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Results on $(LCS)_n$-Manifolds

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Abstract

We obtain a some result on the $(LCS)_n$ manifolds satisfying certain conditions on the W_2-curvature tensor.

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1. Introduction

An n-dimensional Lorentzian manifold M is a smooth connected paracompact Hausdorff manifold with a Lorentzian metric g, that is, M admits a smooth symmetric tensor field g of type (0, 2) such that for each point $p \in M$, the tensor $g_p : T_pM \times T_pM \to R$ is a non-degenerate inner product of signature $(-, +, \ldots, +)$, where T_pM denotes the tangent vector space of M at p and R is the real number space.

Definition 1.1. *In a Lorentzian manifold* (M, g) *a vector field* P *defined by*

$$g(X, P) = A(X),$$

for any vector field $X \in \chi(M)$ *is said to be a concircular vector field if*

$$(\nabla_X A)(Y) = \alpha[g(X, Y) + \omega(X)A(Y)],$$

where α *is a non-zero scalar function,* A *is a 1-form and* ω *is a closed 1-form.*

2. Preliminaries

Let M be a n-dimensional Lorentzian manifold admitting a unit timelike concircular vector field ξ, called the characteristic vector field of the manifold. Then we have

(2.1) $\quad g(\xi, \xi) = -1.$

Since ξ is a unit concircular vector field, there exists a non-zero 1-form η such that,

(2.2) $\quad g(X, \xi) = \eta(X),$

the equation of the following form holds

(2.3) $\quad (\nabla_X \eta)(Y) = \alpha[g(X, Y) + \eta(X)\eta(Y)], \ (\alpha \neq 0)$

for all vector fields X, Y, where ∇ denotes the operator of covariant differentiation with respect to Lorentzian metric g and α is a non-zero scalar function satisfying

(2.4) $\quad \nabla_X \alpha = (X\alpha) = d\alpha(X) = \rho\eta(X),$

ρ being a certain scalar function given by $\rho = -(\xi\alpha)$. If we put

(2.5) $\quad \varphi X = \frac{1}{\alpha} \nabla \times \xi,$

then from (2.3) and (2.5), we have

(2.6) $\quad \varphi X = X + \eta(X)\xi,$

from which it follows that φ is a symmetric (1, 1) tensor. Thus the Lorentzian manifold M together with the unit timelike concircular vector field ξ, its associated 1-form η and (1, 1) tensor field φ is said to be a Lorentzian concircular structure manifold (briefly *(LCS)n*-manifold)[13]. Especially, if we take $\alpha = 1$, then we can obtain the Lorentzian para-Sasakian structure of Matsumoto [7]. In a $(LCS)_n$-manifold, the following relations hold [13]:

(2.7) $\quad \eta(\xi) = -1, \quad \varphi\xi = 0, \quad \eta(\varphi X) = 0$

(2.8) $\quad \varphi^2 = I + \eta \otimes \xi,$

(2.9) $\quad g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y),$

(2.10) $\quad \eta(R(X, Y)Z) = (\alpha^2 - \rho)(g(Y, Z)\eta(X) - g(X, Z)\eta(Y)),$

(2.11) $\quad R(\xi, X)Y = (\alpha^2 - \rho)(g(X, Y)\xi - \eta(Y)X),$

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(2.12) $R(X, Y)\xi = (\alpha^2 - \rho)(\eta(Y)X - \eta(X)Y),$

(2.13) $R(\xi, X)\xi = (\alpha^2 - \rho)(\eta(X)\xi + X),$

(2.14) $S(X, \xi) = (n - 1)(\alpha^2 - \rho)\eta(X),$

(2.15) $S(\varphi X, \varphi Y) = S(X, Y) + (n - 1)(\alpha^2 - \rho)\eta(X)\eta(Y),$

for all vecotr fields X, Y, Z, where S is the Ricci tensor and Q is the Ricci operator given by

$$S(X, Y) = g(QX, Y).$$

An $(LCS)_n$-manifold M is said to be Einstein if its Ricci tensor S is of the form

(2.16) $S(X, Y) = ag(X, Y),$

for any vector fields X, Y, where a is a function on M.

In [9], Pokhariyal and Mishra have defined the curvature tensor W_2, given by

(2.17) $W_2(X, Y, U, V) = R(X, Y, U, V) + 1/(n-1) [g(X, U)S(Y, V) - g(Y, U)S(X, V)],$

Consider a $(LCS)_n$-manifold satisfying $W_2 = 0$ in (2.17), then we have

(2.18) $R(X, Y, U, V) = 1/(n-1) [g(Y, U)S(X, V)$

$- g(X, U)S(Y, V)].$

Putting $X = U = \xi$ in (2.18) then using (2.11) and (2.14), we obtain

(2.19) $S(Y, V) = (\alpha^2 - \rho)(n - 1)g(Y, V).$

Thus, M is an Einstein manifold.

Definition 2.2. *An $(LCS)_n$ manifold is called W_2-semisymmetric if it satisfies*

(2.20) $R(X, Y) \cdot W_2 = 0,$

where $R(X, Y)$ is to be considered as a derivation of the tensor algebra at each point of the manifold for tangent vectors X, Y.

In a $(LCS)_n$-manifold the W_2-curvature tensor satisfies the condition

(2.21) $\eta(W_2(X, Y)Z) = 0.$

3. $(LCS)_n$-Manifolds Satisfying $P(X, Y) \cdot W_2 = 0$

The projective curvature tensor P is defined as [19]

(3.1) $P(X, Y)Z = R(X, Y)Z - 1/(n-1)$

$[S(Y, Z)X - S(X, Z)Y].$

Using (2.11) and (2.14), equation (3.1) reduces to

(3.2) $P(\xi, Y)Z = (\alpha^2 - \rho)g(Y, Z)\xi - 1/(n-1)\, S(Y, Z)\xi.$

Let us suppose that in $(LCS)_n$-manifold

$P(X, Y) \cdot W_2 = 0.$

This condition implies that

(3.3) $P(X, Y)W_2(U, V)Z - W_2(P(X, Y)U, V)Z$

$-W_2(U, P(X, Y)V)Z - W_2(U, V)P(X, Y)Z = 0.$

Put $X = \xi$ in (3.3) and then taking the inner product with ξ, we obtain

(3.4) $g(P(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(P(\xi, Y)U, V)Z, \xi)$

$-g(W_2(U, P(\xi, Y)V)Z, \xi)$

$- g(W_2(U, V)P(\xi, Y)Z, \xi) = 0.$

Using (3.2) in (3.4), we obtain

(3.5) $(\alpha^2 - \rho)[-g(Y, W_2(U, V)Z)$

$- g(Y, U)\eta(W_2(\xi, V)Z) - g(Y, V)\eta(W_2(U, \xi)Z)$

$-g(Y, Z)\eta(W_2(U, V)\xi)] + \frac{1}{1-n}[S(Y, (W_2(U, V)Z)$

$+ S(Y, U)\eta(W_2(\xi, V)Z) + S(Y, V)\eta(W_2(U, \xi)Z)$

$+ S(Y, Z)\eta(W_2(U, V)\xi)].$

By using (2.21) in (3.5), we get

(3.6) $(\alpha^2 - \rho)[g(Y, W_2(U, V)Z)]$

$- 1/(n-1)\, [S(Y, W_2(U, V)Z)] = 0.$

Taking $U = Z = \xi$ in (3.6) and using (2.17) and (2.11), we have

(3.7) $S(QY, V) = 2(n - 1)(\alpha^2 - \rho)S(Y, V)$

$- (\alpha^2 - \rho)^2(n - 1)^2 g(Y, V).$

This implies that

(3.8) $QY = (n - 1)(\alpha^2 - \rho)Y.$

From this, we get

(3.9) $S(Y, V) = (n - 1)(\alpha^2 - \rho)g(Y, V).$

Thus, we can state the following:

Theorem 3.1. *An* $(LCS)_n$*-manifold M satisfying the condition* $P(X, Y).W_2 = 0$*, then M is an Einstein manifold.*

4. $(LCS)_n$-Manifolds Satisfying $\tilde{Z}(X, Y) \cdot W_2 = 0$

The concircular curvature tensor $\tilde{Z}$ is defined as [18]

(4.1) $\tilde{Z}(X, Y)Z = R(X, Y)Z - r/n(n-1)\,[g(Y, Z)X - g(X, Z)Y]$.

Using (2.11) and (2.14), equation (4.1) reduces to

(4.2) $\tilde{Z}(\xi, Y)Z = [(\alpha^2 - \rho) - r/n(n-1)\,][g(Y, Z)\xi - \eta(Z)Y]$.

Now consider in an $(LCS)_n$-manifold

This condition implies that

$$\tilde{Z}(X, Y) \cdot W_2 = 0.$$

(4.3) $\tilde{Z}(X, Y)W_2(U, V)Z - W_2(\tilde{Z}(X, Y)U, V)Z$

$-W_2(U, \tilde{Z}(X, Y)V)Z - W_2(U, V)\tilde{Z}(X, Y)Z = 0.$

Put $X = \xi$ in (4.3) and then taking the inner product with ξ, we obtain

(4.4) $g(\tilde{Z}(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(\tilde{Z}(\xi, Y)U, V)Z, \xi)$

$-g(W_2(U, \tilde{Z}(\xi, Y)V)Z, \xi)$

$- g(W_2(U, V)\tilde{Z}(\xi, Y)Z, \xi) = 0.$

Using (4.2) in (4.4), we obtain

(4.5) $[(\alpha^2 - \rho) - r/n(n-1)\,]\;[-g(Y, W_2(U, V)Z)$

$- g(Y, U)\eta(W_2\xi, V)Z) - g(Y, V)\eta(W_2(U, \xi)Z)$

$- g(Y, Z)\eta(W_2(U, V)\xi) - \eta(Y)\eta(W_2(U, V)Z)$

$+\eta(U)\eta(W_2(V, Y)Z) + \eta(V)\eta(W_2(U, Y)Z)]$

$+ \eta(Z)\eta(W_2(U, V)Y)] = 0.$

By using (2.21) in (4.5), we get

(4.6) $[(\alpha^2 - \rho) - r/n(n-1)][g(Y, W(U, V)Z)] = 0.$

But $(\alpha^2 - \rho) - r/n(n-1) \neq 0$. So

(4.7) $W_2(U, V, Z, Y) = 0.$

In view of (2.17) and (4.7), it follows that

(4.8) $R(U, V, Z, Y) = \frac{1}{1-n}[g(V, Z)S(U, Y) - g(U, Z)S(V, Y)].$

Contracting (4.8), we have (4.9)

$S(V, Z) = (n-1)g(V, Z).$

Therefore, M is an Einstein manifold.

Theorem 4.2. *If on a* $(LCS)_n$*-manifold M, the condition* $\tilde{Z}(X, Y) \cdot W_2 = 0$*, then M is an Einstein manifold.*

5. $(LCS)_n$-manifolds satisfying $N(X, Y) \cdot W_2 = 0$

The conhormonic curvature tensor N is defined as [3]

(5.1) $N(X, Y)Z = R(X, Y)Z - \frac{1}{n-2}[S(Y, Z)X - S(X, Z)Y$

$+ g(Y, Z)QX - g(X, Z)QY].$

Using (2.11) and (2.14), equation (5.1) reduces to

(5.2) $N(\xi, Y)Z = -(\alpha^2 - \rho)/(n-2)\,[g(Y, Z)\xi - \eta(Z)Y]$

$- 1/(n-2)\,[S(Y, Z)\xi - \eta(Z)QY].$

Now consider in an $(LCS)_n$-manifold

This condition implies that

$$N(X, Y) \cdot W_2 = 0.$$

(5.3) $N(X, Y)W_2(U, V)Z - W_2(N(X, Y)U, V)Z$

$-W_2(U, N(X, Y)V)Z - W_2(U, V)N(X, Y)Z = 0.$

Put $X = \xi$ in (5.3) and then taking the inner product with ξ, we obtain

(5.4) $g(N(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(N(\xi, Y)U, V)Z, \xi)$

$-g(W_2(U, N(\xi, Y)V)Z, \xi) - g(W_2(U, V)N(\xi, Y)Z, \xi) = 0.$

Using (5.2) in (5.4), we obtain

(5.5) $\frac{(\alpha 2 - \rho)}{n-2}[-g(Y, W_2(U, V)Z) - g(Y, U)\eta(W_2(\xi, V)Z)$

$- g(Y, V)\eta(W_2(U, \xi)Z) - g(Y, Z)\eta(W_2(U, V)\xi)$

$- \eta(Y)\eta(W_2(U, V)Z) + \eta(U)\eta(W_2(Y, V)Z)$

$+\eta(V)\eta(W_2(U, Y)Z) + \eta(Z)\eta(W_2(U, V)Y)]$

$- \frac{1}{n-2}[-S(Y, W_2(U, V)Z) - S(Y, U)\eta(W_2(\xi, V)Z)$

$- S(Y, V)\eta(W_2(U, \xi)Z) - S(Y, Z)\eta(W_2(U, V)\xi)$

$-\eta(QY)\eta(W_2(U, V)Z) + \eta(U)\eta(W_2(QY, V)Z)$

$+\eta(V)\eta(W_2(U, QY)Z) + \eta(Z)\eta(W_2(U, V)QY)] = 0.$

By using (2.21) in (5.5), we get

(5.6) $\{(\alpha^2 - \rho)/(n-2)\}g(Y, W_2(U, V)Z)$

$+ 1/(n-2)\, S(Y, W_2(U, V)Z) = 0.$

Taking $U = Z = \xi$ in (5.6) and then using (2.17) and (2.13), we have

(5.7) $S(QY, V) = (\alpha^2 - \rho)(n-2)S(Y, V) + (\alpha^2 - \rho)^2(n-1)g(Y, V).$

Thus, we can state the following.

Theorem 5.3. *If on a* $(LCS)_n$*-manifold M, the condition* $N(X, Y) \cdot W_2 = 0$*, then equation* (5.7) *is satisfied on M.*

6. $(LCS)_n$-manifolds satisfying $\tilde{M}(X, Y) \cdot W_2 = 0$

The M-projective curvature tensor $\tilde{M}$ is defined as [10]

(6.1) $\tilde{M}(X, Y)Z = R(X, Y)Z - 1/2(n-1)\,[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY].$

Using (2.11) and (2.14), equation (6.1) reduces to

(6.2) $\tilde{M}(\xi, Y)Z = (\alpha^2 - \rho)/2\,[g(Y, Z)\xi - \eta(Z)Y] - 1/2(n-1)\,[S(Y, Z)\xi - \eta(Z)QY]$

Now consider in an $(LCS)_n$-manifold

$$\tilde{M}(X, Y) \cdot W_2 = 0.$$

This condition implies that

(6.3) $\tilde{M}(X, Y)W_2(U, V)Z - W_2(\tilde{M}(X, Y)U, V)Z - W_2(U, \tilde{M}(X, Y)V)Z - W_2(U, V)\tilde{M}(X, Y)Z = 0.$

Put $X = \xi$ in (6.3) and then taking the inner product with ξ, we obtain

(6.4) $g(\tilde{M}(\xi, Y)W_2(U, V)Z, \xi) - g(W_2(\tilde{M}(\xi, Y)U, V)Z, \xi) - g(W_2(U, \tilde{M}(\xi, Y)V)Z, \xi) - g(W_2(U, V)\tilde{M}(\xi, Y)Z, \xi) = 0.$

Using (6.2) in (6.4), we obtain

(6.5) $\frac{(\alpha 2 - \rho)}{2}[-g(Y, W_2(U, V)Z) - g(Y, U)\eta(W_2(\xi, V)Z) - g(Y, V)\eta(W_2(U, \xi)Z) - g(Y, Z)\eta(W_2(U, V)\xi) - \eta(Y)\eta(W_2(U, V)Z) + \eta(U)\eta(W_2(Y, V)Z) + \eta(V)\eta(W_2(U, Y)Z) + \eta(Z)\eta(W_2(U, V)Y)] + \frac{1}{2(n-1)}[S(Y, W_2(U, V)Z) + S(Y, U)\eta(W_2(\xi, V)Z) + S(Y, V)\eta(W_2(U, \xi)Z) + S(Y, Z)\eta(W_2(U, V)\xi) + \eta(QY)\eta(W_2(U, V)Z) - \eta(U)\eta(W_2(QY, V)Z) - \eta(V)\eta(W_2(U, QY)Z) - \eta(Z)\eta(W_2(U, V)QY)] = 0.$

By using (2.21) in (6.5), we get

(6.6) $(\alpha^2 - \rho)/2\; g(Y, W_2(U, V)Z) - 1/2(n-1)\; S(Y, W_2(U, V)Z) = 0.$

Taking $U = Z = \xi$ in (6.6) and then using (2.17) and (2.13), we have

(6.7) $S(QY, V) = (\alpha^2 - \rho)2(n-1) - (\alpha^2 - \rho)^2(n-1)^2 g(V, Y).$

This implies that

(6.8) $QY = (n-1)(\alpha^2 - \rho)Y.$

Which gives

(6.9) $S(Y, V) = (n-1)(\alpha^2 - \rho)g(Y, V).$

Thus, we can state the following.

Theorem 6.4. *If on a* $(LCS)_n$*-manifold M, the condition* $\tilde{M}(X, Y) \cdot W_2 = 0$*, then M is an Einstein manifold.*

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