsiBM then based on Theorem 3.3, $\mathcal{B}^f$ is a complex-valued time-change Brownian motion with variance $\sigma^f = \sigma_X^f + \sigma_Y^f$, where $\sigma^f = \sigma \circ f$, for all continuous flows $f : [a, b] \to \mathbf{A}(\mathrm{u})$ (i.e., there exist continuous functions $\theta_X, \theta_Y : [0, \infty) \to [a, b]$ such that $\mathcal{B}^{f, \theta_X, \theta_Y} = X^{f \circ \theta_X} + iY^{f \circ \theta_Y}$ is a complex-valued Brownian motion). Thus, $\langle \mathcal{B}^{f, \theta_X, \theta_Y} \rangle$ is deterministic and in particular

$$\langle \mathcal{B}^{f, \theta_X, \theta_Y} \rangle_t = \sigma_X \Big(\big(f(\theta_X((t)) \big) \Big) + \sigma_Y \Big(\big(f(\theta_Y((t)) \big) \Big) \text{ for all } 0 \le t.$$

Let $f_1, f_2 : [a, b] \to \mathbf{A}(\mathrm{u})$ be two continuous flows when $f_1(a) = f_2(a)$ and $f_1(b) = f_2(b)$. Then there exist continuous functions $\theta_X^1, \theta_X^2, \theta_Y^1, \theta_Y^2 : [0, \infty) \to [a, b]$ and $0 \le t_1, t_2$ such that

$$f_1(b) = f_1(\theta_X^1(t_1)) = f_1(\theta_Y^1(t_1)) = f_2(b) = f_2(\theta_X^2(t_2)) = f_2(\theta_Y^2(t_2)).$$

Thus,

$$\langle \mathcal{B}^{f_1} \rangle(b) = \langle \mathcal{B}^{f, \theta_X^1, \theta_Y^1} \rangle_{t_1} = \langle \mathcal{B}^{f, \theta_X^2, \theta_Y^2} \rangle_{t_2} = \langle \mathcal{B}^{f_2} \rangle(b)$$

Let $A \in \mathbf{A}$. Based on Lemma 3.2 , there exist a continuous flow $f:[a,b]\to \mathbf{A}(\mathrm{u})$ and $a\le s\le b$ such that $f(s)=A$ (i.e., there exist continuous functions $\theta_X,\theta_Y:[0,\infty)\to[a,b]$ and $0\le r$ such that $f(s)=f(\theta_X(r))=f(\theta_Y(r))=A$). Then,

$$\langle\mathcal{B}\rangle(A)=\langle\mathcal{B}^f\rangle_s=\langle\mathcal{B}^{f,\theta_X,\theta_Y}\rangle_r=\sigma_X\left(\left(f\left(\theta_X\left((r)\right)\right)\right)+\sigma_Y\left(\left(f\left(\theta_Y\left((r)\right)\right)\right)=\sigma(f(s))=\sigma(A).\ \square$$

Corollary 4.1. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A:A\in\mathbf{A}\}$ be a complex-valued set-indexed square-integrable which is inner- and outer-continuous. If $\mathcal{B}^f$ has independent increment and also has time-changed stationary increments then $\mathcal{B}$ has $p.i.v.$

The main idea of the proof is: $\mathcal{B}$ is inner- and outer-continuous then $\mathcal{B}^f$ continuous. The process $\mathcal{B}^f$ has independent increment and there exist continuous functions $\theta_X,\theta_Y:[0,\infty)\to[a,b]$ such that $\mathcal{B}^{f,\theta_X,\theta_Y}=X^{f\circ\theta_X}+iY^{f\circ\theta_Y}$ has stationary increments, then $\mathcal{B}^{f,\theta_X,\theta_Y}$ is a time-change Brownian motion. Based on Theorem 4.3, $\mathcal{B}$ has $p.i.v.$

Theorem 4.4. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A:A\in\mathbf{A}\}$ be a complex-valued set-indexed square-integrable then $\mathcal{B}$ has $p.i.v.$ if and only if $(X^2)^f-\langle X\rangle^f,(Y^2)^f-\langle Y\rangle^f$ are martingales, for all continuous flows $f:[a,b]\to\mathbf{A}(\mathrm{u})$.

Proof.

(if) Clear.

(only if) Let $A\subseteq B$, $A,B\in\mathbf{A}(\mathrm{u})$. According to Lemma 3.1, there exist a continuous flow $f:[a,b]\to\mathbf{A}(\mathrm{u})$ and $a\le r\le s\le b$ such that $f(r)=A, f(s)=B$. But, $(X^2)^f-\langle X\rangle^f,(Y^2)^f-\langle Y\rangle^f$ are martingales for all continuous flows $f:[a,b]\to\mathbf{A}(\mathrm{u})$ then

$$E\left[X_{f(r)}^2-X_{f(s)}^2\middle|F_{f(r)}\right]=E\left[\langle X\rangle_{f(r)}-\langle X\rangle_{f(s)}\middle|F_{f(r)}\right]$$

$$E\left[Y_{f(r)}^2-Y_{f(s)}^2\middle|F_{f(r)}\right]=E\left[\langle X\rangle_{f(r)}-\langle Y\rangle_{f(s)}\middle|F_{f(r)}\right]$$

and

$$E[X_A^2-X_B^2|F_A]=E\left[X_{f(r)}^2-X_{f(s)}^2\middle|F_{f(r)}\right]=E\left[\langle X\rangle_{f(r)}-\langle X\rangle_{f(s)}\middle|F_{f(r)}\right]=E[\langle X\rangle_A-\langle X\rangle_B|F_A]$$

$$E[Y_A^2-Y_B^2|F_A]=E\left[Y_{f(r)}^2-Y_{f(s)}^2\middle|F_{f(r)}\right]=E\left[\langle Y\rangle_{f(r)}-\langle Y\rangle_{f(s)}\middle|F_{f(r)}\right]=E[\langle Y\rangle_A-\langle Y\rangle_B|F_A]$$

Thus, X,Y have $p.i.v.$ (i.e., $\mathcal{B}$ has $p.i.v$). $\square$

Corollary 4.2. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A:A\in\mathbf{A}\}$ be a cvsiBM with variance $\sigma=\sigma_X+\sigma_Y$.

Then

$$\|\mathcal{B}\|^2-\langle Re(\mathcal{B})\rangle-\langle Im(\mathcal{B})\rangle \text{ is a martingale}$$

(i.e., $\mathcal{B}\overline{\mathcal{B}}-\langle X\rangle-\langle Y\rangle$ is a martingale) where $\sigma_X,\sigma_Y\in M(\mathbf{A})$.

The main idea of the proof is: $\mathcal{B}$ is a cvsiBM. Based on proof of Theorem 4.4, $X^2-\langle X\rangle$ and $Y^2-\langle Y\rangle$ are martingales. Thus, $\|\mathcal{B}\|^2-\langle Re(\mathcal{B})\rangle-\langle Im(\mathcal{B})\rangle=X^2-\langle X\rangle+Y^2-\langle Y\rangle$ is a martingale.

Theorem 4.5. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A:A\in\mathbf{A}\}$ be an inner- and outer-continuous complex-valued set-indexed strong martingale such that $\sup_{A\in\mathbf{A}}\mathrm{E}\|\mathcal{B}_A\|^4<\infty$ then $\mathcal{B}$ has $p.i.v.$ (Definition of inner- and outer-continuous can be found in [13])

Proof.

$\mathcal{B}$ is an inner- and outer-continuous complex-valued set-indexed strong martingale then X is an inner- and outer-continuous complex-valued set-indexed strong martingale. It is clear that

$$\sup_{A\in\mathbf{A}}\mathrm{E}\|X_A\|^4\le\sup_{A\in\mathbf{A}}\mathrm{E}\|\mathcal{B}_A\|^4<\infty$$

Then based on [12], X has $p.i.v.$ Similarly, it can be shown that Y has $p.i.v.$ Then $\mathcal{B}$ has $p.i.v.$ $\square$

5. SAMPLE PATH PROPERTIES

In order to study the Holder-continuity of set-indexed processes, we consider a distance on the indexing collection. We may sometimes specify the distance on $\mathbf{A}$ that we are using. The pseudo-distance $d_{\mathbf{A}}$ defined by $\forall A,B\in\mathbf{A}$, $d_{\mathbf{A}}(A,B)=\sigma(A\Delta B)$ where σ is the measure on T and Δ denotes the symmetric difference of sets. Without further assumptions than those of Definition 2.1, $\mathbf{A}$ is not totally bounded.

Assumption $\mathrm{H}_{\mathbb{A}}$. [12] Let $d_{\mathbf{A}}$ be a pseudo-distance on the indexing collection $\mathbf{A}$. Let us suppose that for $\mathbb{A}=\{\mathbf{A}_n\}_{n\in\mathbb{N}}$, $\mathbf{A}_n=\{A_1^n,\dots,A_{k_n}^n\}\subseteq\mathbf{A}$, there exist positive real numbers $q_{\mathbb{A}}$ and $M>0$ such that:

a. For all $n\in\mathbb{N}$, $\sup_{A\in\mathbf{A}_n}d_{\mathbf{A}}(A,g_n(A))\le Mk_n^{-\frac{1}{q_{\mathbb{A}}}}$

b. The collection $\{\mathbf{A}_n\}_{n\in\mathbb{N}}$ is minimal in the sense that: setting for all $n\in\mathbb{N}$ and all $A\in\mathbf{A}_n$,

$$\mathcal{V}_n(A)=\left\{B\in\mathbf{A}_n:B\supset A,d_{\mathbf{A}}(A,g_n(A))\le 3Mk_n^{-\frac{1}{q_{\mathbb{A}}}}\right\}\text{ and }\forall\delta>0,\ \sum_{n=1}^{\infty}k_n^{-\delta}\max_{A\in\mathbf{A}_n}|\mathcal{V}_n(A)|<\infty.$$

(For more details see [12])

Continuity

Definition 5.1. Let $d_{\mathbf{A}}$ be a pseudo-distance on the indexing collection $\mathbf{A}$, whose subclasses $\mathbb{A}=\{\mathbf{A}_n\}_{n\in\mathbb{N}}$, $\mathbf{A}_n=\{A_1^n,\dots,A_{k_n}^n\}\subseteq\mathbf{A}$ satisfy Assumption $\mathrm{H}_{\mathbb{A}}$. Let $\mathcal{B}=\{\mathcal{B}_A=X_A+iY_A:A\in\mathbf{A}\}$ be a cvsiBM.

a. $\mathcal{B}$ is said to be (α,ε)-Holder continuous at $A\in\mathbf{A}$ if there exist $M>0$ such that $\|\mathcal{B}_{A^\varepsilon}-\mathcal{B}_A\|\le M\varepsilon^\alpha$ for all $0<\varepsilon$, for all $A^\varepsilon \in D_A^\varepsilon$ and $0 < \alpha \leq 1$.

b. $\mathcal{B}$ is said to be α-Holder continuous at $A \in \mathbf{A}$ if there exist $M > 0, \delta > 0$ such that for all $B \in \mathbf{A}$ with $d_{\mathbf{A}}(A, B) < \delta$, $\|\mathcal{B}_A - \mathcal{B}_B\| \leq M d_{\mathbf{A}}(A, B)^\alpha$.

Theorem 5.1. (α-Holder continuity) Let $d_{\mathbf{A}}$ be a pseudo-distance on the indexing collection $\mathbf{A}$, whose subclasses $\mathbb{A} = \{\mathbf{A}_n\}_{n\in\mathbb{N}}, \mathbf{A}_n = \{A_1^n, \dots, A_{k_n}^n\} \subseteq \mathbf{A}$ satisfy Assumption $\mathrm{H}_{\mathbb{A}}$. Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ be a cvsiBM.

a. If $0 < \alpha < \frac{1}{2}$ then $\mathcal{B}_A$ is a (α, ε)-Holder continuous at $A \in \mathbf{A}$, almost everywhere.

b. If $0 < \alpha < \frac{1}{2}$ and $\forall B \in \mathbf{A}, E[\|\mathcal{B}_A - \mathcal{B}_B\|^2] \leq K d_{\mathbf{A}}(A, B)$ where $K > 0$ then $\mathcal{B}_A$ is a α-Holder continuous at $A \in \mathbf{A}$, almost everywhere.

Proof.

(a) Let $A \in \mathbf{A}$ and $A^\varepsilon \in D_A^\varepsilon$. Based on Theorem 3.3, there exists a flow $f : [0, \infty) \to \mathbf{A}(\mathrm{u})$ and $0 \leq t \leq t^\varepsilon$ such that X^f, Y^f are time-changed Brownian motions and $A^\varepsilon = f(t^\varepsilon), A = f(t)$. Thus, there exist $\theta_X, \theta_Y : [0, \infty) \to [a, b], 0 \leq \beta \leq \beta^\varepsilon$ such that $X^{f\circ\theta_X}, Y^{f\circ\theta_Y}$ are Brownian motions and $A^\varepsilon = f(t^\varepsilon) = f\big(\theta_X(\beta^\varepsilon)\big) = f(\theta_Y(\beta^\varepsilon)),\ A = f(t) = f(\theta_X(\beta) = f(\theta_Y(\beta))$.

$X^{f\circ\theta_X}$ is a Brownian motion (We recall that, if $W = \{W_t : t \geq 0\}$ is a Brownian motion and $\alpha < \frac{1}{2}$ then W is a α-Holder continuous path almost everywhere), then there exist a $M_X > 0$ such that

$$\left|X_{\beta^\varepsilon}^{f\circ\theta_X} - X_\beta^{f\circ\theta_X}\right| = |X_{A^\varepsilon} - X_A| \leq M_X \varepsilon^\alpha \text{ for all } 0 < \varepsilon.$$

Similarly, $Y^{f\circ\theta}$ is a Brownian motion, then there exist $M_Y > 0$ such that

$$\left|Y_{\beta^\varepsilon}^{f\circ\theta_Y} - Y_\beta^{f\circ\theta_X}\right| = |Y_{A^\varepsilon} - Y_A| \leq M_Y \varepsilon^\alpha \text{ for all } 0 < \varepsilon.$$

Then,

$$\|\mathcal{B}_{A^\varepsilon} - \mathcal{B}_A\| = \|X_{A^\varepsilon} - X_A - i(Y_{A^\varepsilon} - Y_A)\| \leq |X_{A^\varepsilon} - X_A| + |Y_{A^\varepsilon} - Y_A| \leq M\varepsilon^\alpha$$

where $M = 2\max\{M_X, M_Y\}$.

(b) Let $A \in \mathbf{A}$ and $\forall B \in \mathbf{A}, E[\|\mathcal{B}_A - \mathcal{B}_B\|^2] \leq K d_{\mathbf{A}}(A, B)$. Then based on [12], we derive:

$$\begin{aligned}\|\mathcal{B}_A - \mathcal{B}_B\| &\leq |X_A - X_B| + |Y_A - Y_B| \\ &\leq M_X d_{\mathbf{A}}(A, B)^\alpha + M_Y d_{\mathbf{A}}(A, B)^\alpha \\ &\leq M d_{\mathbf{A}}(A, B)^\alpha\end{aligned}$$

where $M = 2\max\{M_X, M_Y\}, M_X, M_Y > 0$.□

Differentiability

To study a set-indexed version of the notion of ε-self-similarity for a cvsiBM, we need some another assumptions about the set $\mathbf{A}$.

Definition 5.1. Let $\mu \in M(\mathbf{A})$ be a positive and continuous measure in $\mathbf{A}$. If $A \in \mathbf{A}$, $\mu(A) > 0$ and $\varepsilon > 0$ then

a. Define $\Gamma_A^\varepsilon = \{B \in \mathbf{A} \colon \frac{\mu(B)}{\mu(A)} = \varepsilon\}$. Denote by $A^{[\varepsilon]}$ an element in Γ_A^ε and assume that $\Gamma_A^\varepsilon \neq \emptyset$.

b. Define $\Pi_A^\varepsilon = \{A^{[\varepsilon]} \in \Gamma_A^\varepsilon \colon \exists B^{[\varepsilon]} \in \Gamma_B^\varepsilon,\ \mu(A^{[\varepsilon]} \backslash B^{[\varepsilon]}) = \varepsilon\mu(A \backslash B)\}$. Denote by $[A]^\varepsilon$ an element in Π_A^ε and assume that $\Pi_A^\varepsilon \neq \emptyset$.

Hereafter, we assume that:

- For all $A \in \mathbf{A}$ and for all $\varepsilon > 0$ there exists $[A]^\varepsilon \in \Pi_A^\varepsilon$, $\mu([A]^\varepsilon) = \varepsilon\mu(A)$.
- For all $A, B \in \mathbf{A}$ and for all $\varepsilon > 0$ there exists $[A]^\varepsilon \in \Pi_A^\varepsilon$, $[B]^\varepsilon \in \Pi_B^\varepsilon$ such that $\mu([A]^\varepsilon \backslash [B]^\varepsilon) = \varepsilon\mu(A \backslash B)$.
- For all $A^\varepsilon \in D_A^\varepsilon$ there exists $[B]^\varepsilon \in \Pi_B^\varepsilon$ such that $\mu([B]^\varepsilon) = \mu(A^\varepsilon)$.

The classical example is: $T = \mathfrak{R}_+^d$ and $\mathbf{A} = \mathbf{A}(\mathfrak{R}_+^d) = \{[0, x] \colon x \in \mathfrak{R}_+^d\}$ when μ is Lebesgue measure.

The following notions have been introduced by Kolwankar and Gangal [15], studied by Ben Adda and Cresson [1] and Thale [25].

Let f be a continuous function on $[a, b]$, and let $0 < \alpha \leq 1$. Call a right (resp., left) local fractional α-derivative of f at $t_0 \in [a, b]$ if:

$$d_+^\alpha f(t_0) = \Gamma(1 + \alpha) \lim_{\varepsilon\to0^+} \frac{f(t_0+\varepsilon) - f(t_0)}{\varepsilon^\alpha},$$
$$\left(d_-^\alpha f(t_0) = -\Gamma(1 + \alpha) \lim_{\varepsilon\to0^-} \frac{f(t_0+\varepsilon) - f(t_0)}{\varepsilon^\alpha}\right)$$

where Γ is the Euler function. The function f is said to be α-differentiable at $t_0 \in [a, b]$ if and only if $d_+^\alpha f(t_0)$ and $d_-^\alpha f(t_0)$ exist and are equal. In this case, denote by $d^\alpha f(t_0)$ the α-derivative of f at t_0.

Definition 5.2. Let $d_{\mathbf{A}}$ be a pseudo-distance on the indexing collection $\mathbf{A}$, whose subclasses $\mathbb{A} = \{\mathbf{A}_n\}_{n\in\mathbb{N}}$, $\mathbf{A}_n = \{A_1^n, \dots, A_{k_n}^n\} \subseteq \mathbf{A}$ satisfy Assumption $\mathrm{H}_{\mathbb{A}}$. Let $X = \{X_A : A \in \mathbf{A}\}$ be a set-indexed stochastic process.

a. X is said to be (α, ε)-differentiable at $A_0 \in \mathbf{A}$ if $d_+^\alpha X_{A_0}$ and $d_-^\alpha X_{A_0}$ exist and are equal. In this case, denote by $d^\alpha X_{A_0}$ the (α, ε)-derivative of X at A_0, where

$$d_+^\alpha X_{A_0} = \Gamma(1 + \alpha) \lim_{\varepsilon\to0^+} \frac{X_{A_0^\varepsilon} - X_{A_0}}{\varepsilon^\alpha},$$
$$d_-^\alpha X_{A_0} = -\Gamma(1 + \alpha) \lim_{\varepsilon\to0^-} \frac{X_{A_0^\varepsilon} - X_{A_0}}{\varepsilon^\alpha}$$

and Γ is the Euler function.

b. X is said to be α-differentiable at $A_0 \in \mathbf{A}$ if

$$\overline{\lim_{\delta\to0}} \sup_{B\in B_{d_{\mathbf{A}}}(A_0,\delta)} \frac{|X_B - X_{A_0}|}{d_{\mathbf{A}}(A_0, B)^\alpha} < +\infty \text{ where } B_{d_{\mathbf{A}}}(A_0, \delta) = \{B \in \mathbf{A} \colon d_{\mathbf{A}}(A_0, B) < \delta\}$$

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Theorem 5.2. The cvsiBM $\mathcal{B} = \{\mathcal{B}_A: A \in \mathbf{A}\} = \{X_A + iY_A: A \in \mathbf{A}\}$ has ε-self-similarity.

(i.e., $\{\mathcal{B}_{[A]^\varepsilon}: A \in \mathbf{A}\} \stackrel{d}{=} \{\sqrt{\varepsilon}\mathcal{B}_A: A \in \mathbf{A}\}$ for all $\varepsilon > 0$ and for all $[A]^\varepsilon \in \Pi_A^\varepsilon$.

Proof.

Let $\varepsilon > 0$, $A, B \in \mathbf{A}$ and $[A]^\varepsilon \in \Pi_A^\varepsilon$, $[B]^\varepsilon \in \Pi_B^\varepsilon$. $\mathcal{B}_{[A]^\varepsilon}$ and $\mathcal{B}_A$ are Gaussian and centered. Therefore, one only has to prove that they have the same covariance function. Based on Theorem 3.1,

$$Cov(\mathcal{B}_{[A]^\varepsilon}, \mathcal{B}_{[B]^\varepsilon}) = \sigma_X([A]^\varepsilon \cap [B]^\varepsilon) + \sigma_Y([A]^\varepsilon \cap [B]^\varepsilon)$$

But,

$$\sigma_X([A]^\varepsilon \cap [B]^\varepsilon) = \sigma_X([A]^\varepsilon) + \sigma_X([B]^\varepsilon) - \sigma_X([A]^\varepsilon \Delta [B]^\varepsilon)$$

and

$$\sigma_X([A]^\varepsilon \Delta [B]^\varepsilon) = \sigma_X([A]^\varepsilon \backslash [B]^\varepsilon) + \sigma_X([B]^\varepsilon \backslash [A]^\varepsilon) = \varepsilon[\sigma_X(A \backslash B) + \sigma_X(B \backslash A)] = \varepsilon\sigma_X(A \Delta B)$$

Thus, $\sigma_X([A]^\varepsilon \cap [B]^\varepsilon) = \varepsilon\sigma_X(A \cap B)$. In the same way, we derive $\sigma_Y([A]^\varepsilon \cap [B]^\varepsilon) = \varepsilon\sigma_Y(A \cap B)$.

Then

$$Cov(\mathcal{B}_{[A]^\varepsilon}, \mathcal{B}_{[B]^\varepsilon}) = \varepsilon\sigma_X(A \cap B) + \varepsilon\sigma_Y(A \cap B) = \varepsilon Cov(\mathcal{B}_A, \mathcal{B}_B) = Cov(\sqrt{\varepsilon}\mathcal{B}_A, \sqrt{\varepsilon}\mathcal{B}_B). \quad \square$$

Theorem 5.3. (Differentiability) Let $d_{\mathbf{A}}$ be a pseudo-distance on the indexing collection $\mathbf{A}$, whose subclasses $\mathbb{A} = \{\mathbf{A}_n\}_{n\in\mathbb{N}}$, $\mathbf{A}_n = \{A_1^n, \dots, A_{k_n}^n\} \subseteq \mathbf{A}$ satisfy Assumption $\mathrm{H}_{\mathbb{A}}$. Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A: A \in \mathbf{A}\}$ be a cvsiBM with variance $\sigma = \sigma_X + \sigma_Y$, $\sigma_X, \sigma_Y \in M(\mathbf{A})$.

a. For all $\alpha \in (0, \frac{1}{2})$, the sample paths of the $\mathcal{B}_A$ are almost surely (α, ε)-differentiable at $A \in \mathbf{A}$ and $P[d^\alpha X_A = 0] = 1$.

b. For all $\alpha \in [\frac{1}{2}, 1]$, the sample paths of the $\mathcal{B}_A$ are nowhere (α, ε)-differentiable at $A \in \mathbf{A}$, almost surely.

c. Almost surely, $d^\alpha\mathcal{B}_A = \begin{cases} 0 & , \alpha < \frac{1}{2} \\ +\infty & , \alpha \geq \frac{1}{2} \end{cases}$

d. For all $\alpha \in (0, \frac{1}{2})$, the sample paths of the $\mathcal{B}_A$ are almost surely ε-differentiable at $A \in \mathbf{A}$.

Proof.

Let $A \in \mathbf{A}$ and $A^\varepsilon \in D_A^\varepsilon$.

a. Based on the Theorem 3.1 (ε-stationary increments: $\mathcal{B}_{A^\varepsilon} - \mathcal{B}_A \stackrel{d}{=} \mathcal{B}_{\emptyset^\varepsilon}$), we have $\frac{\mathcal{B}_{A^\varepsilon} - \mathcal{B}_A}{\varepsilon^\alpha} \stackrel{d}{=} \frac{\mathcal{B}_{\emptyset^\varepsilon}}{\varepsilon^\alpha}$. But $\emptyset^\varepsilon \in D_A^\varepsilon$ then there exists a $[D]^\varepsilon \in \Pi_D^\varepsilon$ such that $\sigma([D]^\varepsilon) = \sigma(\emptyset^\varepsilon)$ and $\mathcal{B}_{\emptyset^\varepsilon} \stackrel{d}{=} \mathcal{B}_{[D]^\varepsilon}$. Thus, based on Theorem 5.2 (ε-self-similarity: $\mathcal{B}_{[D]^\varepsilon} \stackrel{d}{=} \sqrt{\varepsilon}\mathcal{B}_D$), we derive

$$\frac{\mathcal{B}_{A^\varepsilon} - \mathcal{B}_A}{\varepsilon^\alpha} \stackrel{d}{=} \frac{\mathcal{B}_{\emptyset^\varepsilon}}{\varepsilon^\alpha} \stackrel{d}{=} \frac{\mathcal{B}_{[D]^\varepsilon}}{\varepsilon^\alpha} \stackrel{d}{=} \frac{\sqrt{\varepsilon}\mathcal{B}_D}{\varepsilon^\alpha} = \varepsilon^{0.5-\alpha}\mathcal{B}_D.$$

In consequence,

$$P[d_+^\alpha\mathcal{B}_A = 0] = P\left[\lim_{\varepsilon\to 0^+} \frac{\mathcal{B}_{A^\varepsilon} - \mathcal{B}_A}{\varepsilon^\alpha} = 0\right] = P\left[\lim_{\varepsilon\to 0^+} \varepsilon^{0.5-\alpha}\mathcal{B}_D = 0\right] = 1.$$

The proof for $d_-^\alpha\mathcal{B}_A$ is the same.

b. Based on Lemma 3.2, there exists a flow $f: [0, \infty) \to \mathbf{A}(\mathrm{u})$ and $0 \leq t \leq t^\varepsilon$ such that X^f, Y^f are time-change Brownian motions and $A^\varepsilon = f(t^\varepsilon), A = f(t)$. Thus, there exist $\theta_X, \theta_Y: [0, \infty) \to [0, \infty)$ and $0 \leq \beta \leq \beta^\varepsilon$ such that $X^{f\circ\theta}, Y^{f\circ\theta}$ are Brownian motions and $A^\varepsilon = f(\theta_X(\beta^\varepsilon)) = f(\theta_Y(\beta^\varepsilon)), A = f(\theta_X(\beta)) = f(\theta_Y(\beta))$, $X^{f\circ\theta_X}, Y^{f\circ\theta_Y}$ are Brownian motions, (We recall that, if $W = \{W_t: t \geq 0\}$ is a Brownian motion and $\alpha \geq 0.5$, then $\overline{\lim_{\varepsilon\to 0^+}} \left|\frac{W_{t+\varepsilon} - W_t}{\varepsilon^\alpha}\right| = +\infty$, with probability 1) then

$$\overline{\lim_{\varepsilon\to 0^+}} \left|\frac{X_{A^\varepsilon} - X_A}{\varepsilon^\alpha}\right| = \overline{\lim_{\varepsilon\to 0^+}} \left|\frac{X_{\beta^\varepsilon}^{f\circ\theta_X} - X_\beta^{f\circ\theta_X}}{\varepsilon^\alpha}\right| = +\infty,$$

$$\overline{\lim_{\varepsilon\to 0^+}} \left|\frac{Y_{A^\varepsilon} - Y_A}{\varepsilon^\alpha}\right| = \overline{\lim_{\varepsilon\to 0^+}} \left|\frac{Y_{\beta^\varepsilon}^{f\circ\theta_Y} - Y_\beta^{f\circ\theta_Y}}{\varepsilon^\alpha}\right| = +\infty$$

with probability 1. Moreover, If $\alpha \geq 0.5$ then $\mathcal{B}_A$ is not (α, ε)-differentiable at $A \in \mathbf{A}$, almost surely.

c. Trivial from (a) and (b).

d. Based of [12], the sample paths of the X_A, Y_A are almost surely ε-differentiable at $A \in \mathbf{A}$ then $\sup_{B\in B_{d_{\mathbf{A}}}(A,\delta)} \frac{\|\mathcal{B}_B - \mathcal{B}_A\|}{d_{\mathbf{A}}(A,B)^\alpha} \leq \sup_{B\in B_{d_{\mathbf{A}}}(A,\delta)} \frac{|X_B - X_A|}{d_{\mathbf{A}}(A,B)^\alpha} + \sup_{B\in B_{d_{\mathbf{A}}}(A,\delta)} \frac{|Y_B - Y_A|}{d_{\mathbf{A}}(A,B)^\alpha} < +\infty$ for all $\delta > 0$ and $\alpha \in (0, \frac{1}{2})$. $\square$

6. STOPPING LINES

We induce a natural topology on $\mathbf{A}$ via the Hausdorff metric d, defined by

$$d(A, B) = \inf\{\epsilon > 0: A \subseteq B_\varepsilon \wedge B \subseteq A_\varepsilon\}, A, B \in \mathbf{A},$$

where $A_\varepsilon = \{x \in T: d(A, x) \leq \varepsilon\}$ for any $\varepsilon > 0$. If $A, B \subseteq T$ are compact, it is straightforward to show that $B \subseteq A^\circ$ implies $B_\varepsilon \subseteq A^\circ$ for some $\varepsilon > 0$.

Definition 6.1. ([24]) A d-closed subset L of $\mathbf{A}$ is a decreasing line if

a) Given $A, B \in L$, if $A \prec B$ or $B \prec A$, then $B = A$ (we write $A \prec B$ if $A \subseteq B^\circ$).

b) Given any domain $A \in \mathbf{A}$ ($A = \bar{A}^\circ$), if $A \notin L$, then either $A \prec L$ or $L \prec A$ (we write $A \prec L$ ($L \prec A$) if there is a set $A' \in L$ such that $A \prec A'$ (respectively, $L \prec A$)).

Let $\mathbf{L}(\mathbf{A})$ denote the collection of all decreasing lines in $\mathbf{A}$. Included in $\mathbf{L}(\mathbf{A})$ is the decreasing line at infinity, denoted L_∞ and characterized by the property, $A \prec L_\infty$ for all $A \in \mathbf{A}$.

Given $L \in \mathbf{L}(\mathbf{A})$ and $A \in \mathbf{A}$, we write $A \leq L$ $(L \leq A)$ if $A \subseteq B$ $(B \subseteq A)$ and $A < L$ $(L < A)$ if $A \subset B$ $(B \subset A)$ for some set $B \in L$. We write $C < L$ $(C \leq L)$ if there exist $A \in \mathbf{A}$ such that $B \subseteq A, A < L$ $(A \leq L)$, for some set $C \in \mathbf{C}$.

Definition 6.2. ([14], [24]) A map $L: \Omega \longrightarrow \mathbf{L}(\mathbf{A})$ is set indexed stopping line (or shortly, $\mathbf{A}$ -stopping line) if $[A \leq L] \in F_A$ for all $A \in \mathbf{A}$. The collection of all $\mathbf{A}$ -stopping lines is denoted **SL**. Equivalently, if we define the random collection, $R_L = \{(A, \omega) \in \mathbf{A} \times \Omega: A \leq L(\omega)\}$, then L is an $\mathbf{A}$-stopping line if $[A \in R_L] = \{\omega \in \Omega: (A, \omega) \in R_L\} \in F_A$ for every $A \in \mathbf{A}$.

Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A: A \in \mathbf{A}\}$ be a cvsiBM with variance $\sigma = \sigma_X + \sigma_Y$, $\sigma_X, \sigma_Y \in M(\mathbf{A})$. For $a>0$ define L_a^X to be a decreasing line in $\mathbf{A}$ such that:

a) If $A < L_a^X$ then $X_A < a$.

b) If $A \in L_a^X$ then $X_A = a$ in the first time on $A \in \mathbf{A}$.

(In other words, L_a^X is a collection of sets A when X reaches the value a for the first time). Similarly, we define L_b^Y to be a collection of sets B when Y reaches the value b for the first time where $b>0$ and $B \in \mathbf{A}$.

Lemma 6.1. $L_a^X, L_b^Y \in \mathbf{SL}$ (i.e., L_a^X, L_b^Y are set-indexed stopping lines).

Proof.

Let $A \in \mathbf{A}$. Clearly, if we know X_B for all $B \subseteq A$ then we know whether the set indexed Brownian motion (X) had the value a before or at A, or not. Thus, we know that $[L_a^X \leq A]$ has occurred or not just by observing the past of the process prior to A. In other words, $[L_a^X \leq A] = [sup_{B \subseteq A} X_B \geq a] \in F_A^X$. In the same way, we can prove that $[L_b^Y \leq A] = [sup_{B \subseteq A} Y_B \geq a] \in F_A^Y$. Then L_a^X, L_b^Y are set-indexed stopping lines. □

Lemma 6.2. (Unboundedness) Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A: A \in \mathbf{A}\}$ be a cvsiBM with variance $\sigma = \sigma_X + \sigma_Y$, $\sigma_X, \sigma_Y \in M(\mathbf{A})$ then $\lim_{A \uparrow T} \|\mathcal{B}_A\| = +\infty$ almost surely

Proof.

Let $A_i \uparrow T$. Clearly, $\lim_{A \uparrow T} |X_A| \leq \lim_{A \uparrow T} \|\mathcal{B}_A\|$. Based on Lemma 3.2, there exists a continuous flow $f: [0, \infty) \to \mathbf{A}(\mathrm{u}), f(0) = \emptyset'$ and $f(i) = A_i$ for all $1 \leq i$, such that $\mathcal{B}^f$ is a complex-valued time-change Brownian motion. Then, there exists $\theta_X: [0, \infty) \to [0, \infty)$ and $0 \leq t_i$ such that $X^{f \circ \theta_X}$ is a Brownian motion such that $A_i = f(i) = f(\theta_X(t_i))$, (We recall that, if $W = \{W_t: t \geq 0\}$ is a Brownian motion then $\lim_{t \to \infty} W_t = +\infty$, with probability 1) then $+\infty = \lim_{t_i \to \infty} |X_{t_i}^{f \circ \theta_X}| \leq \lim_{A \uparrow T} |X_A| \leq \lim_{A \uparrow T} \|\mathcal{B}_A\|$. □

Lemma 6.3. $L_a^X < L_\infty^X, L_b^Y < L_\infty^Y$ (i.e., L_a^X, L_b^Y are bounded stopping lines).

Proof.

Sufficient to prove that X is almost surely unbounded, since if X hits some level b $(b \geq a)$ almost surely, then by continuity and since $X_{\emptyset'}$, it hits level a almost surely. Based on Lemma 6.2, $\lim_{A \uparrow T} |X_A| = +\infty$ almost surely. Hence, we obtain that $L_a^X < L_\infty^X$. □

Theorem 6.1. (Reflection principle) Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A: A \in \mathbf{A}\}$ be a cvsiBM with $\sigma = \sigma_X + \sigma_Y$, $\sigma_X, \sigma_Y \in M(\mathbf{A})$ then $\widehat{\mathcal{B}} = \{\widehat{\mathcal{B}}_A: A \in \mathbf{A}\}$ is a cvsiBM where

$$\widehat{\mathcal{B}}_A = \begin{cases} \mathcal{B}_A & , A < L_a^X \text{ and } A < L_b^Y \\ 2bi + \overline{\mathcal{B}}_A & , A < L_a^X \text{ and } A \geq L_b^Y \\ 2a - \overline{\mathcal{B}}_A & , A \geq L_a^X \text{ and } A < L_b^Y \\ 2a + 2bi - i\mathcal{B}_A & , A \geq L_a^X \text{ and } A \geq L_b^Y \end{cases}$$

Proof.

It must be shown that if $\{C_i\}_{i=1}^k \in \mathbf{C}$ are disjoint, then $\{\hat{X}_{C_i}\}_{i=1}^k, \{\hat{Y}_{C_i}\}_{i=1}^k$ are independent normal random variables with variances $\{\sigma_X(C_i)\}_{i=1}^k, \{\sigma_Y(C_i)\}_{i=1}^k$, respectively, where

$$\hat{X}_A = \begin{cases} X_A & , A < L_a^X \\ 2a - X_A & , A \geq L_a^X \end{cases} \text{ and } \hat{Y}_A = \begin{cases} Y_A & , A < L_b^Y \\ 2b - Y_A & , A \geq L_b^Y \end{cases}.$$

Let $A \in L_a^X$, without loss of generality, we may assume that the sets $\{C_i\}_{i=1}^k$ are the left neighborhoods of the sub semi-lattice A^{ss} of $\mathbf{A}$ equipped with a numbering consistent with the strong past. According to Lemma 3.2, there exists a continuous flow $f: [0, \infty) \to \mathbf{A}(\mathrm{u})$ and $0 \leq t_a^X$ such that X^f is a time-change Brownian motion (i.e., there exists a increasing function $\theta_X: [0, \infty) \to [0, \infty)$ such that $X^{f \circ \theta_X}$ is a Brownian motion) and each left-neighborhood generated by A^{ss} is of the form $C_i = f(i) \backslash f(i-1)$, $i = 1, \ldots, k$, and $A = f(\theta_X(t_a^X)) \in L_a^X$. We recall that, if $W = \{W_t: t \geq 0\}$ is a classical Brownian motion and $T_a = \inf\{t \geq 0: W_t = a\}$, then $Z_t = \begin{cases} W_t & , t < T_a \\ 2a - W_t & , T_a \leq t \end{cases}$ is a Brownian motion [8]. Thus, if we define

$$\hat{X}_t^{f \circ \theta_X} = \begin{cases} X_t^{f \circ \theta_X} & , t < t_a \\ 2a - X_t^{f \circ \theta_X} & , t_a \leq t \end{cases}$$

$\hat{X}_t^{f \circ \theta_X}$ turns out to be a Brownian motion, and so $\{\hat{X}_{C_i}\}_{i=1}^k$ are independent normal random variables with variances $\{\sigma_X(C_i)\}_{i=1}^k$, respectively. Then $\hat{X}_A$ is a set-indexed Brownian motion. In the same way, we can prove that $\hat{Y}_A$ is a set-indexed Brownian motion. X, Y are independent then $\hat{X}, \hat{Y}$ are independent.

Thus,

$$\widehat{\mathcal{B}}_A = \hat{X}_A + i\hat{Y}_A = \begin{cases} \mathcal{B}_A & , A < L_a^X \text{ and } A < L_b^Y \\ 2bi + \overline{\mathcal{B}}_A & , A < L_a^X \text{ and } A \geq L_b^Y \\ 2a - \overline{\mathcal{B}}_A & , A \geq L_a^X \text{ and } A < L_b^Y \\ 2a + 2bi - i\mathcal{B}_A & , A \geq L_a^X \text{ and } A \geq L_b^Y \end{cases}$$ is a cvsiBM. □

Theorem 6.2: (Hitting time) Let $\mathcal{B} = \{\mathcal{B}_A = X_A + iY_A : A \in \mathbf{A}\}$ be a cvsiBM with variance $\sigma = \sigma_X + \sigma_Y, \sigma_X, \sigma_Y \in M(\mathbf{A})$ then

$$P[L_a^X \le A, L_b^Y \le B] = 4\left[1 - \Phi\left(\frac{a}{\sqrt{\sigma_X(A)}}\right)\right]\left[1 - \Phi\left(\frac{b}{\sqrt{\sigma_Y(B)}}\right)\right] \text{ for all } A, B \in \mathbf{A}$$

(Φ - standard Gaussian distribution function).

Proof.

Let $A, B \in \mathbf{A}$. Based on a Law of Total Probability:

$$P[X_A \ge a] = P[X_A \ge a | L_a^X \le A] P[L_a^X \le A] + P[X_A \ge a | A < L_a^X] P[A < L_a^X]$$

According to definition of L_a^X, we imply that if $A < L_a^X$ then $X_A < a$, and then $P[X_A \ge a | A < L_a^X] = 0$. Based on Lemma 3.2, there exists a continuous flow $f: [0, \infty) \to \mathbf{A}(\mathrm{u})$ and there exists $0 < t$ such that $f(t) = A$ and X^f is a time-change Brownian motion (i.e., there exists $\theta_X: [0, \infty) \to [0, \infty)$ such that $X^{f \circ \theta_X}$ is a Brownian motion). Since, $X^{f \circ \theta_X}$ is symmetric then clearly:

$$P[X_A \ge a | L_a^X \le A] = P\left[X_t^{f \circ \theta_X} \ge a | L_a^X \le A\right] = \frac{1}{2}$$

Thus, $P[L_a^X \le A] = 2P[X_A \ge a] = 2 - 2\Phi\left(\frac{a}{\sqrt{\sigma_X(A)}}\right)$

In the same way, we derive

$$P[L_b^Y \le B] = 2P[X_B \ge b] = 2 - 2\Phi\left(\frac{b}{\sqrt{\sigma_Y(B)}}\right)$$

But X, Y are independent then

$$P[L_a^X \le A, L_b^Y \le B] = P[L_a^X \le A] P[L_b^Y \le B] = 4\left[1 - \Phi\left(\frac{a}{\sqrt{\sigma_X(A)}}\right)\right]\left[1 - \Phi\left(\frac{b}{\sqrt{\sigma_Y(B)}}\right)\right]. \quad \square$$

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