International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 9 (2019) pp. 2167-2170

1-movable Connected 2-domination in Graphs

Jocecar Lomarda-Hinampas

College of Teacher Education, Bohol Island State University-Main Campus,
CPG North Avenue, 6300 Tagbilaran City, Bohol, Philippines

Abstract :
This paper characterized of 1-movable connected 2-dominating sets in the join of two connected graphs and some bounds or exact values of the 1-movable connected 2-domination number were determined.

Mathematics Subject Classification: 05C69
Keywords: 1-movable connected 2-domination, connected 2-domination, 1-movable domination, 2-domination,domination,

1. INTRODUCTION

Let $G = (V(G), E(G))$ be a graph and $v \in V(G)$. The *open neighborhood* of v is the set $N_G(v) = N(v) = \{u \in V(G) : uv \in E(G)\}$ and the *closed neighborhood* of v is the set $N_G[v] = N[v] = N(v) \cup \{v\}$.

A subset S of $V(G)$ is a *dominating set* of G if for every $v \in V(G)\backslash S$, there exists $u \in S$ such that $uv \in E(G)$. The *domination number* of G is denoted by $\gamma(G)$ is the smallest cardinality of a dominating set of G. A dominating set of G with cardinality equal to $\gamma(G)$ is called a γ*-set* of G.

A nonempty subset S of $V(G)$ is a *1-movable dominating set* of G if S is a dominating set of G and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of G or there exists a vertex $u \in (V(G)\backslash S)\cap N(v)$ such that $(S\backslash\{v\})\cup\{u\}$ is a dominating set of G. The *1-movable domination number* of G denoted by $\gamma_m^1(G)$ is the smallest cardinality of a 1-movable dominating set of G. A 1-movable dominating set with cardinality equal to $\gamma_m^1(G)$ is called γ_m^1*-set* of G. This concept was investigated by Blair, Gera and Horton in [1] and Hinampas and Canoy in [4].

A nonempty subset S of $V(G)$ is a *2-dominating set* of G if for every $v \in V(G)\backslash S$, has at least two neighbors in S. The *2-domination number* of G denoted by $\gamma_2(G)$ is the smallest cardinality of a 2-dominating set of G. A 2-dominating set of G with cardinality equal to $\gamma_2(G)$ is called a γ_2*-set* of G. This concept was studied in [2] and in [3]. A 2-dominating set S of G is called a *connected 2-dominating set* of G if the subgraph $\langle S\rangle$ is connected. The *connected 2-domination number* of G denoted by $\gamma_{2c}(G)$ is the smallest cardinality of a connected 2-dominating set of G. A connected 2-dominating set of G with cardinality equal to $\gamma_{2c}(G)$ is called a γ_{2c}*-set* of G. This concept was studied by Leonida in [5].

A nonempty subset S of $V(G)$ is a *1-movable connected 2-dominating set* of G if S is a connected 2-dominating set of G and for every $v \in S$, $S \setminus \{v\}$ is a connected 2-dominating set of G or there exists $u \in (V(G) \setminus S) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a connected 2-dominating set of G. The *1-movable connected 2-domination number* of a graph G, denoted by $\gamma_{m2c}^1(G)$, is the smallest cardinality of a 1-movable connected 2-dominating set of G. A 1-movable connected 2-dominating set of G with cardinality equal to $\gamma_{m2c}^1(G)$ is called γ_{m2c}^1*-set* of G.

2. RESULTS

Define a collection $\mathscr{R}_{m2c}^1$ be the family of all graphs with 1-movable connected 2-dominating sets. This paper focuses only to those graphs which belong to $\mathscr{R}_{m2c}^1$.

Remark 2.1. *For every connected graph G of order $n \geq 3$, $2 \leq \gamma_{m2c}^1(G) \leq n$.*

Theorem 2.2. *Let G be a connected graph G of order $n \geq 3$. Then $\gamma_{m2c}^1(G) = 2$ if and only if there exist vertices $w, x, y, z \in V(G)$ with $xy, xw, yz \in E(G)$ and the sets $\{x, y\}$, $\{x, w\}$ and $\{y, z\}$ are 2-dominating sets of G .*

Proof. Suppose $\gamma_{m2c}^1(G) = 2$. Then G has a γ_{m2c}^1-set say $\{x, y\}$ for some $x, y \in V(G)$. Hence $xy \in E(G)$ and $\{x, y\}$ is a 2-dominating set of G. Since S is a 1-movable connected 2-dominating set of G, there exists $w \in (V(G) \setminus S) \cap N_G(y)$ such that $S \setminus \{y\} \cup \{w\} = \{x, w\}$ is a connected 2-dominating set of G. Hence, $xw \in E(G)$ and $\{x, w\}$ is a 2-dominating set of G . Also, there exists $z \in (V(G) \setminus S) \cap N_G(x)$ such that $S \setminus \{x\} \cup \{z\} = \{y, z\}$ is a connected 2-dominating set of G. Hence, $yz \in E(G)$ and $\{y, z\}$ is a 2-dominating set of G .

For the converse, suppose there exist vertices $w, x, y, z \in V(G)$ with $xy, xw, yz \in E(G)$ and the sets $\{x, y\}$, $\{x, w\}$ and $\{y, z\}$ are 2-dominating sets of G. Then the sets $S = \{x, y\}$, $S \setminus \{x\} \cup \{z\} = \{y, z\}$ and $S \setminus \{y\} \cup \{w\} = \{x, w\}$ are connected 2-dominating sets of G. Hence, S is a 1-movable connected 2-dominating set of G. Since $|S| = 2$, S is a γ^1_{m2c}-set of G. Hence,$\gamma^1_{m2c}(G) = |S| = 2$. □

Corollary 2.3. *For every complete graph K_n of order $n \geq 3$, $\gamma^1_{m2c}(K_n) = 2$.*

Remark 2.4. *For every cycle C_n of order $n \geq 4$,$\gamma^1_{m2c}(C_n) = n$.*

Theorem 2.5. *Let G and H be connected nontrivial graphs. A subset S of $V(G+H)$ is a 1-movable connected 2-dominating set of $G+H$ if and only if one of the following statements is true:*

(i) *S is a 1-movable connected 2-dominating set of G or S is a connected 2-dominating set of G and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of G and $\gamma(H) = 1$ whenever $|S| = 2$;*

(ii) *S is a 1-movable connected 2-dominating set of H or S is a connected 2-dominating set of H and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of H and $\gamma(G) = 1$ whenever $|S| = 2$;*

(iii) *$S = D_1 \cup D_2$ where $D_1 \subseteq V(G)$,$D_2 \subseteq V(H)$, $|D_1| \geq 2$, $|D_2| \geq 2$ and for every $v \in D_1$, $N_G(v) \cap (V(G) \setminus D_1) \neq \varnothing$ or D_2 or $D_2 \cup \{u\}$ is a dominating set of H for some $u \in V(H) \setminus D_2$ whenever $|D_1| = 2$ and for every $v \in D_2$, $N_H(v) \cap (V(H) \setminus D_2) \neq \varnothing$ or D_1 or $D_1 \cup \{a\}$ is a dominating set of G for some $a \in V(G) \setminus D_1$whenever $|D_2| = 2$;*

(iv) *$S = D_1 \cup D_2$ where $D_1 \subseteq V(G)$,$D_2 \subseteq V(H)$, $|D_1| = 1$, $|D_2| \geq 2$, D_2 is dominating set of H and either $N_G(v) \cap (V(G) \setminus D_1) \neq \varnothing$ or D_2 or $D_2 \cup \{u\}$ is a connected 2-dominating set of H for some $u \in V(H) \setminus D_2$ and for all $v \in D_2$,D_1 and $D_2 \setminus \{v\}$ are dominating sets of G and H, respectively or $D_1 \cup \{a\}$ is a dominating set of G for some $a \in V(G) \setminus D_1$ or $D_2 \setminus \{v\} \cup \{u\}$ is a dominating set of H for some $u \in V(H) \setminus D_2$ whenever $|D_2| = 2$;*

(v) *$S = D_1 \cup D_2$ where $D_1 \subseteq V(G)$,$D_2 \subseteq V(H)$, $|D_1| \geq 2$, $|D_2| = 1$, D_1 is a dominating set and either $N_H(v) \cap (V(H) \setminus D_2) \neq \varnothing$ or D_1 or $D_1 \cup \{u\}$ is a connected 2-dominating set of G for some $u \in V(G) \setminus D_1$ and for all $v \in D_1$, $D_1 \setminus \{v\}$ and D_2 are dominating sets of G and H, respectively or $D_2 \cup \{z\}$ is a dominating set of H for some $z \in V(H) \setminus D_2$ or $D_1 \setminus \{v\} \cup \{u\}$ is a dominating set of H for some $u \in V(G) \setminus D_1$ whenever $|D_1| = 2$;*

(vi) *$S = D_1 \cup D_2$ where $D_1 \subseteq V(G)$,$D_2 \subseteq V(H)$, $|D_1| = 1 = |D_2|$, D_1 and D_2 are dominating sets of G and H, respectively, for $v \in D_1$, either D_1 is a 1-movable dominating set of G or $D_2 \cup \{u\}$ is a connected 2-dominating set of H for some $u \in V(H) \setminus D_2$ and for $v \in D_2$, either D_2 is a 1-movable dominating set of H or $D_1 \cup \{a\}$ is connected 2-dominating set of G for some $a \in V(G) \setminus D_1$.*

Proof. Suppose S is a 1-movable connected 2-dominating set of $G+H$ and suppose $S \subseteq V(G)$. Then S is a connected 2-dominating set of G. Let $v \in S$. Since S is a 1-movable connected 2-dominating set of $G+H$, $S \setminus \{v\}$ is a connected 2-dominating set of G. Suppose $S \setminus \{v\}$ is not a connected 2-dominating set of G. By assumption, there exists $u \in V(G+H) \setminus S \cap N_{G+H}(v)$ such that $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G+H$. If $u \in (V(G) \setminus S) \cap N_G(v)$, then $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of G. Hence, S is a 1-movable connected 2-dominating set of G. If $u \in V(H)$, then $S \setminus \{v\}$ must be a dominating set of G. Suppose $|S| = 2$. Since $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G+H$, $\{u\}$ must also be a dominating set of H. Thus, $\gamma(H) = |\{u\}| = 1$. Hence, (i) holds. Similar arguments follow if $S \subseteq V(H)$. Thus, (ii) holds. Suppose $S = D_1 \cup D_2$ where $D_1 \subseteq V(G)$, $D_2 \subseteq V(H)$, $|D_1| \geq 2$ and $|D_2| \geq 2$. Suppose further that $|D_1| = 2$. Let $v \in D_1$. Since S is a 1-movable connected 2-dominating set of $G+H$, $S \setminus \{v\} = D_1 \setminus \{v\} \cup D_2$ is a connected 2-dominating set of $G+H$. This implies that D_2 must be a dominating set of H. Suppose $S \setminus \{v\}$ is a not connected 2-dominating set of $G+H$. By assumption, there exists $u \in V(G+H) \setminus S \cap N_{G+H}(v)$ such that $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G+H$. If $u \in V(G)$, then $N_G(v) \cap (V(G) \setminus D_1) \neq \varnothing$. If $u \in V(H)$, then $S \setminus \{v\} \cup \{u\} = D_1 \setminus \{v\} \cup (D_2 \cup \{u\})$ is a connected 2-dominating set of $G+H$. Hence, $D_2 \cup \{u\}$ must be a dominating set of H. Similar arguments follow if $v \in D_2$ and $|D_2| = 2$. Hence, (iii) holds. Suppose $|D_1| = 1$ and $|D_2| \geq 2$. Then D_2 must be a dominating set of H. Let $v \in S$ and suppose $v \in D_1$. Suppose $S \setminus \{v\} = (D_1 \setminus \{v\}) \cup D_2 = D_2$ is a connected 2-dominating set of $G+H$. Then D_2 is a connected 2-dominating set of H. Suppose $S \setminus \{v\} = D_2$ is not a connected 2-dominating set of $G+H$. By assumption, there exists $u \in (V(G+H) \setminus S) \cap N_{G+H}(v)$ such that $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G+H$. If $u \in V(H) \setminus D_2$, then $D_2 \cup \{u\}$ is a connected 2-dominating set of H. If $u \in V(G)$, then $N_G(v) \cap (V(G) \setminus D_1) \neq \varnothing$. Suppose $v \in D_2$ and $|D_2| = 2$. Suppose $S \setminus \{v\} = D_1 \cup D_2 \setminus \{v\}$ is a connected 2-dominating set of $G+H$. Hence,D_1 and $D_2 \setminus \{v\}$ must be dominating sets of G and H, respectively. Suppose $S \setminus \{v\} = D_1 \cup D_2 \setminus \{v\}$ is not a connected 2-dominating set of $G+H$. By assumption, there exists $u \in (V(G+H) \setminus S) \cap N_{G+H}(v)$ such that $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G+H$. Suppose $u \in V(G) \setminus D_1$.

Take $u = a$ and $S \setminus \{v\} \cup \{u\} = S \setminus \{v\} \cup \{a\} = (D_1 \cup \{a\}) \cup (D_2 \setminus \{v\})$ is a connected 2-dominating set of $G + H$. This implies that $D_1 \cup \{a\}$ is a dominating set of G. Suppose $u \in (V(H) \setminus D_2) \cap N(v)$. Then, $S \setminus \{v\} \cup \{u\} = D_1 \cup (D_2 \setminus \{v\} \cup \{u\})$ is a connected 2-dominating set of $G + H$. Hence, $(D_2 \setminus \{v\}) \cup \{u\}$ is a dominating set of H. Thus, (iv) holds. Similar arguments follow if $|D_1| \geq 2$ and $|D_2| = 1$. Hence, (v) also holds. Suppose $|D_1| = 1 = |D_2|$. Then D_1 and D_2 must be dominating sets of G and H, respectively. Let $v \in S$ and suppose $v \in D_1$. Since S is a 1-movable connected 2-dominating set of $G + H$, there exists $u \in (V(G + H) \setminus S) \cap N_{G+H}(v)$ such that $(S \setminus \{v\}) \cup \{u\}$ is a connected 2-dominating set of $G + H$. If $u \in V(G) \setminus D_1$, then $(S \setminus \{v\}) \cup \{u\} = (D_1 \setminus \{v\} \cup \{u\}) \cup D_2$ is a connected 2-dominating set of $G + H$. Hence, $D_1 \setminus \{v\} \cup \{u\}$ is a dominating set of G. Thus, D_1 is a 1-movable dominating set of G. If $u \in V(H) \setminus D_2$, then $(S \setminus \{v\}) \cup \{u\} = (D_1 \setminus \{v\}) \cup (D_2 \cup \{u\}) = D_2 \cup \{u\}$ is a connected 2-dominating set of $G + H$ and hence a connected 2-dominating set of H. Similarly if $v \in D_2$, then either D_2 is a 1-movable dominating set of H or $D_1 \cup \{a\}$ is dominating set of G for some $a \in V(G) \setminus D_1$. Thus, (vi) holds.

For the converse, suppose (i) holds. Suppose first that S is a 1-movable connected 2-dominating set of G. By definition of the join of graphs, S is also a 1-movable connected 2-dominating set of $G + H$. Suppose S is a connected 2-dominating set of G. Then S is also a connected 2-dominating set of $G+H$. Let $v \in S$. By assumption, $S \setminus \{v\}$ is a dominating set of G. If $|S| \geq 3$, then $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G+H$ for some $u \in V(H)$. Suppose $|S| = 2$. By assumption, $\gamma(H) = 1$. Hence, there exists $u \in V(H)$ such that $\{u\}$ is a dominating set of H. Thus, $S \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $G + H$. Therefore, S qualifies to be a 1-movable connected 2-dominating set of $G+H$. Similarly, if (ii) holds, then S is a 1-movable connected 2-dominating set of $G + H$. Suppose (iii) holds. By definition of the join of two graphs, S is a connected 2-dominating set of $G + H$. Let $v \in S$ and suppose $v \in D_1$. If $|D_1| \geq 3$, then $|D_1 \setminus \{v\}| \geq 2$. Hence, $S \setminus \{v\} = (D_1 \setminus \{v\}) \cup D_2$ is a connected 2-dominating set of $G + H$. Suppose $|D_1| = 2$. By assumption, D_2 is a dominating set of H. Hence, $S \setminus \{v\} = (D_1 \setminus \{v\}) \cup D_2$ is a connected 2-dominating set of $G + H$. Suppose D_2 is not a dominating set of H. By assumption, $D_2 \cup \{u\}$ is a dominating set of H for some $u \in V(H) \setminus D_2$. Hence, $(S \setminus \{v\}) \cup \{u\} = (D_1 \setminus \{v\}) \cup (D_2 \cup \{u\})$ is a connected 2-dominating set of $G + H$. Suppose $D_2 \cup \{u\}$ is not a dominating set of H for some $u \in V(H) \setminus D_2$. By assumption, $N_G(v) \cap (V(G) \setminus D_1) \neq \varnothing$. Hence, there exists $u \in (V(G) \setminus D_1) \cap N_G(v)$ Thus, $|D_1 \setminus \{v\} \cup \{u\}| \geq 2$. Thus, $(S \setminus \{v\}) \cup \{u\} = (D_1 \setminus \{v\} \cup \{u\}) \cup D_2$ is a connected 2-dominating set of $G + H$. Similar arguments follow if $v \in D_2$. Therefore, S is a 1-movable connected 2-dominating set of $G + H$. Suppose (iv) holds. Since D_2 is a dominating set of H, $S = D_1 \cup D_2$ is a connected 2-dominating set of $G + H$. Let $v \in S$ and suppose $v \in D_1$. By assumption, $N_G(v) \cap (V(G) \setminus D_1) \neq \varnothing$. Hence, there exists $u \in (V(G) \setminus D_1) \cap N_G(v)$ such that $(S \setminus \{v\}) \cup \{u\} = D_1 \setminus \{v\} \cup \{u\} \cup D_2$ is a connected 2-dominating set of $G+H$. Suppose $N_G(v) \cap (V(G) \setminus D_1) = \varnothing$. By assumption, D_2 is a connected 2-dominating set of H and hence of $G + H$. Then $S \setminus \{v\} = D_1 \setminus \{v\} \cup D_2 = D_2$ is a connected 2-dominating set of $G + H$. Suppose D_2 is not a connected 2-dominating set of H. By assumption, $D_2 \cup \{u\}$ is a connected 2-dominating set of H and hence of $G + H$ for some $u \in V(H) \setminus D_2$. Thus, $S \setminus \{v\} \cup \{u\} = D_1 \setminus \{v\} \cup (D_2 \cup \{u\}) = D_2 \cup \{u\}$ is a connected 2-dominating set of $G + H$. Suppose $v \in D_2$ and $|D_2| \geq 3$. Then $|D_1 \cup \{a\}| = 2$ for some $a \in V(G) \setminus D_1$ and $|D_2| \geq 2$. Hence, $S \setminus \{v\} \cup \{a\} = D_1 \cup \{a\} \cup D_2 \setminus \{v\}$ is a connected 2-dominating set of $G + H$. Suppose $|D_2| = 2$. By assumption, D_1 and $D_2 \setminus \{v\}$ are dominating sets of G and H, respectively. Hence, $S \setminus \{v\} = D_1 \cup D_2 \setminus \{v\}$ is a connected 2-dominating set of $G + H$. Suppose D_1 and $D_2 \setminus \{v\}$ are not dominating sets of G and H, respectively. By assumption, $D_1 \cup \{a\}$ is a dominating set of G for some $a \in V(G) \setminus D_1$. Hence, $S \setminus \{v\} \cup \{a\} = D_1 \cup \{a\} \cup D_2 \setminus \{v\}$ is a connected 2-dominating set of $G + H$. Suppose $D_1 \cup \{a\}$ is not a dominating set of G for all $a \in V(G) \setminus D_1$. By assumption, $D_2 \setminus \{v\} \cup \{u\}$ is a dominating set of H for some $u \in V(H) \setminus D_2$. Hence, $S \setminus \{v\} \cup \{u\} = D_1 \cup (D_2 \setminus \{v\} \cup \{u\})$ is a connected 2-dominating set of $G + H$. Therefore, S is a 1-movable connected 2-dominating set of $G + H$. Similar arguments follow if (v) holds. Hence,S is also a 1-movable connected 2-dominating set of $G + H$. Suppose (vi) holds. Then $S = D_1 \cup D_2$ is a connected 2-dominating set of $G + H$. Let $v \in S$ and suppose $v \in D_1$. Suppose D_1 is a 1-movable dominating set of G. Then there exists $u \in V(G) \setminus D_1 \cap N(v)$ such that $D_1 \setminus \{v\} \cup \{u\}$ is a dominating set of G. Hence, $S \setminus \{v\} \cup \{u\} = (D_1 \setminus \{v\} \cup \{u\}) \cup D_2$ is a connected 2-dominating set of $G + H$. Suppose D_1 is not a 1-movable dominating set of G. By assumption, $D_2 \cup \{u\}$ is a connected 2-dominating set of H for some $u \in V(H) \setminus D_2$. Hence, $S \setminus \{v\} \cup \{u\} = (D_1 \setminus \{v\}) \cup (D_2 \cup \{u\}) = D_2 \cup \{u\}$ is a connected 2-dominating set of $G + H$. Similar arguments follow if $v \in D_2$. Therefore, S is a 1-movable connected 2-dominating set of $G + H$. □

Corollary 2.6. *Let G and H be a connected nontrivial graphs, then $\gamma^1_{m2c}(G + H) = 2$ if one of the following holds:*

(i) $\gamma^1_{m2c}(G) = 2$;

(ii) $\gamma^1_{m2c}(H) = 2$;

(iii) $\gamma^1_{m}(G) = 1 = \gamma^1_{m}(H)$.

Theorem 2.7. *Let H be a connected graph of order $n \geq 2$. Then $S \subseteq V(K_1 + H)$ is a 1-movable connected 2-dominating set of $K_1 + H$ if and only if one of the following statements holds:*

(*i*) *$S = V(K_1) \cup D$, D is a dominating set of H, and either D or $D \cup \{z\}$ is a connected 2-dominating set of H for some $z \in V(H) \setminus D$ and for every $v \in D$, $D \setminus \{v\}$ or $D \setminus \{v\} \cup \{u\}$ is a dominating set of H for some $u \in V(H) \setminus D \cap N_H(v)$;*

(*ii*) *S is a 1-movable connected 2-dominating set of H or S is a connected 2-dominating set of H and for every $v \in S$, $S \setminus \{v\}$ is a dominating set of H.*

Proof. Suppose S is a 1-movable connected 2-dominating set of $K_1 + H$. Suppose $S = V(K_1) \cup D$ where $D \subseteq V(H)$. Then D must be a dominating set of H. Let $v \in S$ and suppose $v \in V(K_1)$. Suppose $S \setminus \{v\} = V(K_1) \setminus \{v\} \cup D = D$ is a connected 2-dominating set of $K_1 + H$. Then D is a connected 2-dominating set of H. Suppose $S \setminus \{v\}$ is not a connected 2-dominating set of $K_1 + H$. By assumption, there exists $z \in V(K_1 + H) \setminus S) \cap N(v)$ such that $S \setminus \{v\} \cup \{z\} = V(K_1) \setminus \{v\} \cup D \cup \{z\} = D \cup \{z\}$ is a connected 2-dominating set of $K_1 + H$. Hence,$D \cup \{z\}$ is a connected 2-dominating set of H. Suppose $v \in D$. Suppose $S \setminus \{v\} = V(K_1) \cup D \setminus \{v\}$ is a connected 2-dominating set of $K_1 + H$. Then $D \setminus \{v\}$ must be a dominating set of H. Suppose $S \setminus \{v\} = V(K_1) \cup D \setminus \{v\}$ is not a connected 2-dominating set of $K_1 + H$. By assumption, $u \in V(K_1 + H) \setminus S) \cap N(v)$ such that $S \setminus \{v\} \cup \{u\} = V(K_1) \cup D \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $K_1 + H$. Hence, $D \setminus \{v\} \cup \{u\}$ must be a dominating set of H. Thus, (i) holds. Suppose $S \subseteq V(H)$. Then S must be a connected 2-dominating set of H. Since S is a 1-movable connected 2-dominating set of $K_1 + H$, S is also a 1-movable connected 2-dominating set of H. Suppose S is not a 1-movable connected 2-dominating set of H. Let $v \in S$. By assumption, there exists $u \in V(K_1) \cap N(v)$ such that $S \setminus \{v\} \cup \{u\} = V(K_1) \cup S \setminus \{v\}$ is a connected 2-dominating set of $K_1 + H$. Hence, $S \setminus \{v\}$ must be a dominating set of H.

For the converse, suppose (i) holds. Since D is a dominating set of H, $S = V(K_1) \cup D$ is a connected 2-dominating set of $K_1 + H$. Let $v \in S$. Suppose $v \in V(K_1)$. By assumption, D is a connected 2-dominating set of H. Hence, $S \setminus \{v\} = D$ is a connected 2-dominating set of $K_1 + H$. Suppose D is not a connected 2-dominating set of H. By assumption, $D \cup \{z\}$ is a connected 2-dominating set of H for some $z \in V(H) \setminus D$. Hence, $S \setminus \{v\} \cup \{z\} = D \cup \{z\}$ is a connected 2-dominating set of $K_1 + H$. Suppose $v \in D$ and $D \setminus \{v\}$ is a dominating set of H. Hence, $S \setminus \{v\} = V(K_1) \cup D \setminus \{v\}$ is a connected 2-dominating set of $K_1 + H$. Suppose $D \setminus \{v\}$ is not a dominating set of H. By assumption, $D \setminus \{v\} \cup \{u\}$ is a dominating set of H for some $u \in V(H) \setminus D \cap N(v)$. Hence, $S \setminus \{v\} \cup \{u\} = V(K_1) \cup D \setminus \{v\} \cup \{u\}$ is a connected 2-dominating set of $K_1 + H$. Therefore, S is a 1-movable connected 2-dominating set of $K_1 + H$. Suppose (ii) holds. Suppose S is a 1-movable connected 2-dominating set of H. By definition of the join of graphs, S is a 1-movable connected 2-dominating set of $K_1 + H$. Suppose S is not a 1-movable connected 2-dominating set of H. By assumption, S is a connected 2-dominating set of H. Hence S is a connected 2-dominating set of $K_1 + H$. Let $v \in S$. By assumption, $S \setminus \{v\}$ is a dominating set of H. Hence, $S \setminus \{v\} \cup V(K_1)$ is a connected 2-dominating set of $K_1 + H$. Therefore, S is a 1-movable connected 2-dominating set of $K_1 + H$. □

Corollary 2.8. *Let H be a connected graph of order $n \geq 2$. Then $\gamma^1_{m2c}(K_1 + H) = 2$ if $\gamma^1_{m2c}(H) = 2$.*

REFERENCES

[1] J. Blair, R. Gera and S. Horton. *Movable dominating sensor sets in networks*. Journal of Combinatorial Mathematics and Combinatorial Computing, 77 (2011) ,103-123.

[2] M. Blidia, M. Chellali and O. Favaron.*Independence and 2-domination in trees*. Australasian Journal of Combinatorics,33 (2005), 317-327.

[3] M. Chellali.*Bounds on the 2-domination number in cactus graphs*. Opuscula Mathematica, 26 (2005),5-12.

[4] R.G. Hinampas,Jr., and S.R. Canoy, Jr. *1-movable domination in graphs*. Applied Mathematical Sciences, 8 (2014), no. 172, 8565-8571.

[5] R.E. Leonida *Connected 2-domination in the join of graphs*. Applied Mathematical Sciences, 9 (2015), no. 64, 3181-3186.