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Properties of the Caputo-Fabrizio Fractional Derivative

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Abstract

In this paper, we investigate some properties related Caputo-Fabrizio fractional derivative. We prove some regularity properties and bounds characterizing the Caputo-Fabrizio derivative operator. Using the method of Laplace transform, we found explicit solutions of some differential equations containing the Caputo-Fabrizio fractional derivative. Different types of inequalities generated by using the Caputo-Fabrizio derivative are also presented.

Keywords: Caputo-Fabrizio fractional derivative, Fractional integral inequalities, Functions with the same sense of variation.

1. INTRODUCTION

A fractional derivative D^{α} is an operator, which generalizes the ordinary derivative. The origins of the fractional derivatives date back to 1695 when L'Hopital raised, by a letter to Leibniz, the question of how the expression

$$D^n f(t) = \frac{d^n}{dt^n} f(t),$$

should be understood if *n* was a fraction [1]. This question is commonly accepted as the first occurrence of what is currently known as fractional calculus. Since then, this branch has been addressed by eminent mathematicians, such as Euler, Laplace, Fourier, Liouville, Riemann, Laurent, Weyl, and Abel, who first applied it in physics to solve the integral equation arising from the tautochrone problem [2]. Fractional derivative has various definitions, which do not generally coincide. This is possible since different researchers attempt to preserve different properties of the classical integer order derivative. Fractional operators are used in various fields of science and engineering to describe some natural phenomena in [3-9], to enhance the contrast in an image [10-12], as well as to prove the existence and uniqueness of fractional differential equations in [13-18].

In 2015, Caputo and Fabrizio introduced a new fractional approach [19]. The interest for this definition was due to the necessity to describe a class of non-local systems, which cannot be well described by classical local theories or by fractional models with singular kernel [19]. In this paper, we present some interesting properties of Caputo-Fabrizio derivative, as follows: In Section 2, we briefly review the basic concepts and definitions. In Section 3, we obtain some regularity properties. Section 4 presents the solutions of some ordinary fractional differential equations with the Caputo-Fabrizio derivative. In Section 5, we deduce new integral inequalities. Section 6 is dedicated to conclusion.

2. PRELIMINARIES

Here, we present basic definitions and theorems, which are used in our subsequent discussion.

Definition 1. We postulate that two functions f and g have the same sense of variation (synchronous) on $[0,\infty)$ if [20]

$$\big(f(\tau)-f(\rho)\big)\cdot\big(g(\tau)-g(\rho)\big)\geq 0,\ \tau,\rho\in(0,t), t>0.$$

For example, one can easily see that functions $f(t)=t$ and $g(t)=t^2$ are synchronous on $[0,\infty)$, i.e.

$$\begin{aligned}(\tau-\rho)\cdot(\tau^2-\rho^2) &= (\tau-\rho)\cdot(\tau-\rho)\cdot(\tau+\rho)\\ &= (\tau-\rho)^2\cdot(\tau+\rho)\geq 0.\end{aligned}$$

Definition 2. Let $f(t)\in L^1_{loc}(\mathbb{R})$. The Laplace transform of $f(t)$ is defined by [21]

$$\mathcal{L}\{f(t)\}(s) := \int_0^{\infty} e^{-st} f(t)\,dt.$$

Definition 3. The Sobolev space $W^{1,p}(a,b)$ is defined by [22]

$$W^{1,p}(a,b) = \left\{\begin{array}{l} f\in L^p(a,b): \exists g\in L^p(a,b): \\ \int_a^b f\varphi' = -\int_a^b g\varphi, \forall\varphi\in C_0^{\infty}(a,b)\end{array}\right\}.$$

We set $H^1(a,b)=W^{1,2}(a,b)$.

Note 1. Let $I=[a,b]$, then $C^1(I)\subset W^{1,p}(I)$ for all $1\leq p\leq\infty$.

Definition 4. Let $a,b,\alpha\in\mathbb{R}, a<b, 0<\alpha<1, f\in H^1(a,b)$. The Caputo-Fabrizio fractional integral of order α is defined by [23]

$$I_{at}^{\alpha} f(t) = (1-\alpha)f(t) + \alpha\int_a^t f(\tau)\,d\tau.$$

Definition 5. Let $a,b,\alpha\in\mathbb{R}, a<b, 0<\alpha<1, f\in H^1(a,b)$.

The Caputo-Fabrizio fractional derivative of order α respect to time variable is defined by [24]

$$D_{at}^{\alpha} f(t) = \frac{1}{1-\alpha} \cdot \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau) d\tau$$
$$= \frac{1}{1-\alpha} \cdot \left(f(t) - e^{-\frac{\alpha}{1-\alpha}t} f(a) \right) - \quad (1)$$
$$- \frac{\alpha}{(1-\alpha)^2} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau.$$

Definition 6. Let $a, b, \in \mathbb{R}, a < b, [a, b] \subset \mathbb{R}$. The space of continuous functions from the subset $[a,b]$ to $\mathbb{R}$ is defined by

$$C([a,b],\mathbb{R}) = \{f: [a,b] \to \mathbb{R}, f \in C[a,b]\},$$

with the norm

$$\|f(x)\|_{C([a,b])} = \max_{x\in[a,b]} |f(x)|. \quad (2)$$

Definition 7. Let $a, b \in \mathbb{R}, a < b$. We denote by $C^1([a,b])$ the space of real-valued functions $f(x)$ whose derivative f' is continuous on $[a,b]$, with the norm

$$\|f(x)\|_{C^1([a,b])} = \max_{x\in[a,b]} |f(x)| + \max_{x\in[a,b]} |f'(x)|. \quad (3)$$

Theorem 1. Let $a, b \in \mathbb{R}, a < b, n \in \mathbb{N}$ and $f \in C^n([a,b])$. Then the equality [25]

$$\frac{d^n}{dt^n}\left(D_{at}^{\alpha} f(t)\right) = \sum_{i=1}^{n} (-1)^{n-i} \frac{\alpha^{n-i}}{(1-\alpha)^{n+1-i}} f^{(i)}(t) +$$
$$+ (-1)^n \left(\frac{\alpha}{1-\alpha}\right)^n D_{at}^{\alpha} f(t),$$

holds true.

3. SOME REGULARITY PROPERTIES OF THE CAPUTO-FABRIZIO DERIVATIVE OPERATOR

In this section, we introduce some theorems that characterize the Caputo-Fabrizio fractional derivative in certain spaces, such as *C* (space of all continuous functions) or *C*¹ (space of functions which first derivative is continuous). Although this class of spaces are considered very restricted, their importance for practical applications is great because the character of the majority of dynamic processes is smooth and has no discontinuities.

Theorem 2. Let $f \in C^1[a,b]$. Then $D_{at}^{\alpha} f(t) \in C^1[a,b]$.

Proof. As the function

$$y_\tau(t) = \frac{1}{1-\alpha} \cdot e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \cdot f'(\tau),$$

is continuous and integrable for all $t, \tau \in [a,b]$, we conclude that the function

$$F(t) = \frac{1}{1-\alpha} \cdot \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} \cdot f'(\tau) d\tau,$$

is differentiable in $[a,b]$. This means that $D_{at}^{\alpha} f(t) \in C^1[a,b]$.

Theorem 3. The operator $D_{at}^{\alpha}: C^1[a,b] \to C^1[a,b]$ is bounded and

$$\left\|D_{at}^{\alpha} f\right\|_{C^1[a,b]} \le \frac{1}{\alpha} \cdot \left(1 - e^{-\frac{\alpha}{1-\alpha}(b-a)}\right) \|f\|_{C^1[a,b]}. \quad (4)$$

Proof. Considering the norm defined in (3), we obtain

$$\left\|D_{at}^{\alpha} f\right\|_{C^1[a,b]} = \left\| \frac{1}{1-\alpha} \cdot \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} f'(\xi) d\xi \right\|_{C^1[a,b]}$$
$$\le \left\| \frac{1}{1-\alpha} \cdot \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} |f'(\xi)| d\xi \right\|_{C^1[a,b]}$$
$$\le \left\| \frac{1}{1-\alpha} \cdot \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} \left(|f(\xi)| + |f'(\xi)|\right) d\xi \right\|_{C^1[a,b]}$$
$$\le \frac{1}{1-\alpha} \|f\|_{C^1[a,b]} \cdot \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\xi)} d\xi$$
$$\le \frac{1}{\alpha} \|f\|_{C^1[a,b]} \cdot \left(1 - e^{-\frac{\alpha}{1-\alpha}(t-a)}\right)$$
$$\le \frac{1}{\alpha} \|f\|_{C^1[a,b]} \cdot \left(1 - e^{-\frac{\alpha}{1-\alpha}(b-a)}\right).$$

Lemma 1. Let $f(t) \in H^1[a,b]$. Then $D_{at}^{\alpha} f(t) \in L^2(a,b)$.

Proof. One can easily see that

$$\left\|D_{at}^{\alpha} f(t)\right\|_{L^2(a,b)}^2 = \frac{1}{1-\alpha} \cdot \int_a^b \left| \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f'(\tau) d\tau \right|^2 dt$$
$$\le \frac{1}{1-\alpha} \int_a^b \left(\int_a^t |f'(\tau)|^2 d\tau \right) dt$$
$$\le \frac{1}{1-\alpha} \int_a^b \left(\int_a^b |f'(\tau)|^2 d\tau \right) dt = \frac{b-a}{1-\alpha} \cdot \|f'\|_{L^2(a,b)}^2 < \infty,$$

as required.

Theorem 4. Let $f \in C^1[a,b]$. Then $D_{at}^{\alpha} f \in W^{1,p}[a,b], 1 \le p \le \infty$.

Proof. As $f \in C^1[a,b]$, we obtain from Theorem 2 that $D_{at}^{\alpha} f \in C^1[a,b]$. Considering the Note 1, we know that $C^1[a,b] \subset W^{1,p}[a,b], \forall p \ge 1$. Therefore $D_{at}^{\alpha} f \in W^{1,p}[a,b]$.

Theorem 5. The Caputo-Fabrizio operator $D_{at}^{\alpha}: C^1[a,b] \to W^{1,1}[a,b]$ satisfies

$$\left\| D_{at}^{\alpha} f \right\|_{W^{1,1}[a,b]} \le \frac{1+(b-a)\left[1+\alpha(b-a)\right]}{(1-\alpha)^2} \left\| f \right\|_{C^1[a,b]}.$$

Proof. Using the Theorem 1 and (1), we obtain

$$\left\| D_{at}^{\alpha} f \right\|_{W^{1,1}[a,b]} = \left\| D_{at}^{\alpha} f \right\|_{L^1[a,b]} + \left\| \frac{d}{dt}\left(D_{at}^{\alpha} f\right) \right\|_{L^1[a,b]}$$

$$= \left\| D_{at}^{\alpha} f \right\|_{L^1[a,b]} + \left\| \frac{1}{1-\alpha} f'(t) - \frac{\alpha}{1-\alpha} D_{at}^{\alpha} f \right\|_{L^1[a,b]}$$

$$\le \frac{1}{1-\alpha} \left\| f'(t) \right\|_{L^1[a,b]} + \frac{1}{1-\alpha} \left\| D_{at}^{\alpha} f \right\|_{L^1[a,b]}$$

$$= \frac{1}{1-\alpha} \left\| f'(t) \right\|_{L^1[a,b]}$$

$$+ \frac{1}{(1-\alpha)^2} \left\| f(t) - e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a) - \frac{\alpha}{1-\alpha} \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau \right\|_{L^1[a,b]}$$

$$\le \frac{1}{1-\alpha} \left\| f'(t) \right\|_{L^1[a,b]} + \frac{1}{(1-\alpha)^2} \left\| f(t) \right\|_{L^1[a,b]}$$

$$+ \frac{1}{(1-\alpha)^2} \left\| e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a) \right\|_{L^1[a,b]}$$

$$+ \frac{\alpha}{(1-\alpha)^3} \int_a^b \left| \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau \right| dt.$$

Combining the last previous inequality with the fact that

$$\left\| e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a) \right\|_{L^1[a,b]} = \int_a^b \left| e^{-\frac{\alpha}{1-\alpha}(t-a)} f(a) \right| dt$$

$$\le \int_a^b \left| f(a) \right| dt \le \int_a^b \max_{t\in[a,b]} \left| f(t) \right| dt = (b-a) \max_{t\in[a,b]} \left| f(t) \right|,$$

$$\int_a^b \left| \int_a^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau \right| dt \le \int_a^b \left(\int_a^t \left| f(\tau) \right| d\tau \right) dt$$

$$\le \int_a^b \left(\int_a^b \max_{\tau\in[a,b]} \left| f(\tau) \right| d\tau \right) dt = (b-a)^2 \max_{t\in[a,b]} \left| f(t) \right|,$$

$$\left\| f'(t) \right\|_{L^1[a,b]} = \int_a^b \left| f'(t) \right| dt \le \int_a^b \max_{t\in[a,b]} \left| f'(t) \right| dt$$

$$= (b-a) \max_{t\in[a,b]} \left| f'(t) \right|,$$

$$\left\| f(t) \right\|_{L^1[a,b]} = \int_a^b \left| f(t) \right| dt \le \int_a^b \max_{t\in[a,b]} \left| f(t) \right| dt$$

$$= (b-a) \max_{t\in[a,b]} \left| f(t) \right|,$$

we obtain

$$\left\| D_{at}^{\alpha} f \right\|_{W^{1,1}[a,b]} \le \frac{b-a}{1-\alpha} \max_{t\in[a,b]} \left| f'(t) \right| + \frac{b-a}{(1-\alpha)^2} \max_{t\in[a,b]} \left| f(t) \right|$$

$$+ \frac{b-a}{(1-\alpha)^2} \max_{t\in[a,b]} \left| f(t) \right| + \frac{\alpha}{(1-\alpha)^3} (b-a)^2 \max_{t\in[a,b]} \left| f(t) \right|$$

$$= \frac{1}{1-\alpha} (b-a) \max_{t\in[a,b]} \left| f'(t) \right|$$

$$+ \left\{ \frac{b-a}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3} \right\} \max_{t\in[a,b]} \left| f(t) \right|$$

$$\le \left\{ \frac{b-a}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3} \right\} \max_{t\in[a,b]} \left| f'(t) \right|$$

$$+ \left\{ \frac{b-a}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3} \right\} \max_{t\in[a,b]} \left| f(t) \right|$$

$$= \left\{ \frac{b-a}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3} \right\}.$$

$$\left[\left\{ \frac{b-a}{(1-\alpha)^2} + \frac{1}{(1-\alpha)^2} + \frac{\alpha(b-a)^2}{(1-\alpha)^3} \right\} \max_{t\in[a,b]} \left| f'(t) \right| + \max_{t\in[a,b]} \left| f(t) \right| \right]$$

$$= \frac{(b-a+1)(1-\alpha) + \alpha(b-a)^2}{(1-\alpha)^3} \left\| f(t) \right\|_{C^1[a,b]},$$

as required.

Theorem 6. The subspace $C^1[a,b] \subset H^1$ is invariant with respect to the Caputo-Fabrizio operator D_{at}^{α}.

Proof. We want to show that for all $f \in C^1[a,b]$, then $D_{at}^{\alpha} f \in C^1[a,b]$. From the Note 1, we know that $C^1[a,b] \subset H^1$. Let $f \in C^1[a,b]$. Then using Theorem 2 we conclude that $D_{at}^{\alpha} f \in C^1[a,b]$.

Theorem 7. Let $a, b \in \mathbb{R}, a < b, \Omega = [a, b]$ and $D(\Omega)$ the space of test functions. The operator $T^{\alpha}: D(\Omega) \to \mathbb{R}$ given by

$$T^{\alpha}(u) = \frac{1}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u'(\tau)\,d\tau,$$

is a distribution.

Proof. We have to proof that operator T^{α} is linear and continuous. For any $u, v \in D(\Omega)$, we obtain the linearity, as follows:

$$T^{\alpha}(\beta u + \gamma v) = \frac{1}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} (\beta u + \gamma v)'(\tau)\,d\tau$$
$$= \frac{\beta}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} (u)'(\tau)\,d\tau$$
$$+ \frac{\gamma}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} (v)'(\tau)\,d\tau = \beta T^{\alpha}(u) + \gamma T^{\alpha}(v).$$

Suppose that u is any element of the space of test function $D(\Omega)$. Then we know that $u \in W_0^{1,p}(\Omega)(1 \le p < \infty)$. Thus, $u(a) = u(b) = 0$ and

$$T^{\alpha}(u) = \frac{1}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u'(\tau)\,d\tau = \frac{1}{1-\alpha}\cdot$$
$$\left[u(b) - e^{-\frac{\alpha}{1-\alpha}(b-a)} u(a) - \frac{\alpha}{1-\alpha}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u(\tau)\,d\tau\right]$$
$$= -\frac{\alpha}{(1-\alpha)^2}\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} u(\tau)\,d\tau.$$

(7)

Now, to show the continuity of T^{α}, let take $u, u_n \in D(\Omega)$ such that $u_n \to u$. Using (7), we obtain

$$\left|T^{\alpha}(u) - T^{\alpha}(u_n)\right| =$$
$$= \frac{\alpha}{(1-\alpha)^2}\left|\int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} \left[u(\tau) - u_n(\tau)\right] d\tau\right|$$
$$\le \frac{\alpha}{(1-\alpha)^2} \sup_{\tau\in[a,b]} \left|u(\tau) - u_n(\tau)\right| \int_a^b e^{-\frac{\alpha}{1-\alpha}(b-\tau)} d\tau,$$

which tends to zero by the uniform convergence of the u_n and grace to that $e^{-\frac{\alpha}{1-\alpha}(b-\tau)} \in L^1(\Omega)$. Thus the operator T^{α} is a distribution.

4. ORDINARY FRACTIONAL DIFFERENTIAL EQUATIONS

Fractional differential equations frequently appear in various areas of engineering applications. Some examples of differential equations containing fractional derivative have been explored [26]. In [27], authors considered the following linear fractional differential equation

$$D_{at}^{\beta} x(t) - \lambda x(t) = f(t),$$

where $\beta = \alpha + 1$ such that $0 < \alpha \le 1, a \in (-\infty, t)$. In this section, we present some examples of the solution of differential equations containing the Caputo-Fabrizio fractional derivative.

Example 1. Let us consider the equation

$$D_{0t}^{\alpha} x(t) + c_2 \cdot D_{0t}^{\beta} x(t) - c_3 \cdot x(t) = 0, \quad (8)$$

$$x(0) = x_0, \quad (9)$$

where $c_2, c_3, \alpha, \beta, x_0 \in \mathbb{R}$ with $0 < \alpha, \beta < 1$.

Applying the definition of fractional Caputo-Fabrizio derivative, we rewrite equation (8) as

$$\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} x'(\tau)\,d\tau +$$
$$+ \frac{c_2}{1-\beta}\int_0^t e^{-\frac{\beta}{1-\beta}(t-\tau)} x'(\tau)\,d\tau = c_3 x(t). \quad (10)$$

Applying the Laplace transform to (10) and taking into account the condition (9), we get

$$\mathcal{L}\{x(t)\} = \frac{p(s)}{q(s)}, \quad (11)$$

where

$$p(s) = [x_0(1-\beta) + c_2 x_0(1-\alpha)]s + x_0\beta + c_2 x_0\alpha,$$
$$q(s) = [(1-\beta) + c_2(1-\alpha) - c_3(1-\alpha)(1-\beta)]s^2$$
$$+ \left[\beta + c_2\alpha - c_3(1-\alpha)\beta - c_3\alpha(1-\beta)\right]s - c_3\alpha\beta.$$

The function $q(s)$ has two roots

$$\gamma_1 = \frac{-\Delta_0 + \sqrt{\Delta}}{2\Delta_1}, \gamma_2 = \frac{-\Delta_0 - \sqrt{\Delta}}{2\Delta_1},$$

with

$$\Delta_0 = \beta + c_2\alpha - c_3(1-\alpha)\beta - c_3\alpha(1-\beta),$$
$$\Delta_1 = (1-\beta) + c_2(1-\alpha) - c_3(1-\alpha)(1-\beta),$$
$$\Delta = \Delta_0^2 + 4c_3 \cdot \alpha \cdot \beta\Delta_1.$$

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Case 1. When $\Delta > 0$: we have from (11) that

$$\mathcal{L}\{x(t)\} = \\ = \frac{\left[x_0(1-\beta)+c_2x_0(1-\alpha)\right]s+x_0\beta+c_2x_0\alpha}{(s-\gamma_1)(s-\gamma_2)} \quad (12) \\ = \frac{A}{s-\gamma_1}+\frac{B}{s-\gamma_2}=\frac{A(s-\gamma_2)+B(s-\gamma_1)}{(s-\gamma_1)(s-\gamma_2)}.$$

Using elementary calculations, we obtain

$$B=\frac{\left[c_1x_0(1-\beta)+c_2x_0(1-\alpha)\right]\gamma_2+c_1x_0\beta+c_2x_0\alpha}{\gamma_2-\gamma_1},$$

$$A=\frac{\left[c_1x_0(1-\beta)+c_2x_0(1-\alpha)\right]\gamma_1+c_1x_0\beta+c_2x_0\alpha}{\gamma_1-\gamma_2}.$$

Applying inverse Laplace transform to (12), one easily obtain

$$x(t)=A\cdot e^{\gamma_1 t}+B\cdot e^{\gamma_2 t}. \quad (13)$$

Case 2. When $\Delta < 0$: we obtain from (11) that

$$\mathcal{L}\{x(t)\}=\frac{p(s)}{q(s)}=\frac{1}{\Delta_1}\cdot\frac{\left[x_0(1-\beta)+c_2x_0(1-\alpha)\right]s-\left[-x_0\beta-c_2x_0\alpha\right]}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2+\left[\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta-\frac{\Delta_0^2}{4\Delta_1}\right)}\right]^2}$$

$$=\frac{x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot\frac{s+\frac{\Delta_0}{2\Delta_1}}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2+\left[\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta-\frac{\Delta_0^2}{4\Delta_1}\right)}\right]^2}$$

$$-\frac{x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot\frac{\frac{-x_0\beta-c_2x_0\alpha}{x_0(1-\beta)+c_2x_0(1-\alpha)}+\frac{\Delta_0}{2\Delta_1}}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2+\left[\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta-\frac{\Delta_0^2}{4\Delta_1}\right)}\right]^2}. \quad (14)$$

Using inverse Laplace transform, we obtain

$$x(t)=\frac{c_1x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot e^{-\frac{\Delta_0}{2\Delta_1}t}\cdot\cos\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta-\frac{\Delta_0^2}{4\Delta_1}\right)}\cdot t-$$

$$-\frac{c_1x_0(1-\beta)+c_2x_0(1-\alpha)}{\Delta_1}\cdot\frac{\frac{-c_1x_0\beta-c_2x_0\alpha}{c_1x_0(1-\beta)+c_2x_0(1-\alpha)}+\frac{\Delta_0}{2\Delta_1}}{\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta-\frac{\Delta_0^2}{4\Delta_1}\right)}}\cdot e^{-\frac{\Delta_0}{2\Delta_1}t}. \quad (15)$$

$$\cdot\sin\sqrt{\frac{1}{\Delta_1}\cdot\left(-c_3\alpha\beta-\frac{\Delta_0^2}{4\Delta_1}\right)}\cdot t.$$

Case 3. $\Delta = 0$: we obtain from (11), the following

$$\mathcal{L}\{x(t)\}=\frac{\left[c_1x_0(1-\beta)+c_2x_0(1-\alpha)\right]s+c_1x_0\beta+c_2x_0\alpha}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2}$$

$$=\frac{A_2}{s+\frac{\Delta_0}{2\Delta_1}}+\frac{B_2}{\left(s+\frac{\Delta_0}{2\Delta_1}\right)^2}.$$

Where A and B are constants given by

$$B_2=-\left[c_1x_0(1-\beta)+c_2x_0(1-\alpha)\right]\frac{\Delta_0}{2\Delta_1}+c_1x_0\beta+c_2x_0\alpha,$$

and

$$A_2=\frac{c_1x_0\beta+c_2x_0\alpha-B_2}{\Delta_0}\cdot 2\cdot\Delta_1,$$

respectively. Applying the inverse Laplace transform, we obtain

$$x(t)=A_2\cdot e^{\frac{\Delta_0}{2\Delta_1}t}+B_2\cdot t\cdot e^{\frac{\Delta_0}{2\Delta_1}t}. \quad (16)$$

This result can be formulated in the following Theorem.

Theorem 8. Let $0<\alpha,\beta<1, c_2, c_3, x_0 \in \mathbb{R}$.

1) If $\Delta > 0$, then the problem (8)-(9) has a unique solution, which is given by (13).

2) If $\Delta < 0$, then a unique solution to problem (8)-(9) exists and is given by (15).

3) If $\Delta = 0$, then the problem (8)-(9) has a unique solution, which is given by (16).

Example 2. Consider the equation of linear vibrations with the fractional dissipation term of order $\alpha \in (0,1)$.

$$f''(t)+c_1\cdot D_{0t}^{\alpha}f(t)+c_2\cdot f(t)=0, \quad (17)$$

$$f'(0)=f_1; f(0)=f_0, t>0, \quad (18)$$

where $c_1, c_2, f_1, f_2, \alpha \in \mathbb{R}$. Inserting the definition of Caputo-Fabrizio derivative into (17), we obtain the equation

$$f'''(t)+\frac{\alpha}{1-\alpha}f''(t)+\frac{(1-\alpha)c_2+c_1}{1-\alpha}f'(t) \\ +\frac{\alpha\cdot c_2}{1-\alpha}f(t)=0. \quad (19)$$

The corresponding characteristic equation of (19) is

$$r^3+\frac{\alpha}{1-\alpha}r^2+\frac{(1-\alpha)c_2+c_1}{1-\alpha}r+\frac{\alpha\cdot c_2}{1-\alpha}=0,$$

which, by using the Cardano's method, the roots are

$$\begin{cases} r_1 = S_1 + S_2 - \frac{a_1}{3}, \\ r_2 = -\frac{S_1+S_2}{2} - \frac{a_1}{3} + \frac{i\sqrt{3}}{2}(S_1 - S_2), \\ r_3 = -\frac{S_1+S_2}{2} - \frac{a_1}{3} - \frac{i\sqrt{3}}{2}(S_1 - S_2), \end{cases}$$

where

$$a_1 = \frac{\alpha}{1-\alpha}, a_2 = \frac{(1-\alpha)c_2 + c_1}{1-\alpha}, a_3 = \frac{\alpha \cdot c_2}{1-\alpha}.$$

$$Q = \frac{3a_2 - a_1^2}{9}, S_1 = \sqrt[3]{R + \sqrt{Q^3 + R^2}}.$$

$$R = \frac{9a_1a_2 - 27a_3 - 2a_1^3}{54}, S_2 = \sqrt[3]{R - \sqrt{Q^3 + R^2}}.$$

Taking into account the conditions (18), we finally obtain

$$f(t) = \frac{r_2(1-f_0) - f_1 + r_3}{r_1 - r_2} e^{r_1 t} + \frac{r_1(f_0 - 1) - (f_1 - r_3)}{r_1 - r_2} e^{r_2 t} + e^{r_3 t}.$$

5. FRACTIONAL INTEGRAL INEQUALITIES INVOLVING THE CAPUTO-FABRIZIO FRACTIONAL DERIVATIVE

In literature, few results have been obtained on fractional integral inequalities using Caputo-Fabrizio fractional operators [28]. Motivated by [29], we propose, using this operator, to establish some new integral inequalities. We write

$$D_{0t}^{\alpha} f(t) = \frac{1}{1-\alpha} \cdot \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f'(s) ds$$
$$= \frac{1}{1-\alpha} \cdot \left(f(t) - e^{-\frac{\alpha}{1-\alpha}t} f(0) \right) \quad (20)$$
$$- \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f(s) ds.$$

Let

$$G(f)(t) = \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)} f(s) ds. \quad (21)$$

Theorem 9. Let $\alpha \in (0,1)$, f and g are two functions that have the same sense of variation on $[0,\infty)$. Then

$$G(f) \cdot G(g) \le \frac{1}{1-\alpha} \cdot \left(1 - e^{-\frac{\alpha}{1-\alpha}t} \right) \cdot G(fg), \quad (22)$$

where G is given by (21).

Proof. Suppose that f and g are functions having the same sense of variation on $[0,\infty)$. Then, for all $\tau \ge 0, \rho \ge 0$, we have

$$(f(\tau) - f(\rho)) \cdot (g(\tau) - g(\rho)) \ge 0.$$

Hence,

$$f(\tau)g(\tau) + f(\rho)g(\rho) \ge f(\tau)g(\rho) + f(\rho)g(\tau). \quad (23)$$

Multiplying both sides of (23) by $\frac{\alpha^2}{(1-\alpha)^4} e^{-\frac{\alpha}{1-\alpha}(t-\tau)} e^{-\frac{\alpha}{1-\alpha}(t-\rho)}$ and integrating with respect to τ and ρ over $(0,t) x (0,t)$, we obtain the inequality

$$\frac{\alpha}{(1-\alpha)^2} \int_0^t \left\{ \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) g(\tau) d\tau \right\} e^{-\frac{\alpha}{1-\alpha}(t-\rho)} d\rho$$
$$+ \frac{\alpha}{(1-\alpha)^2} \int_0^t \left\{ \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)} f(\rho) g(\rho) d\rho \right\} e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau$$
$$\ge \frac{\alpha}{(1-\alpha)^2} \int_0^t \left\{ \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} f(\tau) d\tau \right\} e^{-\frac{\alpha}{1-\alpha}(t-\rho)} g(\rho) d\rho$$
$$+ \frac{\alpha}{(1-\alpha)^2} \int_0^t \left\{ \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)} f(\rho) d\rho \right\} e^{-\frac{\alpha}{1-\alpha}(t-\tau)} g(\tau) d\tau,$$

which is equivalent to

$$G(fg) \cdot \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)} d\rho$$
$$+ G(fg) \frac{\alpha}{(1-\alpha)^2} \int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)} d\tau \quad (24)$$
$$\ge G(f) \cdot G(g) + G(g) \cdot G(f).$$

Inequality (22) follows from (24).

Theorem 10. Let $0 < \alpha < 1$ and f, g two non negative functions having the same sense of variation on $[0,\infty)$ such that

1) $f(0) = g(0) = 0,$

2) $fg \ge D_{0t}^{\alpha}(fg).$

Then

$$D_{0t}^{\alpha}(fg)\le f(t)\cdot D_{0t}^{\alpha}(g)+g(t)\cdot D_{0t}^{\alpha}(f) -(1-\alpha)D_{0t}^{\alpha}(f)D_{0t}^{\alpha}(g). \quad (25)$$

Proof. From Theorem 9, we get

$$\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(f)(s)ds\right\}\cdot \cdot\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(g)(s)ds\right\} \le\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-s)}(fg)(s)ds\right\}\cdot \cdot\frac{1}{1-\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right). \quad (26)$$

Combining the expression (20) with the conditions $f(0)=g(0)=0,$ we obtain from (26) that

$$\left\{\frac{1}{1-\alpha}f(t)-D_{0t}^{\alpha}(f)(t)\right\}\cdot \cdot\left\{\frac{1}{1-\alpha}g(t)-D_{0t}^{\alpha}(g)(t)\right\} \le\left\{\frac{1}{1-\alpha}fg(t)-D_{0t}^{\alpha}(fg)(t)\right\}\cdot \cdot\frac{1}{1-\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right). \quad (27)$$

Using the condition $fg\ge D_{0t}^{\alpha}(fg)$, one easily obtain

$$0\le(fg)-D_{0t}^{\alpha}(fg)\le\frac{1}{1-\alpha}(fg)-D_{0t}^{\alpha}(fg). \quad (28)$$

Combining (28) with (27), we achieve

$$\left\{\frac{1}{1-\alpha}f(t)-D_{0t}^{\alpha}(f)(t)\right\}\cdot \cdot\left\{\frac{1}{1-\alpha}g(t)-D_{0t}^{\alpha}(g)(t)\right\} \le\frac{1}{1-\alpha}\left\{\frac{1}{1-\alpha}(fg)(t)-D_{0t}^{\alpha}(fg)(t)\right\}. \quad (29)$$

Inequality (25) follows from (29).

Theorem 11. Let $\alpha\in(0,1)$ and f,g,f',g' (f',g' are derivative of functions f,g, respectively) non-negative functions on $[0,\infty)$. Moreover let f,g have the same sense of variation on $[0,\infty)$ such that

1) $f(0)=g(0)=0,$

2) $fg\ge D_{0t}^{\alpha}(fg).$

Then

$$D_{0t}^{\alpha}(fg)\le f(t)\cdot D_{0t}^{\alpha}(g)+g(t)\cdot D_{0t}^{\alpha}(f). \quad (30)$$

Proof. Combining the fact that functions *f* and *g* have the same sense of variation with the conditions 1) and 2), we obtain from Theorem 10 the inequality

$$D_{0t}^{\alpha}(fg)+(1-\alpha)D_{0t}^{\alpha}(f)D_{0t}^{\alpha}(g) \le f(t)\cdot D_{0t}^{\alpha}(g)+g(t)\cdot D_{0t}^{\alpha}(f). \quad (31)$$

On the other hand, considering that functions f,g,f',g' are non-negatives, we conclude that $D_{0t}^{\alpha}f\ge0, D_{0t}^{\alpha}g\ge0, D_{0t}^{\alpha}fg\ge0$. Therefore, all terms of inequality (31) are non-negatives. Then, deleting the second term of the left hand side of (31), we obtain (30).

Theorem 12. Let $0<\alpha<1$ and f',g' two non-negative functions having the same sense of variation on $[0,\infty)$. Then,

$$D_{0t}^{\alpha}f(t)\cdot D_{0t}^{\alpha}g(t) \le\frac{1-\alpha}{\alpha^2}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}t}\right)\cdot G(f'g'). \quad (32)$$

Proof. As f',g' are two non-negative functions having the same sense of variation on $[0,\infty)$, then for all $\tau\ge0,\rho\ge0$ we have

$$f'(\tau)g'(\tau)+f'(\rho)g'(\rho) \ge f'(\tau)g'(\rho)+f'(\rho)g'(\tau).$$

(33)

Multiplying both sides of (33) by $\frac{\alpha}{(1-\alpha)^3}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}$ and integrating with respect to τ and ρ over $(0,t)\text{x}(0,t)$, we obtain the inequality

$$\frac{1}{1-\alpha}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f'(\tau)g'(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho +\frac{1}{1-\alpha}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}f'(\rho)g'(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau \ge\frac{\alpha}{1-\alpha}\frac{1}{1-\alpha}\int_0^t\left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f'(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}g'(\rho)d\rho +\frac{\alpha}{1-\alpha}\frac{1}{1-\alpha}\int_0^t\left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}f'(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}g'(\tau)d\tau,$$

which is equivalent to

$$\frac{1}{1-\alpha}G(f'g')\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho \geq \frac{\alpha}{1-\alpha}D_{at}^{\alpha}f(t)D_{at}^{\alpha}g(t). \quad (34)$$

Inequality (32) follows from (34).

Theorem 13. Let $0<\alpha<1$ and f',g two functions having the same sense of variation on $[0,\infty)$. Then,

$$D_{0t}^{\alpha}f(t)\cdot G(g)\leq \frac{1}{\alpha}\cdot\left(1-e^{-\frac{\alpha}{1-\alpha}(t-a)}\right)\cdot G(f'g). \quad (35)$$

Proof. As f',g are two functions having the same sense of variation on $[0,\infty)$, then for all $\tau\geq 0,\rho\geq 0$ we have

$$f'(\tau)g(\tau)+f'(\rho)g(\rho)\geq f'(\tau)g(\rho)+f'(\rho)g(\tau). \quad (36)$$

Multiplying both sides of (36) by $\frac{\alpha}{(1-\alpha)^3}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}$ and integrating with respect to τ and ρ over $(0,t)\,x\,(0,t)$, we obtain the inequality

$$\frac{1}{1-\alpha}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f'(\tau)g(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho$$
$$+\frac{1}{1-\alpha}\int_0^t\left\{\frac{\alpha}{(1-\alpha)^2}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}f'(\rho)g(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}d\tau$$
$$\geq\frac{\alpha}{(1-\alpha)^2}\int_0^t\left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\tau)}f'(\tau)d\tau\right\}e^{-\frac{\alpha}{1-\alpha}(t-\rho)}g(\rho)d\rho$$
$$+\frac{\alpha}{(1-\alpha)^2}\int_0^t\left\{\frac{1}{1-\alpha}\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}f'(\rho)d\rho\right\}e^{-\frac{\alpha}{1-\alpha}(t-\tau)}g(\tau)d\tau,$$

which is equivalent to

$$\frac{1}{1-\alpha}G(f'g)\int_0^t e^{-\frac{\alpha}{1-\alpha}(t-\rho)}d\rho\geq D_{at}^{\alpha}f(t)\cdot G(g). \quad (37)$$

Inequality (35) follows from (37).

6. CONCLUSION

In this paper, we have investigated some linear differential equations involving Caputo-Fabrizio fractional derivative. In addition, some theorems to characterize the Caputo-Fabrizio derivative in certain spaces have been proven. Statements on fractional inequalities containing the Caputo-Fabrizio derivative were also presented.

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