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Haar Wavelet Collocation Method for Solving the Telegraph Equation with Variable Coefficients

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ABSTRACT:

The telegraph equation is one of the important equations of mathematical physics. In the current work, we implement the Haar wavelet collocation method (HWCM) to find numerical solution of one-dimensional telegraph equation with variable coefficients. Two illustrative numerical examples are given to show the stability and accuracy of the present method. The obtained numerical results are found to be in good agreement with known analytical solutions and also with earlier studies.

Keywords: Telegraph equations; Variable coefficients; Haar wavelets; Collocation method.

1. INTRODUCTION

In the present work we are dealing with the numerical approximation of the second order telegraph equation with variable coefficients problem has the following form [19]:

$$\frac{\partial^2 g}{\partial t^2}(x,t)+2\alpha(x,t)\frac{\partial g}{\partial t}(x,t)+\beta^2(x,t)\,g(x,t)$$
$$=B(x,t)\frac{\partial^2 g}{\partial x^2}(x,t)+k(x,t),\ c\le x<d,\ t\ge 0, \qquad (1.1)$$

provided with the following initial conditions,

$$g(x,0)=f(x),\ c\le x<d,$$
$$\frac{\partial g}{\partial t}(x,0)=f_1(x),\ c\le x<d,$$

and the boundary conditions as follows,

$$g(c,t)=q(t),\ g(d,t)=q_1(t),\ t\ge 0.$$

Where $f(x), f_1(x)$ and $q(t), q_1(t)$ are known functions, the function $g(x,t)$ is unknown and where $\alpha(x,t),\beta(x,t)$, $B(x,t)$ are variable coefficients. The telegraph equation is applied in the modeling for transmission and propagation of electrical signals in a cable transmission line [1, 30]. The telegraph equation plays a very important role in communication systems because by using this equation we can able to analyze signal failure in the system, consequently it leads to maximum output and minimum error. The mathematical derivation for the telegraph equation has been given in [20] in terms of current and voltage. Telegraph equation also finds its applications in radio frequency and microwaves fields [29]. In fact the telegraph equation is more convenient than ordinary diffusion equation in modeling reaction–diffusion for many branches of sciences. For example biologists find these equations in the study of pulsate blood flow in arteries and in one- dimensional random motion of bugs along a hedge [21]. Also parallel flows of viscous Maxwell fluids [4] and the propagation of acoustic waves in Darcy-type porous media [8] are just some of the phenomena governed [7,14] by equation (1.1). Clearly the equation (1.1) and its solution have applications in science and engineering.

In recent times, much concentration has been given by the various researchers for the analysis, evolution and employment of the numerical schemes for solving the telegraph equations. Aloy, R. et al. proposed discrete eigenfunction method [1] for solving the telegraph equation with variable coefficients. Mohanty, R. K. et al. [22] established higher order difference schemes for solving nonlinear hyperbolic equations with variable coefficients. Shivanian, E. et al. implemented the meshless local radial point interpolation for solving 1D linear telegraph equation with variable coefficients [32]. Tchebyshev-Galerkin operational matrix method is introduced by Abd-Elhameed, W. M. et al. [2] for finding the solution of linear and nonlinear hyperbolic telegraph type equations. Dehghan, M. et al. [5] studied the He’s variational iteration method for solving the telegraph equation. Lakestani, M. et al. [16], Mittal, R. C. et al. [24] and Saadatmandi, A. et al. [31] worked on evaluating the numerical schemes for solving hyperbolic telegraph equations. Also authors in [3,5,8,13] are implemented a numerical techniques for solving the telegraph equation with variable coefficients.

Wavelet analysis is a modern branch of mathematics which is extensively applied in image processing, signal analysis and numerical analysis etc. In 1910, A. Haar founded the term wavelet [10]. His primary theory has been developed newly into a broad number of applications, but initially it supports for the recoganization of different functions by a combination of step functions and wavelets over particular interval widths. A brief note on establishment of the Haar wavelets and its applications can be found in [17,12,9,15]. Numerical techniques based on Haar wavelet are well recognized methods for solving different types of ordinary differential equations (ODEs) and partial differential equations (PDEs) because of its property of localization. The advantages of the Haar wavelets are representation of sparse matrix, implementation of fast algorithms, more precise and less computation time. Many researchers have studied on Haar wavelet method to solve different types of ODEs and PDEs which are listed in [10,18,25,28].

In order to explain our arguments in a systematic way, we have look upon the article in the following sections: in Section 2, we discussed about the notations of Haar wavelets and the evaluation of their integrals. Section 3 deals with numerical algorithm based on HWCM is introduced. We have solved two test examples by using HWCM and compared obtained numerical results with exiting method in Section 4. In section 5 explanations about the obtained numerical results are given. Finally in Section 6 the conclusion of article is given.

2. HAAR WAVELETS AND THEIR INTEGRALS

Let us consider the interval $x \in [c,d]$ divided into 2^{J+1} subintervals of equal length $h_{dx} = \frac{d-c}{2^{J+1}}$. Where 'J' indicates the maximum level of resolution. The Haar wavelet family $h_b(x)$ is defined as follows,

$$h_b(x) = \begin{cases} 1, & \text{for } x \in \left[\frac{r}{m}, \frac{r+0.5}{m}\right), \\ -1, & \text{for } x \in \left[\frac{r+0.5}{m}, \frac{r+1}{m}\right), \\ 0, & \text{otherwise.} \end{cases} \qquad (2.1)$$

Where $m=2^i$ and $i = 0,1,2,\ldots,J$ The parameter $r=0,1,2,\ldots,m-1$ is referred as the translation parameter. Where 'b' indicates the number of daughter Haar wavelets which are evaluated using the formula $b=m+r+1$. When $b=1,2.$ $h_1(x)$ and $h_2(x)$ are called father and mother wavelet respectively [18]. The value of 'b' lies between $2 < b \le 2^{J+1}$. The collocation points $x_a = \frac{a-0.5}{2L}$, where $a = 1,2,....,2L$. Where $L = 2^J$.

We introduce the following notations are as follows

$$P_{1,b}(x) = \int_0^x h_b(x)\,dx, \qquad (2.2)$$

Integrating the Haar functions for 'λ' times. We get

$$P_{\lambda,b}(x) = \frac{1}{(\lambda-1)!}\int_0^x (x-s)^{\lambda-1} h_b(s)\,ds. \qquad (2.3)$$

All these integrals are evaluated directly as explained by Lepik, U. [18]. When $b=1$ the equation (2.3) gives

$$P_{\lambda,1}(x) = \frac{1}{(\lambda)!}(x)^{\lambda}. \qquad (2.4)$$

For $b \ge 2$ from the equation (2.3), we get

$$P_{\lambda,b}(x) = \frac{1}{\lambda!}\begin{cases} 0, & \text{if } x \in [0, \alpha(b)), \\ [x-\alpha(b)]^{\lambda}, & \text{if } x \in [\alpha(b), \beta(b)), \\ \left[[x-\alpha(b)]^{\lambda} - 2[x-\beta(b)]^{\lambda}\right], & \text{if } x \in [\beta(b), \gamma(b)), \\ \left[[x-\alpha(b)]^{\lambda} - 2[x-\beta(b)]^{\lambda} + [x-\gamma(b)]^{\lambda}\right], & \text{if } x \in [\gamma(b), 1). \end{cases} \qquad (2.5)$$

Where $\alpha(b) = \frac{r}{m}$, $\beta(b) = \frac{r+0.5}{m}$ and $\gamma(b) = \frac{r+1}{m}$.

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3. EXPLANATION OF THE PROPOSED NUMERICAL SCHEME

Consider the telegraph equation with variable coefficients problem (1.1) along with the initial and boundary conditions in the interval $x \in [0,1)$ and $t \geq 0$.

Step 1: Let us approximate the highest derivative in the equation (1.1) in terms of Haar wavelets as follows,

$$\ddot{g}''(x,t) = \sum_{b=1}^{2L} e_b h_b(x). \quad (3.1)$$

Where '..' and '″' means differentiation with respect to 't' and 'x' respectively. Where e_b's are Haar wavelet coefficients in the interval $t \in [t_n, t_{n+1}]$ and $h_b(x)$ is Haar wavelet family.

Step 2: On twice integration of equation (3.1) w.r.t. to t from t_n to t and w.r.t. x from 0 to x, following equations are obtained.

$$\dot{g}''(x,t) = \dot{g}''(x,t_n) + (t-t_n)\sum_{b=1}^{2L} e_b h_b(x), \quad (3.2)$$

$$g''(x,t) = g''(x,t_n) + \frac{(t-t_n)^2}{2}\sum_{b=1}^{2L} e_b h_b(x) + \dot{g}''(x,t_n)(t-t_n), \quad (3.3)$$

$$g'(x,t) = \frac{(t-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{1,b}(x) + (t-t_n)\left[\dot{g}'(x,t_n) - \dot{g}'(0,t_n)\right] + g'(x,t_n) - g'(0,t_n) + g'(0,t), \quad (3.4)$$

$$g(x,t) = \frac{(t-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{2,b}(x) + (t-t_n)\left[\dot{g}(x,t_n) - \dot{g}(0,t_n) - x\dot{g}'(0,t_n)\right] + g(x,t_n) - g(0,t_n) - x\left[g'(0,t_n) - g'(0,t)\right] + g(0,t), \quad (3.5)$$

$$\dot{g}(x,t) = (t-t_n)\sum_{b=1}^{2L} e_b P_{2,b}(x) + \left[\dot{g}(x,t_n) - \dot{g}(0,t_n) - x\dot{g}'(0,t_n)\right] + x\dot{g}'(0,t) + \dot{g}(0,t), \quad (3.6)$$

$$\ddot{g}(x,t) = \sum_{b=1}^{2L} e_b P_{2,b}(x) + x\ddot{g}'(0,t) + \ddot{g}(0,t). \quad (3.7)$$

Step 3: By substituting the boundary conditions in the equations (3.2) to (3.7), we obtain the following equations which are,

$$g'(0,t_n) - g'(0,t) = \frac{(t-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{2,b}(1) + (t-t_n)\left[\dot{g}(1,t_n) - \dot{g}(0,t_n) - \dot{g}'(0,t_n)\right] + g(1,t_n) - g(0,t_n) + g(0,t) - g(1,t), \quad (3.8)$$

$$\ddot{g}'(0,t) = -\sum_{b=1}^{2L} e_b P_{2,b}(1) - \ddot{g}(0,t) + \ddot{g}(1,t). \quad (3.9)$$

Step 4: Substituting the equations (3.8) and (3.9) into (3.2) to (3.7) and discritizing these results $x \to x_a$ and $t \to t_{n+1}$, we obtain the following equations.

$$g(x_a,t_{n+1}) = \frac{(t_{n+1}-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{2,b}(x_a) + (t_{n+1}-t_n)\left[\dot{g}(x_a,t_n) - \dot{g}(0,t_n)\right] + g(x_a,t_n) - g(0,t_n) - x_a\frac{(t_{n+1}-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{2,b}(1) - x_a(t_{n+1}-t_n)\left[\dot{g}(1,t_n) - \dot{g}(0,t_n)\right] - x_a\left[g(1,t_n) - g(0,t_n) + g(0,t_{n+1}) - g(1,t_{n+1})\right] + g(0,t_{n+1}), \quad (3.10)$$

$$g'(x_a,t_{n+1}) = \frac{(t_{n+1}-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{2,b}(x_a) + (t_{n+1}-t_n)\left[\dot{g}'(x_a,t_n)\right] + g'(x_a,t_n)$$
$$-\frac{(t_{n+1}-t_n)^2}{2}\sum_{b=1}^{2L} e_b P_{1,b}(1) - (t_{n+1}-t_n)\left[\dot{g}(1,t_n)-\dot{g}(0,t_n)\right]$$
$$-\left[g(1,t_n)-g(0,t_n)+g(0,t_{n+1})-g(1,t_{n+1})\right], \quad (3.11)$$

$$g''(x_a,t_{n+1}) = \frac{(t_{n+1}-t_n)^2}{2}\sum_{b=1}^{2L} e_b h_b(x_a) + (t_{n+1}-t_n)\dot{g}''(x_a,t_n) + g''(x_a,t_n), \quad (3.12)$$

$$\dot{g}(x_a,t_{n+1}) = (t_{n+1}-t_n)\sum_{b=1}^{2L} e_b P_{2,b}(x_a) + \left[\dot{g}(x_a,t_n)-\dot{g}(0,t_n)\right] - x_a(t_{n+1}-t_n)\sum_{b=1}^{2L} e_b P_{2,b}(1) -$$
$$x_a\left[\dot{g}(1,t_n)-\dot{g}(0,t_n)\right] - x_a\left[\dot{g}(0,t_{n+1})-\dot{g}(1,t_{n+1})\right] + \dot{g}(0,t_{n+1}), \quad (3.13)$$

$$\ddot{g}(x_a,t_{n+1}) = \sum_{b=1}^{2L} e_b\left[P_{2,b}(x_a) - x_a P_{2,b}(1)\right] - x_a\left[\ddot{g}(0,t_{n+1})-\ddot{g}(1,t_{n+1})\right] + \ddot{g}(0,t_{n+1}). \quad (3.14)$$

Step 5: Where e_b's (Haar wavelet coefficients) are unknowns in the equation (3.10). In order to find these unknowns, let us substitute the equations (3.10) to (3.14) into the equation (1.1). We will get the system of linear algebraic equations. The obtained systems of linear algebraic equations were solved by using the MATLAB software. Then substituting obtained coefficients into the equation (3.10). We can successively evaluate the approximate solution at different values of $x \in [0,1)$ and time $'t'$.

4. NUMERICAL RESULTS

In this section, we employ the HWCM on two well-studied numerical examples with known analytical solutions. The efficiency of the numerical scheme is measured by evaluating the L_2, L_∞, RMS errors and ROC (Rate of convergence) which we discuss in this article has the following form.

$$L_\infty = \max_{1\le p\le Z}|g(p)-g_{exat}(p)|,\ L_2 = \sqrt{\sum_{p=1}^{Z}|g(p)-g_{exat}(p)|^2},$$
$$ROC = \frac{\ln\left(\frac{F(Z_1)}{F(Z_2)}\right)}{\ln\left(\frac{Z_2}{Z_1}\right)},\ RMS = \sqrt{\frac{1}{Z}\left(\sum_{p=1}^{Z}|g(p)-g_{exat}(p)|^2\right)}.$$

Where $Z =$ number of nodes, $g(p)$ = approximate solution at $x = x_p$, $g_{exat}(p)$ =exact solution at $x = x_p$, $F(Z_P)$ = L_2 or L_∞ error norm with Z_P node points at $x = x_p$ for $p = 1, 2, .., Z$ respectively.

Example 1. Consider the non-homogeneous telegraph equation (1.1) with variable coefficients $\alpha(x,t) = e^x$, $\beta(x,t) = \sin(x)$, $B(x,t) = (1+x^2)$ and $k(x,t) = e^{-t}\sinh(x)\left[-2e^x + \sin^2(x) - x^2\right]$ with the following initial and boundary conditions [19],

$$g(x,0) = \sinh(x),\ \frac{\partial g}{\partial t}(x,0) = -\sinh(x),\ 0 \le x < 1,$$
$$g(0,t) = 0,\ g(1,t) = e^{-t}\sinh(1),\ t \ge 0.$$

The analytical solution is given by,

$$g(x,t) = e^{-t}\sinh(x).$$

Table 1: Comparison of L_2, L_∞ and RMS error norms for variable coefficients problem in example 1 with J = 2 and dt = 0.005.

t	L_∞ **MLS [19]**	L_∞ **HWCM**	L_2 **MLS [19]**	L_2 **HWCM**	RMS **MLS [19]**	RMS **HWCM**
1	5.1696E-4	2.7196E-6	1.1991E-3	4.7459E-6	3.6153E-4	1.6779E-6
3	5.5395E-5	7.5744E-5	1.2544E-4	1.5328E-4	3.7823E-5	5.4193E-5
5	6.8645E-6	1.1395E-4	1.5484E-5	3.2229E-4	4.6685E-6	1.1395E-4

Table 2 : ROC calculations for $\mathrm{dt} = 0.005$ at $\mathrm{t} = 3$.

Z	L_∞	**ROC**	L_2	**ROC**
2	5.449412292157646E-004	-	6.517271057135045E-004	-
4	2.773057519898231E-004	0.974623104942524	3.947626546197787E-004	0.723282484482927
8	7.574444366980834E-005	1.872265574814479	1.532807898353114E-004	1.364808614034608
16	1.870672029698206E-005	2.017583417427364	5.357345735001329E-005	1.516586592072160
32	4.035321549647475E-006	2.212801101725775	1.628583270334394eE-005	1.717900915645318
64	4.158070594992291E-007	3.278697480128173	2.001889246032279E-006	3.024183422598357

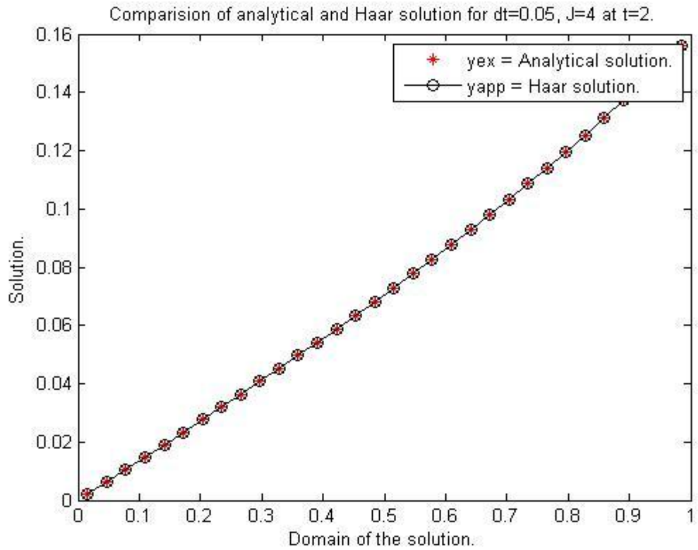

Fig 1: Observation of Haar and analytical solution for dt=0.05, J=4 at t=2 of example 1.

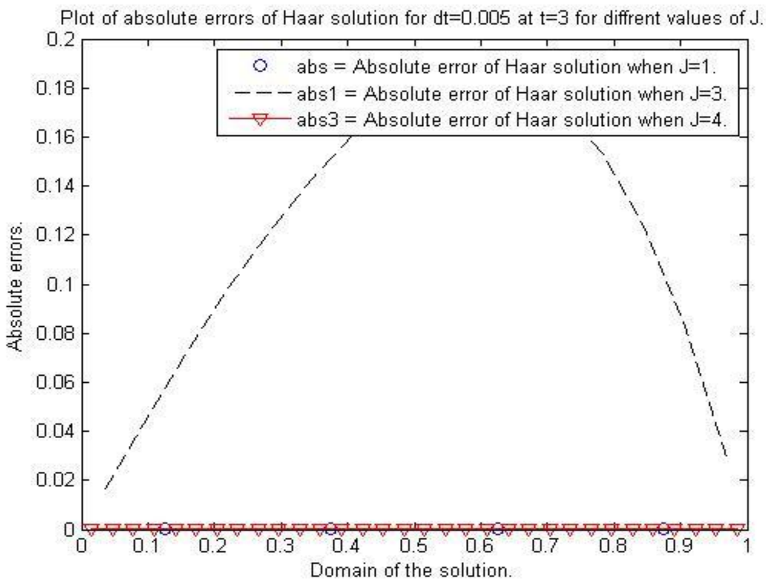

Fig 2: Absolute errors by HWCM with dt=0.005, J=1, 3, 4 at t=3 of example 1.

Example 2. Consider the singular telegraph equation (1.1) with variable coefficients $\alpha(x,t)=\frac{2}{x^2}$, $\beta(x,t)=\frac{1}{x}$, $B(x,t)=(1+x^2)$.

The exact solution is given by [19]

$$g(x,t)=e^{-t}\sinh(x).$$

The function $k(x,t)$ and the initial and boundary conditions can be evaluated utilizing the exact solution.

Table 3: Comparison of L_2, L_∞ and RMS error norms for singular problem in example 2 with J=2 and dt=0.005.

t	L_∞ **MLS [19]**	L_∞ **HWCM**	L_2 **MLS [19]**	L_2 **HWCM**	RMS **MLS [19]**	RMS **HWCM**
1	4.9830E-4	3.51518E-6	1.1365E-3	5.30302E-6	3.4268E-4	1.87490E-6
3	1.5081E-4	5.33041E-5	3.4775E-4	9.79643E-5	1.0485E-4	3.46356E-5
5	3.4549E-5	1.13700E-4	7.7775E-5	2.13886E-4	2.3450E-5	7.56201E-5

Table 4: ROC calculations for dt=0.001 at t=4.

Z	L_∞	ROC	L_2	ROC
2	3.188185660429046E-004	-	4.374463570745805E-004	-
4	2.989736032598493E-004	0.092717532747766	3.820865164474553E-004	0.195206766851366
8	8.331891807558493E-005	1.843302102032065	1.555184346649119E-004	1.296813743931502
16	2.151997188325205E-005	1.952967912661640	5.667483119858301E-005	1.456305508879609
32	5.372193570048894E-006	2.002092999174645	1.999550600459546E-005	1.503032398692035
64	1.300648598788479E-006	2.046280052409185	6.796315763532320E-006	1.556851002103322

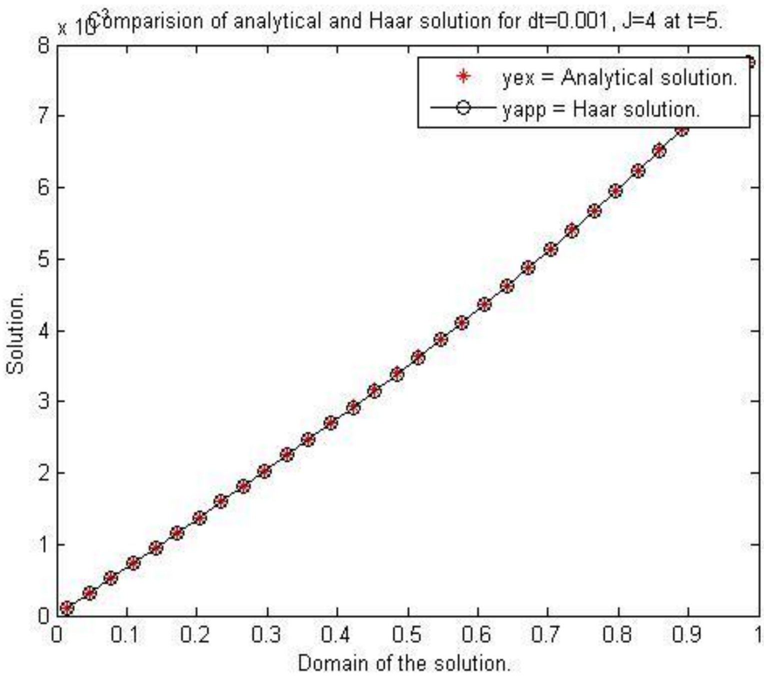

Fig 3: Observation of Haar and analytical solution for dt=0.001, J=4 at t=5 of example 2.

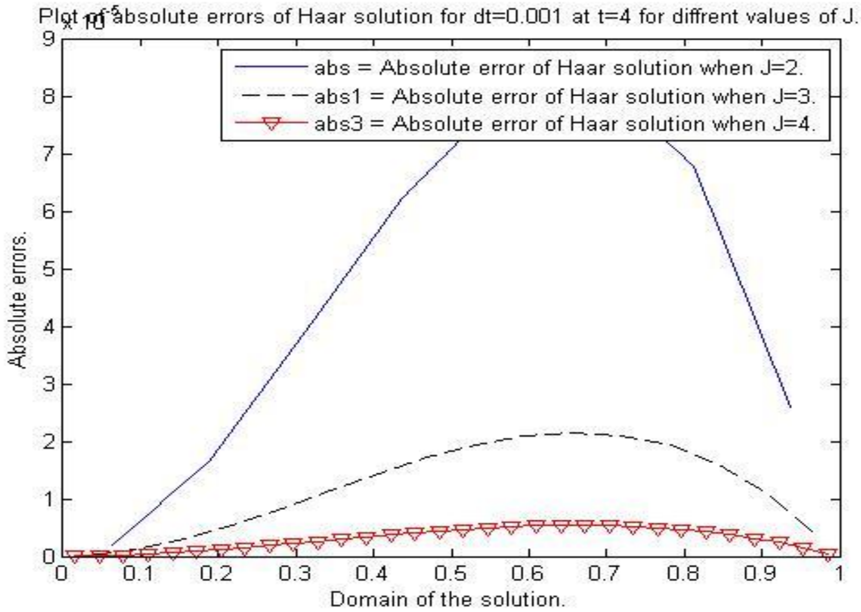

Fig 4: Absolute errors by HWCM with dt=0.001, J=2, 3, 4 at t=4 of example 2.

5. RESULTS AND DISCUSSIONS

The results of the example 1 and 2 are reported in Tables 1-4 and Figs. 1-4. Table 1 and 3 demonstrates the comparison of the L_2, L_∞, RMS errors of HWCM for t = 1, 3, 5, dt=0.005 with the MLS method. The Table 2 and 4 shows the order of convergence of the proposed scheme. But we solved both the example 1 and 2 with Z = 8, but the same problems solved in [19] with Z=11. From the Tables 1 and 3, it can be found that L_2, L_∞ and RMS errors of HWCM is smaller than MLS method hence we may conclude that HWCM is more accurate than MLS method while only need much less grid points than MLS method. Figs. 1 and 3 displays the comparison of the analytical and numerical solutions for J=4, dt=0.05 at t=2 and J=4, dt=0.001 at t=5 of example 1 and 2 respectively. In Figs. 2 and 4, we plotted the absolute errors of the Haar solution for different values of J. It is observed from the Figs. 2 and 4 that as the value J increases the absolute error decreases.

6. CONCLUSION

In this study, we implemented a HWCM for obtaining the approximate solution for two different types of telegraph equations i.e. non-homogeneous telegraph equation with variable coefficients and singular telegraph equations. The conclusion of the article is outlined as follows.

1) The obtained L_2, L_∞ and RMS errors are compared with MLS method and also acquired results display that this technique can solve the problem beneficially.
2) Moreover, only a small number of grid points are required to get acceptable results. The acquired numerical results support this claim.
3) The important advantage of the method is to show the nature of the solution for singular problem where maximum number of methods fails.
4) The displayed results through the tables and figures conform that the proposed scheme finds numerical results with a good accuracy.
5) An excellent agreement among the results of the HWC technique and the analytical solutions was represented nicely in the Figures 1 and 3.
6) From the Figure 2 and 4 it is observed that as the level of resolution (J) increases the absolute error decreases this shows that we can get accurate solution with lesser error by increasing the collocation points.
7) One of the drawback of the proposed method is that it may me impossible to solve these numerical examples without using mathematical software.
8) At the end of this article we conclude that HWCM is accurate, fast, simple and efficient technique for solving telegraph type problems which are arises from the various areas of science and engineering fields.

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