International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 9 (2019) pp. 2176-2178

Relationships of Some Variants of Domination

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Abstract:
This paper shows that every pair of positive integers there corresponds a domination number and an outer-independent domination number and there corresponds an outer-independent domination number and a 1-movable outer-independent domination number from some graph G.

Mathematics Subject Classification: 05C69

Keywords: Domination number, outer-independent domination number, 1-movable outer-independent domination number

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. If $S \subseteq V(G)$, then the *open neighborhood* of S is the set $N_G(S) = N(S) = \bigcup_{v\in S} N_G(v)$ and the *closed neighborhood* of S is the set $N_G[S] = N[S] = S \cup N(S)$.

A subset S of $V(G)$ is an *independent set* of G if for every two elements $x, y \in S$, $xy \notin E(G)$. A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$, that is, $N_G[S] = V(G)$. It is an *outer-independent dominating set* of G if S is a dominating set of G and the set $V(G) \setminus S$ is an independent set. The *domination number* $\gamma(G)$ (resp. *outer-independent domination number* $\gamma^{oi}(G)$), is the smallest cardinality of a dominating(resp. outer-independent dominating) set of G. An outer-independent dominating set of G with cardinality equal to $\gamma^{oi}(G)$ is called γ^{oi}-set of G. Some types of outer-independent dominating sets were established in [3] and in [4].

A nonempty set $S \subseteq V(G)$ is a *1-movable dominating set* of G if S is a dominating set of G and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of G or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The *1-movable domination number* denoted by $\gamma^1_m(G)$ is the smallest cardinality of a 1-movable dominating set of G. Movable dominating sets were investigated in [1] and in [2].

A nonempty set $S \subseteq V(G)$ is a *1-movable outer-independent dominating set* of G if S is an outer-independent dominating set of G and for every $v \in S$, $S \setminus \{v\}$ is an outer-independent dominating set of G or there exists a vertex $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is an outer-independent dominating set of G. The *1-movable outer-independent domination number* denoted by $\gamma^{1oi}_m(G)$ is the smallest cardinality of a 1-movable outer-independent dominating set of G. A 1-movable outer-independent dominating set of G with cardinality equal to $\gamma^{1oi}_m(G)$ is called γ^{1oi}_m-set of G.

This paper established some relationships between domination number and outer-independent domination number and between outer-independent domination number and 1-movable outer-independent domination number of a graph G.

2. RESULTS

Theorem 2.1. *For every pair of positive integers a and b, where $1 \leq a \leq b$, there exists a connected graph G such that $\gamma(G) = a$ and $\gamma^{oi}(G) = b$.*

Proof. Consider the following cases:
Case 1: $1 = a = b$
Consider the star $G = K_{1,n}$, where $n \geq 1$. Then $\gamma(G) = 1 = \gamma^{oi}(G)$.
Case 2: $1 = a < b$
Consider the graph $G = K_n$ where $n \geq 3$. Then $\gamma(G) = 1 < n - 1 = \gamma^{oi}(G)$.
Case 3: $1 < a = b$
Consider the graph G as shown in Figure 1.

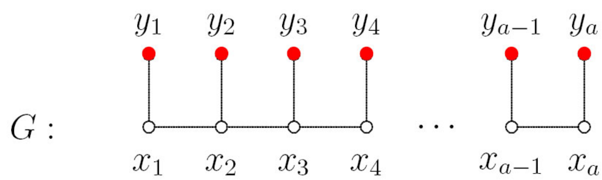

Figure 1: A graph with 1<$\gamma(G) = a = \gamma^{oi}(G)$

The set $S_1 = \{y_1, y_2, ..., y_{a-1}, y_a\}$ is a γ-set of G. Hence, $\gamma(G) = |S_1| = a$. Also, the set $S_2 = \{x_1, x_2, ..., x_{a-1}, x_a\}$ is an outer-independent dominating set of G since S_2 is a dominating set of G and $V(G) \setminus S_2$ is an independent set. It can be verified further that S_2 is a γ^{oi}-set of G. Hence, $\gamma^{oi}(G) = |S_2| = a$. Thus, $1 < \gamma(G) = |S_1| = a = |S_2| = \gamma^{oi}(G)$.

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 9 (2019) pp. 2176-2178

Case 4: $1 < a < b$
Consider the graph G as shown in Figure 2.

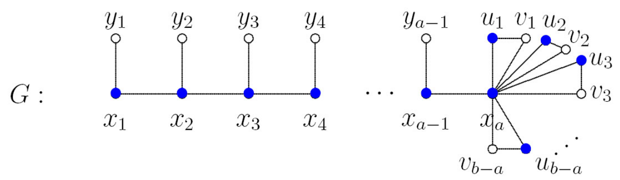

Figure 2: A graph with 1<$\gamma(G) = a < b = \gamma^{oi}(G)$

The set $S_3 = \{x_1, x_2, x_3, ..., x_{a-1}, x_a\}$ is a γ-set of G. Hence, $\gamma(G) = |S_3| = a$. Now, the set $S_4 = S_3 \cup \{u_1, u_2, u_3, ..., u_{b-a}\}$ is an outer-independent dominating set of G since S_4 is a dominating set of G and and $V(G) \setminus S_4$ is an independent set. It can be checked that S_4 is a γ^{oi}-set of G. Hence, $\gamma^{oi}(G) = |S_4| = a + (b - a) = b$. Thus, $1 < \gamma(G) = |S_3| = a < b = |S_4| = \gamma^{oi}(G)$. □

Corollary 2.2. *The difference $\gamma^{oi} - \gamma$ can be made arbitrarily large.*

Proof. Let $n \in \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers . By case 2 of Theorem 2.1, there exists a complete graph K_{n+2} such that $\gamma(K_{n+2}) = 1$ and $\gamma^{oi}(K_{n+2}) = (n+2) - 1 = n+1$. Thus, $\gamma^{oi}(K_{n+2}) - \gamma(K_{n+2}) = (n + 1) - 1 = n$. □

Theorem 2.3. *For every pair of positive integers a and b, where $1 \leq a \leq b$, there exists a connected graph G such that $\gamma^{oi}(G) = a$ and $\gamma_m^{1oi}(G) = b$.*

Proof. Consider the following cases:
Case 1: $1 = a = b$
Consider $G = K_2$. Then it can be verified that $\gamma^{oi}(G) = 1 = \gamma_m^{1oi}(G)$.
Case 2: $1 = a < b$
Consider the graph $G = K_{1,b}$ where $b > 1$ as shown in Figure 3.

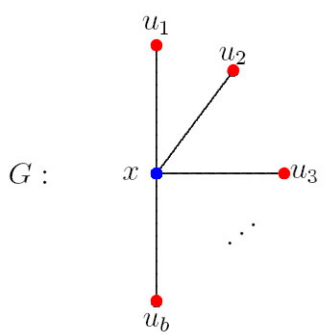

Figure 3: A graph with $\gamma^{oi}(G) = 1 < b = \gamma_m^{1oi}(G)$

The set $S_5 = \{x\}$ is clearly a γ^{oi}-set of G since S_5 is a dominating set with the smallest cardinality and $V(G) \setminus S_5$ is an independent set. Hence, $\gamma^{oi}(G) = |S_5| = 1$. Now for each $i = 1, 2, ..., b-1, b$, the set $S_6 = \{u_1, u_2, ..., u_b\}$ is a dominating set of G and $V(G) \setminus S_6 = \{x\}$ is an independent set. Hence, S_6 is an outer-independent dominating set of G. For each $i = 1, 2, ..., b-1, b$, $S_6 \setminus \{u_i\} \cup \{x\}$ is a dominating set of G and $V(G) \setminus (S_6 \setminus \{u_i\} \cup \{x\}) = \{u_i\}$ is an independent set. Hence, $S_6 \setminus \{u_i\} \cup \{x\}$ is an outer-independent dominating set of G. Thus, S_6 is a 1-movable outer-independent dominating set of G. It can be verified further that S_6 is a γ_m^{1oi}-set of G. Hence, $\gamma_m^{1oi}(G) = |S_6| = b$. Therefore, $\gamma^{oi}(G) = |S_5| = 1 < b = |S_6| = \gamma_m^{1oi}(G)$.
Case 3: $1 < a = b$
Consider the graph G as shown in Figure 4.

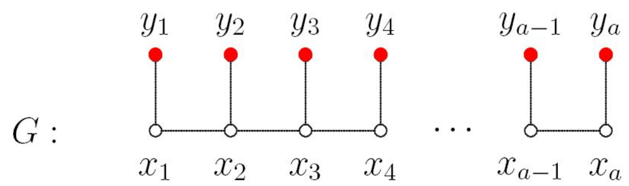

Figure 4: A graph with 1<$\gamma^{oi}(G) = a = \gamma_m^{1oi}(G)$

noindent The set $S_7 = \{x_1, x_2, ..., x_{a-1}, x_a\}$ is a γ^{oi}-set of G. Hence, $\gamma^{oi}(G) = |S_7| = a$. Moreover, for each $i = 1, 2, ..., a$, the set $(S_7 \setminus \{x_i\}) \cup \{y_i\}$ is an outer-independent dominating set of G. Hence, S_7 is a 1-movable outer-independent dominating set of G. It can be verified that S_7 is a γ_m^{1oi}-set of G. Thus, $\gamma_m^{1oi}(G) = |S_7| = a$. Therefore, $\gamma^{oi}(G) = a = \gamma_m^{1oi}(G)$.
Case 4: $1 < a < b$
Consider the graph G as shown in Figure 5.

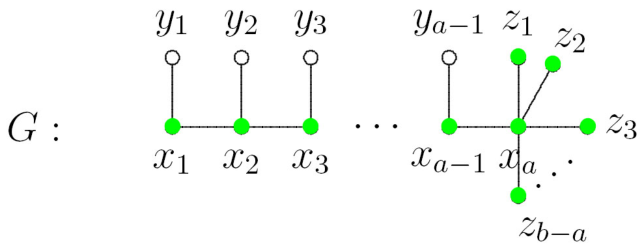

Figure 5: A graph with 1<$\gamma^{oi}(G) = a < b = \gamma_m^{1oi}(G)$

The set $S_8 = \{x_1, x_2, , , , x_a\}$ is a γ^{oi}-set of G. Hence, $\gamma^{oi}(G) = |S_8| = a$. Now, the set $S_9 = \{x_1, x_2, ..., x_{a-1}, x_a\} \cup \{z_1, z_2, ..., z_{b-a}\}$ is dominating set of G and $V(G) \setminus S_9$ is an independent set. Hence, S_9 is an outer-independent dominating set of G. For each $i = 1, 2, ..., a-1$, $S_9 \setminus \{x_i\} \cup \{y_i\}$ is an outer-independent dominating set of G. Also, $S_9 \setminus \{x_a\}$ is outer-independent dominating set of G. For each $j = 1, 2, ..., b-a$, $S_9 \setminus \{z_j\}$ is outer-independent dominating set of G. Hence, S_9 is a

1-movable outer-independent dominating set of G. It can be verified that S_9 is a γ_m^{1oi}-set of G. Thus, $\gamma_m^{1oi}(G) = |S_9| = a + (b - a) = b$. Therefore, $\gamma^{oi}(G) = a < b = \gamma_m^{1oi}(G)$. □

Corollary 2.4. *The difference $\gamma_m^{1oi} - \gamma^{oi}$ can be made arbitrarily large.*

Proof. Let $n \in \mathbb{N}$ where $\mathbb{N}$ is the set of natural numbers . By case 2 of Theorem 2.3, there exists a star graph $K_{1,n+1}$ such that $\gamma^{oi}(K_{1,n+1}) = 1$ and $\gamma_m^{1oi}(K_{1,n+1}) = n + 1$. Thus, $\gamma_m^{1oi}(K_{1,n+1}) - \gamma^{oi}(K_{1,n+1}) = (n + 1) - 1 = n$. □

REFERENCES

[1] J. Blair, R. Gera and S. Horton. Movable dominating sensor sets in networks. Journal of Combinatorial Mathematics and Combinatorial Computing, 77 (2011) ,103-123.

[2] R.G. Hinampas,Jr., and S.R. Canoy, Jr. 1-movable domination in graphs. Applied Mathematical Sciences, 8 (2014), no. 172, 8565-8571.

[3] M. Krzywkowski. A lower bound on the total outer-independent domination number of a tree. Comptes Rendus Mathematique, Vol. 349, Issue 1-2, (2011), 7-9.

[4] N.J. Rad and M. Krzywkowski. 2-outer-independent domination in graphs. National Academy Science Letters. Vol. 38 (2015), Issue 3, 263-269.