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Distance matrices and Adjacency Matrices of Some families of Graphs

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Abstract
This paper describes how the distance matrices of some families of graphs are obtained from their distance formula and then the corresponding adjacency matrices. The spectrum and energy of these classes of graphs are based on the adjacency matrices. We considered mainly two families of graphs - SM sum graphs and SM Balancing graphs. SM sum graphs are associated with the intrinsic combinatorial relationship between the powers of 2 and the positive integers, which is used in binary number system. SM balancing graphs are related with the balanced ternary number system. SM family of graphs is vertex labelled graphs. We provide criteria for finding the distance matrices of these families of graphs. Some results connecting the sum of entries of these distance matrices and adjacency matrices are obtained.
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INTRODUCTION

The distances in a graph are very important concept in Graph theory. The topological indices of a graph are related to the distances in a graph. The distance between two vertices v_i and v_j, is denoted by $d(v_i, v_j)$, is defined as the length of the shortest path between v_i and v_j. The distance matrix of a graph G having n vertices is a symmetric matrix $D = [d_{ij}]$ whose entry d_{ij} is defined as

$$d_{ij} = \begin{cases} d(v_i, v_j) & ,\text{if } i \neq j \\ 0 & ,\text{if } i = j. \end{cases}$$

The two number systems used in computer are the binary number systems and the balanced ternary number systems. The fixed length group of binary bits is generally called a computer word. Each k bit computer word can store a number as large as $2^k - 1$, positive or negative. Usually modern processors including embedded systems have a word length of 8, 16, 24, 32 or 64 bits, while modern general purpose computers use 32 or 64 bits. Any positive integer less than 2^n and not in $P = \{2^m : \text{m is an integer, } 0 \leq m \leq n-1\}$, for a fixed positive integer $n \geq 2$, can be expressed as the sum of two or more distinct elements of P. If $a \notin P$, a is less than 2^n and $a = \sum x_i$, with distinct $x_i \in P$, then each x_i is called an additive component of a. The simple graph, $SM(\sum_n)$ is a graph with vertex set $\{v_1, v_2, \ldots, v_{2^n-1}\}$ and adjacency of vertices defined by two distinct vertices v_x and v_y are adjacent if either x is an additive component of y or y is an additive component of x. Also this expression of $'a'$ is related to the low weight polynomial form of integers [1] which was used in the theory of elliptical curve cryptography. Let $M = \{1, 2, 3, \ldots, 2^n - 1\}$, then $P^c = M - P$. The Hamming weight of a string was defined as the number of 1s in the strings of 0 and 1. Here the number of additive components is the Hamming weight of string (binary) representation of all integers in P^c. Also any positive integer $p \leq \frac{1}{2}(3^n - 1)$, which is not a power of 3 can be expressed as a linear combination of two or more distinct elements of the set $T = \{3^m : \text{m is an integer, } 0 \leq m \leq n-1\}$ with coefficients $-1,\ 0\ or\ 1$. The relationship between p and the elements of T are used to form a new class of graphs called $n^{th}SM$ Balancing graphs, $SM(B_n)$ and $SMD(B_n)$. Some preliminaries are given below.

1. PRELIMINARY

Definition 1.1. *[6] Consider the set* $T = \{3^m : \text{m is an integer, } 0 \leq m \leq n-1\}$ *for a fixed integer* $n \geq 2$. *Let* $I = \{-1, 0, 1\}$. *Any positive integer* $x \leq \frac{1}{2}(3^n - 1)$ *which is not a power of 3 can be expressed as*

$$x = \sum_{j=1}^{n} \alpha_j y_j \tag{1}$$

for some $\alpha_i \in I$ *and* $y_j \in T$ *. If* $\alpha_i \neq 0$ *, then each* y_j *is called a balancing component of x.*

Definition 1.2. *[6] Let T be the set* $T = \{3^m : \text{m is an integer, } 0 \leq m \leq n-1\}$ *for a fixed integer* $n \geq 2$. *Consider the simple directed graph G=(V,E), where the vertex set* $V = \{v_1, v_2, \ldots, v_{\frac{1}{2}(3^n-1)}\}$ *and adjacency of vertices defined by, for two distinct vertices* v_x *and* v_{y_j}, v_x *is adjacent to* v_{y_j} *if (1) holds and* $\alpha = -1$ *and the vertex* v_{y_j} *is adjacent to* v_x *if (1) holds and* $\alpha = 1$. *This graph G is called the* $n^{th}SMD$ *Balancing Graph,* $SMD(B_n)$. *The underlying undirected graph is called* n^{th} *SM Balancing Graphs,* $SM(B_n)$.

Definition 1.3. *[5] If* $p < 2^n$, *is a positive integer which is not a power of 2, then* $p = \sum_1^n x_i$, *with* $x_i = 0$ *or* 2^m, *for some integer m,* $0 \leq m \leq n-1$ *and* x_i*s are distinct. Here we call each* $x_i \neq 0$ *as an additive component of p.*

Definition 1.4. *[5] For a fixed integer* $n \geq 2$, *define a simple graph* $SM(\sum_n)$, *called* n^{th} *SM sum graph, with vertex set* $\{v_1, v_2, \ldots, v_{2^n-1}\}$ *and adjacency of vertices defined by,* v_i *and* v_j *are adjacent if either i is an additive component of j or j is an additive component of i.*

Note: For a fixed integer $n \geq 2$, let $T = \{3^m : \text{m is an integer, } 0 \leq m \leq n-1\}$, $N = \{1, 2, 3, \ldots, t\}$, where $t = \frac{1}{2}(3^n - 1)$. Also let $P = \{2^m : \text{m is an integer, } 0 \leq m \leq n-1\}$, $M = \{1, 2, 3, \ldots, 2^n - 1\}$. Then consider $P^c = M - P$, $T^c = N - T$ throughout this paper unless otherwise specified.

2. DISTANCES IN THE GRAPHS $SM(\sum_N)$ AND $SM(B_N)$

It is very clear from the SM sum graph that any two odd prime numbers are at distance 2. The distance between 1 and any odd number is 1. Some of the distance related results from the previous work is given below.

Lemma 2.1. *[5] If $G = SM(\sum_n)$, $P = \{2^m$: m is an integer, $0 \leq m \leq n-1\}$, $n \geq 2$, then*

$$d(v_i, v_j) = \begin{cases} 1 & \text{, if } i \text{ is an additive component of } j \text{ or } j \text{ is an additive component of } i \\ 2 & \text{, if } i,j \in P \text{ or } i,j \notin P,\ i \text{ and } j \text{ have atleast one common additive component} \\ 3 & \text{, neither } i \text{ nor } j \text{ is an additive component but exactly one of them belongs to } P \\ 4 & \text{, } i,j \notin P,\ i \text{ and } j \text{ have no common additive component.} \end{cases}$$

Proposition 2.2. *[5] Let $G = SM(\sum_n)$ be an n^{th} SM sum graph. Let $d_r(v_i, v_j)$ denote the number of unordered pairs of vertices for which $d(v_i, v_j) = r$. Then*

$$d_r(v_i, v_j) = \begin{cases} n.(2^{n-1}-1) & \text{, if } r = 1 \\ \dfrac{n(n-1)}{2} + \big[\dfrac{(2^n-n-2)(2^n-n-1)}{2} - \delta\big] & \text{, if } r = 2 \\ (n+1).2^n - (n+2)2^{n-1} - n^2 & \text{, if } r = 3 \\ \delta & \text{, if } r = 4. \end{cases}$$

where $\delta = \frac{1}{2}\sum_{r=2}^{n-2}\big[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}\big]$.

Remark 2.3. $\delta = 0$ *for* $n = 2$ *or* 3

The value of δ is the number of pairs of pair wise disjoint subsets of P excluding the empty set and singleton sets. Here the diameter of the graph $SM(\sum_n)$ is given as follows.

$$diam(G) = \begin{cases} 2 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ 4 & \text{if } n \geq 4 \end{cases}$$

Definition 2.4. *[7] Let $P = \{2^m$: m is an integer, $0 \leq m \leq n-1\}$ for a fixed integer $n \geq 2$. Let x be a positive integer $< 2^n$. Then $x = \sum_1^n x_i$, with $x_i = 0$ or 2^m, for some integer m, $0 \leq m \leq n-1$ and x_is are distinct. Each $x_i \neq 0$ and $x_i \in P$ is an additive component of x. Let N be the set of all natural numbers. We define a transformation $T_s : N' \to N$ such that*

$$T_s(x) = \sum_1^n x_i^* \qquad (2)$$

where $N' = \{1, 2, 3, 4, ..., n\}$ and each x_i^* is obtained by changing the base 2 of x_i to base 3. This transformation is called $2S3$ transformation.

Example 2.5. *Let $x = 7 = 2^0 + 2^1 + 2^2$. Then $T_s(x) = 3^0 + 3^1 + 3^2 = 13$.*
Also when $x = 20 = 2^2 + 2^4$. Then $T_s(x) = 3^2 + 3^4 = 90$.

Definition 2.6. *Let P be the set $P = \{2^m$: m is an integer, $0 \leq m \leq n-1\}$ for a fixed integer $n \geq 2$. Let x be a positive integer $< 2^n$. Then $x = \sum_1^n x_i$, with $x_i = 0$ or 2^m, for some integer m, $0 \leq m \leq n-1$ and x_is are distinct. Each $x_i \neq 0$ and $x_i \in P$ is an additive component of x. Let $x = \sum_1^t a_i x_i$ and $y = \sum_1^r b_i x_i$ be two positive integers with $a_i = 0$ or 1 and $b_i = 0$ or 1, for some positive integers t and r. If the terms in the additive component expansion of x are different from that of y, then x and y are called additive distinct integers.*

Theorem 2.7. *Let $T_s(x)$ be a $2S3$ transformation function. Let x and y be two distinct positive integers $< 2^n$ and $v_x, v_y \in V(G)$, where $G = SM(\sum_n)$. Then the number of cases in which $T_s(x+y) = T_s(x) + T_s(y)$ is $\dfrac{n2^n - n^2 - n}{2} + \delta$, where $\delta = \frac{1}{2}\sum_{r=2}^{n-2}\big[\binom{n}{r}\sum_{k=2}^{n-2}\binom{n-r}{k}\big]$.*

Proof. Let $x = \sum_1^t a_i x_i$ and $y = \sum_1^r b_i x_i$ be two positive integers with $a_i = 0$ *or* 1 and $b_i = 0$ *or* 1, for some integers $t > 0$ and $r > 0$. When x and y are additive distinct integers, the terms in the expansion of x are different from that of y. Then we get $T_s(x + y) = T_s(x) + T_s(y)$. From lemma 2.1, we get that when $d(x, y)$ is either 3 or 4, then x and y are additive distinct integers. Also when $x, y \in P$, then x and y are additive distinct integers. Therefore the number of cases in which $T_s(x+y) = T_s(x)+T_s(y)$ is $\dfrac{n2^n - n^2 - n}{2} + \delta$. This completes the proof. ☐

Lemma 2.8. *[6] If the graph $G = SM(B_n)$, $T = \{3^m : m \text{ is an integer}, 0 \leq m \leq n-1\}$, $v_i, v_j \in V(G)$, then*

$$d(v_i, v_j) = \begin{cases} 1 & \text{, if } i \text{ is a balancing component of } j \text{ or } j \text{ is a balancing component of } i. \\ 2 & \text{, if } i, j \in T \text{ or } i, j \notin T, \, i \text{ and } j \text{ have atleast one common balancing component.} \\ 3 & \text{, neither } i \text{ nor } j \text{ is a balancing component but exactly one of them belongs to } T. \\ 4 & \text{, } i, j \notin T, \, i \text{ and } j \text{ have no common balancing component.} \end{cases}$$

Proposition 2.9. *[6] Let the graph $G = SM(B_n)$ be an n^{th} SM Balancing graph. Let $d_r(v_i, v_j)$ denote the number of unordered pairs of vertices for which $d(v_i, v_j) = r$. Let $t = \frac{1}{2}(3^n - 1)$. Then*

$$d_r(v_i, v_j) = \begin{cases} n.(3^{n-1} - 1) & \text{, if } r = 1 \\ \dfrac{n(n-1)}{2} + [\dfrac{(t-n)(t-n-1)}{2} - \sigma] & \text{, if } r = 2 \\ \frac{1}{2}(n.3^{n-1} + n - 2n^2) & \text{, if } r = 3 \\ \sigma & \text{, if } r = 4. \end{cases}$$

where $\sigma = \dfrac{1}{2} \sum_{r=2}^{n-2} \left[\binom{n}{r} \sum_{k=2}^{n-2} \binom{n-r}{k} 2^{r+k-2} \right]$

Remark 2.10. $\sigma = 0$ *for* $n = 2$ *or* 3

2.1. Distance Matrices of $SM(\sum_n)$ and $SM(B_n)$

Consider the graph $G = SM(\sum_n)$, for $n \geq 2$ with vertex set $V = \{v_i : 1 \leq i \leq 2^n - 1\}$. The distance matrix of the graph G having $2^n - 1$ vertices is a symmetric matrix $D_n = [d_{ij}]$ of order $p = 2^n - 1$, whose entry d_{ij} is defined as

$$d_{ij} = \begin{cases} d(v_i, v_j) & \text{, if } i \neq j \\ 0 & \text{, if } i = j, \end{cases}$$

where $d(v_i, v_j)$ is given in Lemma 2.1.

Theorem 2.11. *Let $G = SM(\sum_n)$ be the n^{th} SM sum graph, $n \geq 2$. Let $D_n = [d_{ij}]$ be the distance matrix of G. Then $\sum_{i=1}^{n} \sum_{j=1}^{i} d_{ij} = 2^{2n} - 3.2^n - n^2 + n + 2 + 2\delta$, where* $\delta = \dfrac{1}{2} \sum_{r=2}^{n-2} \left[\binom{n}{r} \sum_{k=2}^{n-2} \binom{n-r}{k} \right]$.

Proof. We have, for the distance matrix, $D_n = [d_{ij}]$ for a graph, the Wiener index of G,
$W(G) = \sum_{i=1}^{n} \sum_{j=1}^{i} d_{ij}$.
Also $W(G) = \sum_{\{u,v\} \subseteq V} d(u, v)$, where $d(u, v)$ is the distance between u and v.
By the Proposition 2.2,

$$\begin{aligned} W(G) &= \sum_{\{u,v\} \subseteq V} d(u, v) \\ &= 4\delta + 3\big[n.2^{n-1} - n^2\big] + 2\big[\frac{n(n-1)}{2} \\ &\quad + \frac{(2^n - n - 2)(2^n - n - 1)}{2} - \delta\big] + 1.n.(2^{n-1} - 1) \\ &= 2^{2n} - 3.2^n - n^2 + n + 2 + 2\delta \end{aligned}$$

Therefore $\sum_{i=1}^{n} \sum_{j=1}^{i} d_{ij} = 2^{2n} - 3.2^n - n^2 + n + 2 + 2\delta$.
Hence proved. ☐

Consider the graph $G = SM(B_n)$, for $n \geq 2$ with vertex set $V = \{v_1, v_2, \ldots, v_{\frac{1}{2}(3^n - 1)}\}$. The distance matrix of the graph G having $\dfrac{1}{2}(3^n - 1)$ vertices is a symmetric matrix $D_n = [d_{ij}]$ whose entry d_{ij} is defined as

$$d_{ij} = \begin{cases} d(v_i, v_j) & \text{, if } i \neq j \\ 0 & \text{, if } i = j. \end{cases}$$

where $d(v_i, v_j)$ is given in Lemma 2.8.

The distance matrix and related matrices of a graph are the sources of many graph invariants like topological indices etc. So these matrices are used in structure property activity modelling by studying the spectra and related polynomials of these graphs. The reciprocal distance matrices (Harary matrices) of SM family of graphs is obtained from the corresponding distance matrices by replacing all non-zero entries by their reciprocals. Therefore the Harary index of the SM sum graphs and SM Balancing graphs are obtained.

Theorem 2.12. *Let the graph $G = SM(\sum_n)$, $n \geq 2$. Let $D'_n = [d'_{ij}]$, where $d'_{ij} = \dfrac{1}{d_{ij}}$, be the Harary Matrix, then*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} d'_{ij} = \frac{n^2}{3} + n - 1 + \frac{2^{2n}}{2} - \frac{3.2^n}{2} + \frac{n.2^n}{3} - \frac{\delta}{2}.$$

Proof. From the proposition 2.2, we get

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n} d'_{ij} &= 2\sum_{\{u,v\}\subseteq V}\frac{1}{d_G(u,v)}\\ &= 2n.(2^{n-1}-1).1\\ &\quad+[\frac{n(n-1)}{2}+\frac{(2^n-n-2)(2^n-n-1)}{2}-\delta]\\ &\quad+\quad[n.2^{n-1}-n^2].\frac{2}{3}+\frac{\delta}{2}\\ &= \frac{n^2}{3}+n-1+\frac{2^{2n}}{2}-\frac{3.2^n}{2}+\frac{n.2^n}{3}-\frac{\delta}{2}.\end{aligned}$$

□

Theorem 2.13. *Let $G = SM(B_n)$, $n \geq 2$. Let $D'_n = [d'_{ij}]$, where $d'_{ij} = \frac{1}{d_{ij}}$, be the Harary Matrix,*

then $$\sum_{i=1}^{n}\sum_{j=1}^{i} d'_{ij} = \frac{n.3^n}{12}-\frac{7n}{12}-\frac{n^2}{6}+\frac{3}{16}+\frac{3^{2n}}{16}-\frac{3^n}{4}+\frac{n.3^{n-1}}{6}-\frac{\sigma}{4}.$$

2.2. Adjacency Matrices of $SM(\sum_n)$ and $SM(B_n)$

Definition 2.14. *[8] The Adjacency matrix of graph G having p vertices is a symmetric matrix $A_n = [a_{ij}]$, of order $p = 2^n - 1$, whose entry a_{ij} is defined as*

$$a_{ij} = \begin{cases} 1 & ,\textit{if } v_i \textit{ is adjcent to } v_j \\ 0 & ,\textit{otherwise.}\end{cases}$$

The Adjacency matrix of the graph $SM(\sum_n)$ with vertex set $V = \{v_i : 1 \leq i \leq 2^n-1\}$ is obtained from the corresponding distance matrix of $SM(\sum_n)$ by replacing all entries which are greater than 1 by 0. Similarly we can get the adjacency matrices of $SM(B_n)$.

Definition 2.15. *Let $A_n = [a_{ij}]$ be the adjacency matrices of the graph $SM(\sum_n)$. For each i, $1 \leq i \leq 2^n-1$, define $R_s(v_i)$ as $R_s(v_i) = \sum_{j=1}^{j=2^n-1} a_{ij}$.*

Proposition 2.16. *Let $G = SM(\sum_n)$, $n \geq 2$.
Then
i) $R_s(v_i)$=$2^{n-1} - 1$, if $i \in P$.
ii) $\sum_{i\in P} R_s(v_i) = \sum_{j\in P^c} R_s(v_j)$.*

Actually the $R_s(v_i)$, $i \in P$ gives the number of times each $i \in P$ is used in the binary - decimal conversion of equivalent positive integer numbers from 1 to $2^n - 1$ which are not in P. Also $\sum_{i\in P} R_s(v_i)$ gives the number of times the elements of P are used in the conversion of binary numbers equivalent to the positive integers i, $i \in P^c$.

Definition 2.17. *Let $A_n = [a_{ij}]$ be the adjacency matrices of the graph $SM(B_n)$. For each i, $1 \leq i \leq t = \frac{3^n-1}{2}$, define $R_s(v_i)$ as $R_s(v_i) = \sum_{j=1}^{j=t} a_{ij}$.*

Proposition 2.18. *Let $G = SM(B_n)$, $n \geq 2$.
Then i) $R_s(v_i)$=$3^{n-1} - 1$, if $i \in T$.
ii) $\sum_{i\in T} R_s(v_i) = \sum_{j\in T^c} R_s(v_j)$.*

Theorem 2.19. *For the SM sum graph $SM(\sum_n)$, $n \geq 2$, $d(v_x, v_y) = 2$ for all $x, y > 2^{n-1}$, $x \neq y$.*

Proof. When $x, y > 2^{n-1}$, $x \neq y$, $n \geq 2$, it is clear that x and y are not in P and they are having one common additive component 2^{n-1}. Therefore by Lemma 2.1, $d(v_x, v_y) = 2$. Hence the theorem. □

The transmission $T_r(v)$[8] of a graph was defined to be the sum of the distances from v to all other vertices.

$$T_r(v) = \sum_{u\in V} d(u,v) \tag{3}$$

A graph is said to be k-transmission regular if its distance matrix has constant row sums equal to k. The graph $SM(\sum_n)$ or $SM(B_n)$ are not transmission regular graph.

Observation 2.20. *Let $G = SM(\sum_n)$ be an n^{th} SM sum graph. Then the graph G' with $V(G') = V(G) - \{v_{2^n-1}\}$, is a 9-transmission regular graph when $n = 3$.*

3. CONCLUSION

In many cases it is difficult to find the distance matrices of family of graphs of bigger dimensions. But here we provide a systematic method for finding the distance matrices of some families of graphs- SM family of graphs. The calculation of distance matrix is much easy by using the distance formula for SM graphs of large n. Probably a computer algorithm can be framed by using these idea to get the distance matrix. Then the calculation of cospectrality and other measures will be easy. Also the distance matrix is related to the Wiener indices of the graphs. So far we considered only the simple graph of SM family of graphs. The distance matrices of the directed graph of SM family of graphs can be found out in this way. Further study of these distance matrices and adjacency matrices of these graphs may lead to very useful results. Also the characteristic polynomial of D_n need to be found.

Conflict of Interest

We hereby declare that the authors have no potential conflict of interest.

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