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A Novel Approach of Homotopy Perturbation Technique to Solution of Non-Linear Fisher Equation

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Abstract

In this paper, we use a powerful an analytical technique, namely known as Homotopy Perturbation technique (HPT), to find solution of non-linear Fisher equation. We constructed that Homotopy method for an imbedding small parameter $0 \leq p \leq 1$. Some examples are provided to take full advantage of the classical perturbation technique. As the result, we obtained from the present technique which is compared with Variational Iteration Method (VIM) and Adomian Decomposition Method (ADM). The present technique is very effective and suitable. The computational symbolic structure such MATLAB permit to implement complex and tedious calculation.

Keywords: Homotopy Perturbation technique, Fisher Equation, Adomian Decomposition Method MATLAB software.

1. INTRODUCTION

Fisher's equation is also known as Kolmogorov-Petrovsky-Piscounov equation, Fisher KPP equation or KPP equation investigated independently by Fisher (1937) and Kolmogorov et al. (1937). This equation has many applications in Physics, Chemistry, biology, heat and mass transfer, physiology, ecology and combustion. Fisher equation defines the process of collaboration between reaction diffusion equations. The concept of non-linear difficulties has recently undertaken many studies, Non-linear study having a significant character in area of application of mathematics. To solve the Fisher equations, many researchers have used several methods. (Wang 1988) considered generalized Fisher equation $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} - \mu\left(\frac{\partial u}{\partial x}\right)^2 = u(1 - u^{\alpha})$ and obtained exact and explicit solitary wave solutions, while the analytic study of Fisher's equation by ADM (Wazwaz et al. 2004). Homotopy perturbation technique was developed by He (1999) and useful for many linear and non-linear problems (He 2000, 2005). This technique is a simple procedure for finding the solution. The Homotopy perturbation technique is still creating large efforts by investigators to explain, the group of equation mainly fisher type equation, brush latter equation, burger equation etc. Mostly researchers have given more attention to the study of the homotopy perturbation technique for obtaining a various type of problem whose mathematical formulations provides a differential equation. Homotopy perturbation technique convert typical problem into a large number of sub problem which is batter to find solution without any requirement to change problem into non liner form. Now, we take some illustrations to solve Variational iteration technique for fisher equation (Matinfar et al. 2009), and employ changed variation iteration technique for generalized fisher equation (Matinfar et al. 2009). (Wazwaz 2004) considered Adomian decomposition technique for nonlinear forms. As a result, it yields a very quick solution and generally a few iteration leads to very careful approximation of the particular solution. In current time, many scholars have given attention for studying the solution of linear and nonlinear equations. In the present paper, we have to established a fisher equation for two and three dimensional nonlinear diffusion equation by using homotopy perturbation technique. Perturbation technique is one of the best well known method to solved non-linear difficulties. (Khan et al.2011) the approximate solutions of Fisher type time-fractional reaction-diffusion equation. Initially HPM used the solution of reaction–diffusion equations (Kumar et al., 2010) and they compared ADM (Kumar et al. 2011) and Differential Transformation Method DTM (Singh et al. 2017). (Rosa et al., 2017) employ generalized Fisher equation by using lie Symmetries. Author suggested potential symmetries which support to the generalized Fisher equation. (Wang et al., 2018) developed a class of compact boundary value methods applied to semi-linear reaction-diffusion equations.

To show the basic idea of homotopy perturbation techniques, let's assume a non-linear differential equation

$$A(v) = \emptyset(r), \ r \in \Omega \quad (1)$$

With primary condition $B\left(v, \frac{\partial v}{\partial n}\right) = 0, r \in \Gamma$,

Where A be a common differential operator, B is represent a primary operator, $\emptyset$ (r) is a well-define analytical function, and Γ is the boundary of the domain Ω. Generally, the operator A break into two portions L and N, where L is called linear and N is called non-linear operator therefore equation (1) can be rewrite the following form.

$$L(v) + N(v) - \emptyset(r) = 0 \quad (2)$$

We constructor a homotopy $u(r, q) : \Omega \times [0,1] \rightarrow R$, that is in accordance

$$H(u, q) = (1 - q)[L(u) - L(v_0)] + q[A(u) - \emptyset(r)] \quad (3)$$
$$= 0, \ q \, \varepsilon \, [0,1], \ r \, \varepsilon \, \Omega,$$

Or, equally,

$$H(u,q) = L(u) - L(v_0) + q\,L(v_0) + q[N(u) - \emptyset(r)] = 0 \quad (4)$$

Where, v_0 is an primary estimate value to the solution of Equation (1) in this technique we usage the homotopy parameter q to develop u as the power series

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$$u = u_0 + qu_1 + q^2u_2 + \ldots\ldots \quad (5)$$

The approximate solution can be obtained by putting $q = 1$,

$$v = \lim_{q\to1} u = u_0 + u_1 + u_2 + \cdots \quad (6)$$

2. FISHER'S EQUATION WITH THE HELP OF HOMOTOPY PERTURBATION TECHNIQUE: -

We take the Fisher's equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \alpha v(1-v) \quad (7)$$

Using the primary condition $v(x,0) = \emptyset(x)$ (8)

Solving equation (7) by homotopy perturbation technique we make a homotopy as

$$(1-q)\left(\frac{\partial U}{\partial t} - \frac{\partial v_0}{\partial t}\right) + q\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - \alpha U - \alpha U^2\right) = 0 \quad (8)$$

$$\frac{\partial U}{\partial t} - \frac{\partial v_0}{\partial t} = \mathrm{q}\left(\frac{\partial^2 U}{\partial x^2} + \alpha U - \alpha U^2 - \frac{\partial v_0}{\partial t}\right) \quad (9)$$

Let solution of equation (9) has the form

$$U(x,t) = U_0(x,t) + qU_1(x,t) + q^2U_2(x,t) + q^3U_3(x,t \quad (10)$$

Substitute (10) into (9) and comparing the terms with same power of q, we get

$$q^0 : \frac{\partial U_0}{\partial t} - \frac{\partial v_0}{\partial t} \to \frac{\partial U_0}{\partial t} = \frac{\partial v_0}{\partial t}, \quad (11)$$

$$q^1 : \frac{\partial U_1}{\partial t} = \frac{\partial^2 U_0}{\partial x^2} + \alpha U_1 - \alpha {U_0}^2 + \frac{\partial v_0}{\partial t} \quad (12)$$

$$q^{k+1} : \frac{\partial U_{k+1}}{\partial t} = \frac{\partial^2 U_k}{\partial x^2} + \alpha U_k - \alpha \sum_{j=0}^{k} U_j\, U_{k-j} \quad (13)$$

Assuming $v_0(x,t) = v(x,0) = \emptyset(x)$ we get

$$U_0(x,t) = v_0(x,t) = \emptyset(x) \quad (14)$$

And $U_{k+1}(x,t) = \int_0^t \left\{\frac{\partial^2 U_k}{\partial x^2}(x,s) + \alpha U_k(x,s) - \alpha \sum_{j=0}^{k} U_j(x,s)\, U_{k-j}(x,s)\right\} ds$ (15)

Where k ≥ 0, not that in this event $\frac{\partial v_0}{\partial t} = 0$ (16)

Therefore, the solution of equation (7) by putting q=1,

i.e. $v(x,t) = \lim_{q\to1} U(x,t) = \sum_{k=0}^{\infty} U_k(x,t)$ (17)

Here some significant cases of non-linear diffusion arise which tally to some actual physical methods, will be studied to show the dependability of the recommended pattern.

2.1 Problem (1):

Now put $\alpha = 1$ in equation (7) with primary condition $v(x,0) = \emptyset(x) = \mu$, where μ is a constant

By equation (14) as well as (16) we obtain

$$U_0(x,t) = \mu \quad (18)$$

$$U_1(x,t) = \mu(1-\mu)t \quad (19)$$

$$U_2(x,t) = \mu(1-\mu)(1-2\mu)\frac{t^2}{2!} \quad (20)$$

$$U_3(x,t) = \mu(1-\mu)(1-6\mu+6\mu^2)\frac{t^3}{3!} \quad (21)$$

$$U_4(x,t) = \mu(1-\mu)(1-2\mu) + (1-12\mu+12\mu^2)\frac{t^4}{4!} \quad (22)$$

And so on

The result of the series form is certain by

$$\mathrm{v(x,t)} = \sum_{k=0}^{\infty} U_k \quad (23)$$

$$v(\mathrm{x,t}) = \mu + \mu(1-\mu)\mathrm{t} + \mu(1-\mu)(1-2\mu)\frac{t^2}{2} + \mu(1-\mu)(1-6\mu+6\mu^2)\frac{t^3}{6} + \ldots$$

$$v(\mathrm{x,t}) = \frac{\mu e^t}{(1-\mu+\mu e^t)} \quad (24)$$

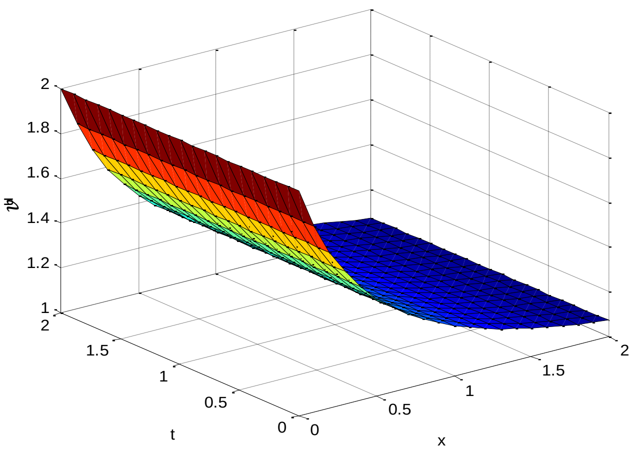

Figure 1.The 3-dimension graph of the solution of equation (7) for taking $\alpha = 1, \mu = 2$

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The figure 1 represent the 3-Dimension solution of equation (7), taking $\alpha = 1$ with the initial condition $\emptyset(x) = \mu = 2$, when $0 \leq t \leq 2$ and $0 \leq x \leq 2$. If the value of t is increasing for all x then $v(x,t)$ decrease continuously. While the figure 2 represent the 3-Dimension solution of equation (7), taking $\alpha = 1$ with the initial condition $\emptyset(x) = \mu = 3$ when $0 \leq t \leq 2$ and $0 \leq x \leq 2$.

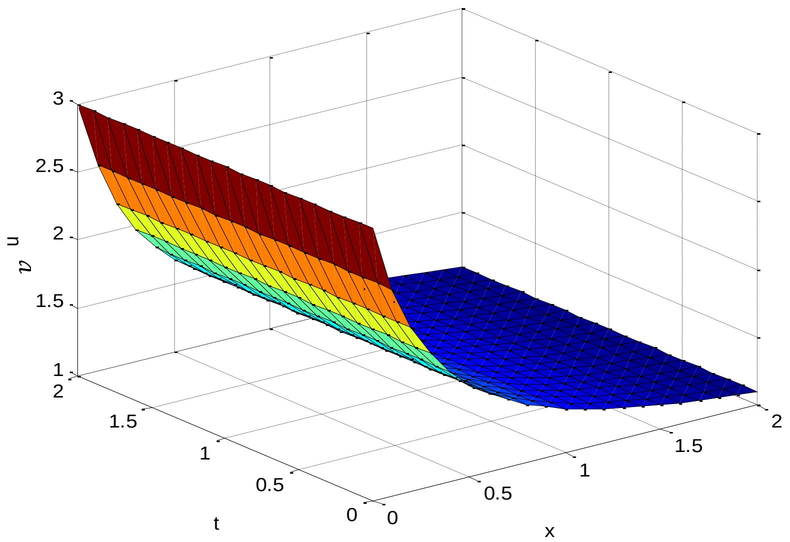

Figure 2. The 3-dimension graph represent the solution of equation (24) for taking $\mu = 3$

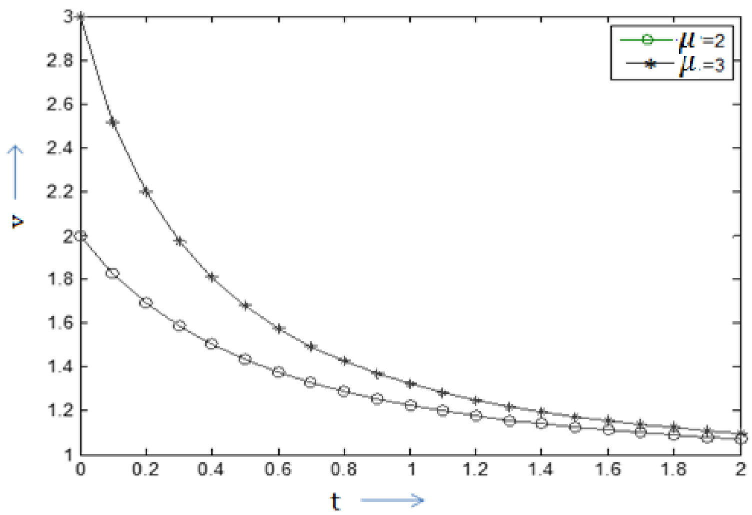

Figure 3.The 2-dimension graph represent the solution of equation (7) taking primary condition $\alpha = 1$ with $\mu = 2$ and 3

The figure 3 denote the 2-dimensional representation of solution equation (7) taking $\alpha = 1$, if t is increases from 0 to 2 then v will be decreases for all value of x with the primary condition $\mu = 2$ and $\mu = 3$. . The approximate solution of problem (1) by using the homotopy perturbation technique as the same result we obtain by Adomain's decomposition technique[20] and Variational iteration technique[17-18] and it is clearly apparent the approximate solution remaining closed form to the analytic solution of the problem.

2.2 Problem 2

In equation (7) we put α=6 and $v(x,0) = \emptyset(x) = \frac{1}{(1+e^x)^2}$ through equation (14) and (15) we obtain

$$U_0(x,t) = \frac{1}{(1+e^x)^2} \tag{25}$$

$$U_1(x,t) = \frac{10e^x t}{(1+e^x)^3} \tag{26}$$

$$U_2(x,t) = \frac{25e^{x}(e^{2x}-1)t^2}{(1+e^{x})^4}\text{.........} \quad (27)$$

So the result of the series form is

$$v(x,t) = \frac{1}{(1+e^{x})^2} + \frac{10e^{x}t}{(1+e^{x})^3} + \frac{25e^{x}(e^{2x}-1)t^2}{(1+e^{x})^4} +$$

$$= \frac{\mathbf{1}}{(\mathbf{1}-\boldsymbol{e}^{(\boldsymbol{x}-\mathbf{5}\boldsymbol{t})})} \quad (28)$$

The equation (28) represent the approximate solution of problem (2) by using the homotopy perturbation technique, it is clearly that the approximate solution remaining closed form, to the analytic solution of the problem.

The figure 4 denoted the 3 dimensional solution of equation (7) taking $\alpha = 6$ with the primary condition $\emptyset(x) = \frac{1}{(1+e^{x})^2}$, when $-4 \leq t \leq 4$ and $-4 \leq x \leq 4$. If the value of t is increasing and x is decrease then $v(x,t)$ is increase and constant after different value of t and x continuously.

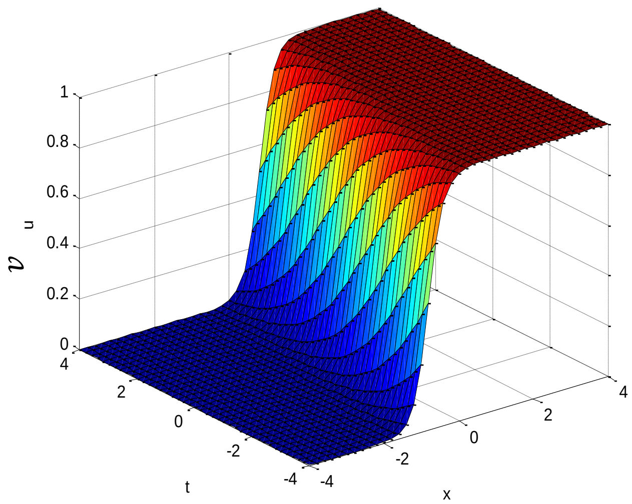

Figure 4. The 3-dimension solution of equation (28) when $-4 \leq t \leq 4$ and $-4 \leq x \leq 4$.

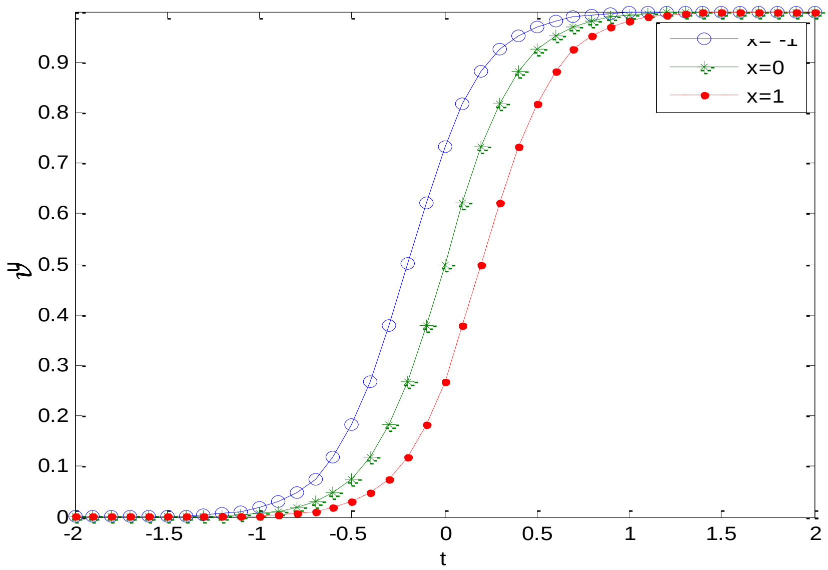

Figure 5. This graph represents t and v for x is equal to $-1 \leq x \leq 1$ if t is increases then v is also increases and after some time v will be constant for other values of x.

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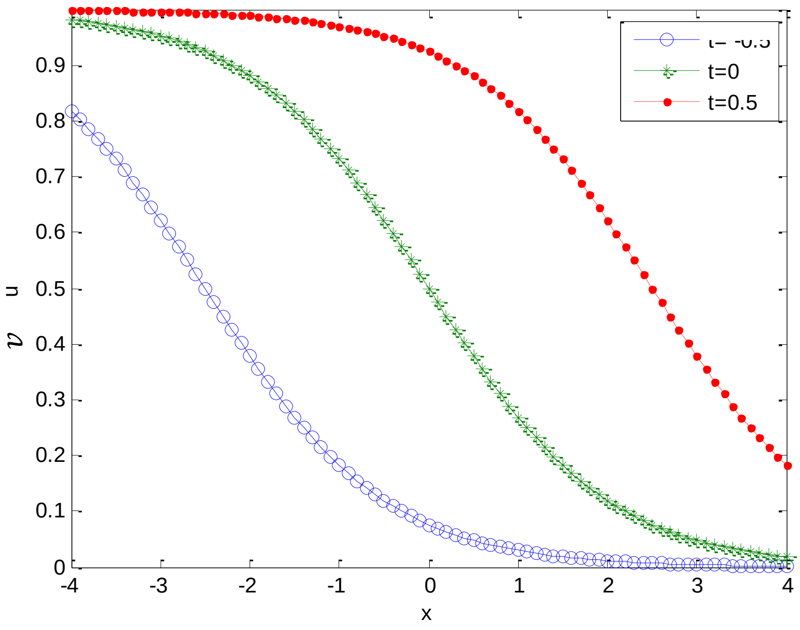

Figure 6.This graph represent for $-0.5 \leq t \leq 0.5$ here x increases and v decreases.

The figure 5 denote the 2-dimensional representation of solution equation (7) taking $\alpha = 6$ and $\mu = \frac{1}{(1+e^x)^2}$, if x varies from for $-1 \leq x \leq 1$, and t is increases then v is also increases and after some time v will be constant for other values of x, while the figure 6 denote the 2-dimensional representation of solution equation (7) taking $\alpha = 6$ and $\mu = \frac{1}{(1+e^x)^2}$, if t varies from for $-0.5 \leq t \leq 0.5$ then v will be decreases for all value of x with the initial condition $\mu = 2$ and $\mu = 3$ The approximate solution of problem (2) by using the homotopy perturbation technique as the same result we obtain by Adomain's decomposition technique[20] and Variational iteration technique[17-18] and it is clearly apparent the approximate solution remaining closed form to the analytic solution of the problem.

3. GENERALIZED FISHER'S EQUATION WITH THE HELP OF HOMOTOPY PERTURBATION TECHNIQUE:-

We consider the Fisher's equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \alpha v(1 - v^{\beta}) \qquad (29)$$

With the primary condition $v(x, 0) = \emptyset(x)$

Solving equation (29) by homotopy perturbation technique we create a homotopy as

$$(1-q)\left(\frac{\partial U}{\partial t} - \frac{\partial v_0}{\partial t}\right) + q\left(\frac{\partial U}{\partial t} - \frac{\partial^2 U}{\partial x^2} - \alpha U - \alpha U^{\beta+1}\right) = 0 \qquad (30)$$

Solving it we get

$$\frac{\partial U}{\partial t} = \frac{\partial v_0}{\partial t} + q\left(\frac{\partial^2 U}{\partial x^2} + \alpha U - \alpha U^{\beta+1} - \frac{\partial v_0}{\partial t}\right) \qquad (31)$$

Let solution of equation (31) has the standard form

$$U(x,t) = U_0(x,t) + qU_1(x,t) + q^2U_2(x,t) + q^3U_3(x,t) + \cdots \qquad (32)$$

Putting (32) into (31) and equate same power of q we get

$$q^0 : \frac{\partial U_0}{\partial t} - \frac{\partial v_0}{\partial t} \rightarrow \frac{\partial U_0}{\partial t} = \frac{\partial v_0}{\partial t}, \qquad (33)$$

$$q^1 : \frac{\partial U_1}{\partial t} = \frac{\partial^2 U_0}{\partial x^2} + \alpha U_1 - \alpha {U_0}^{\beta+1} + \frac{\partial v_0}{\partial t} \qquad (34)$$

$$q^{k+1} : \frac{\partial U_{k+1}}{\partial t} = \frac{\partial^2 U_k}{\partial x^2} + \alpha U_k - \alpha \sum_{j_1=0}^{k} \sum_{j_2=0}^{J_1} \cdots \sum_{J_\beta=0}^{J_{\beta-1}} \left(U_{j\beta} U_{j\beta-1-j\beta} \cdots U_{k-j}\right) ds = 0 \qquad (35)$$

Where $U_{k+1}(x, 0) = 0$, $k \geq 0$

As $v_0(x,t) = v(x,0) = \emptyset(x)$ we obtain

$$U_0(x,t) = v_0(x,t) = \emptyset(x) \qquad (36)$$

$$U_{k+1} = \int_0^t \frac{\partial^2 U_k}{\partial x^2} + \alpha U_k - \alpha \sum_{j_1=0}^{k} \sum_{j_2=0}^{J_1} \cdots \sum_{J_\beta=0}^{J_{\beta-1}} \left(U_{j\beta} U_{j\beta-1-j\beta} \cdots U_{k-j}\right) ds = 0 \qquad (37)$$

Where $k \geq 0$, therefore the solution of equation (29) obtain by putting q

i.e. $v(x,t) = \lim_{q \to 1} U(x,t) = \sum_{k=0}^{\infty} U_k(x,t)$

3.1 Problem-3

In equation (29) we put $\alpha = 1, \beta = 6$ and $v(x,0) = \emptyset(x) = \frac{1}{(1+e^{\frac{3x}{2}})^{\frac{1}{3}}}$ by equation (36) and (37) we have

$$U_0(x,t) = \frac{1}{(1+e^{\frac{3x}{2}})^{\frac{1}{3}}} \qquad (38)$$

$$U_1(x,t) = \frac{5te^{\frac{3x}{2}}}{4(1+e^{\frac{3x}{2}})^{\frac{2}{3}}} \qquad (39)$$

$$U_2(x,t) = \frac{25t^2(e^{\frac{3x}{2}}-3)e^{\frac{3x}{2}}}{16(1+e^{\frac{3x}{2}})^{\frac{5}{3}}2!} \qquad (40)$$

$U_3(x,t) = \frac{125t^3(16e^{\frac{3x}{2}}-e^{3x}-9)e^{\frac{3x}{2}}}{64(1+e^{\frac{3x}{2}})^{\frac{8}{3}}3!}$ and so on (41)

So the answer in the closed procedure is given by

$$v(x,t) = \frac{1}{(1+e^{\frac{3x}{2}})^{\frac{1}{3}}} + \frac{5te^{\frac{3x}{2}}}{4(1+e^{\frac{3x}{2}})^{\frac{2}{3}}} + \frac{25t^2(e^{\frac{3x}{2}}-3)e^{\frac{3x}{2}}}{16(1+e^{\frac{3x}{2}})^{\frac{5}{3}}2!} + \frac{125t^3(16e^{\frac{3x}{2}}-e^{3x}-9)e^{\frac{3x}{2}}}{64(1+e^{\frac{3x}{2}})^{\frac{8}{3}}3!} + \cdots\cdots$$

$$= \left\{\frac{1}{2}tanh\left[\frac{-3}{4}\left(x-\frac{5}{2}t\right)+\frac{1}{2}\right]\right\}^{\frac{1}{3}} \qquad (42)$$

The figure 7 represent the 3-dimension graph represents the solution of equation (42) taking initial condition $\alpha = 1, \beta = 6\ and\ \mu = \frac{1}{(1+e^{\frac{3x}{2}})^{\frac{1}{3}}}$ and also taking $0 \le t \le 2\ and\ 0 \le x \le 2$ respectivly when the value of t and x as well as increasing then v will be increase continuously.

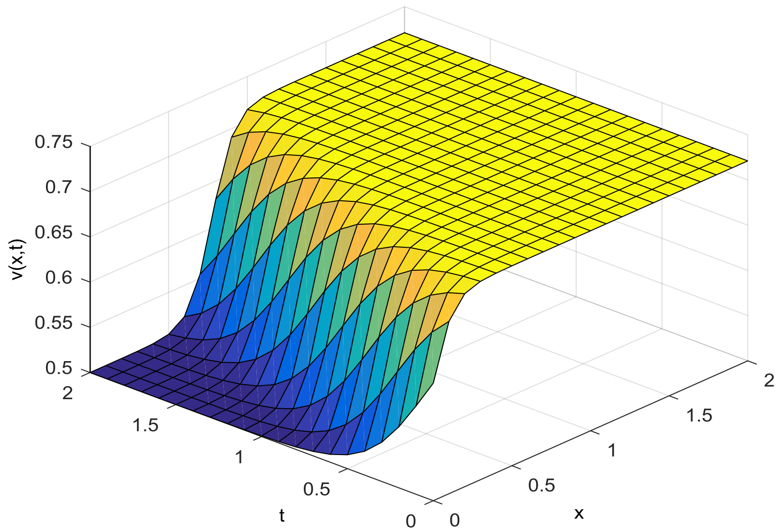

Figure 7. The 3-dimension graph represents the solution of equation (42) taking initial condition $\alpha = 1, \beta = 6\ and\ \mu = \frac{1}{(1+e^{\frac{3x}{2}})^{\frac{1}{3}}}$

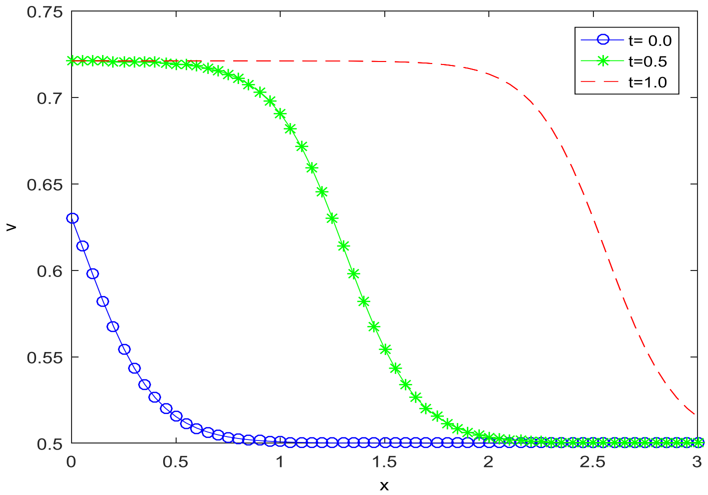

Figure 8. This graph represent t and v for $0 \le x \le 1$ if t is increases then v is also decrease for all values of x.

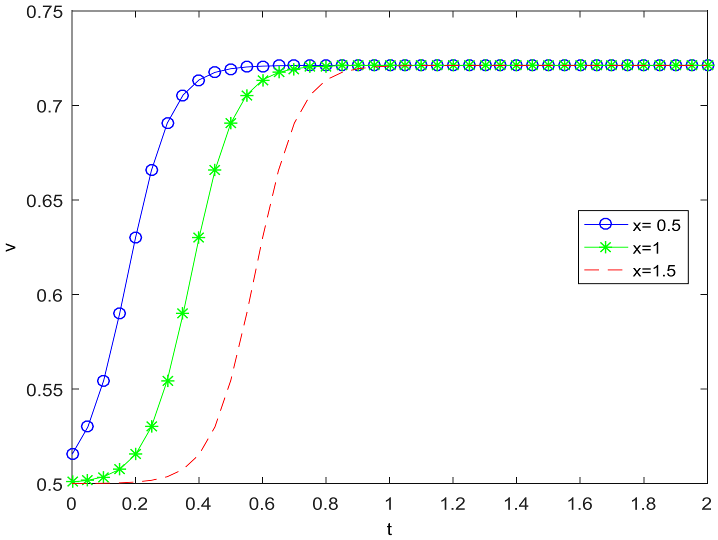

Figure 9.Represent for $0 \leq t \leq 2$ here x increases and v increase

The figure 9 denote the 2-dimensional representation of solution equation (29) taking $\alpha = 1, \beta = 6$ *and* $\mu = \frac{1}{(1+e^{\frac{3x}{2}})^{\frac{1}{3}}}$, if t varies from for $0 \leq t \leq 2$ then v will be increase for all value of x with the initial condition $\mu = 2$ and $\mu = 3$ and after some term v will be constant The approximate solution of general form of fisher equation by using the homotopy perturbation technique as the same result we obtain by Adomain's decomposition technique[20] and Variational iteration technique[17-18] and it is clearly apparent the approximate solution remaining closed form to the analytic solution of the problem.

4. CONCLUSIONS

In this paper, we have developed the solution of non-linear Fisher's equation, generalized Fisher's equation and reaction-diffusion equation of the Fisher type by homotopy perturbation technique. The present study advocates that the HTP is a significant scientific technique for solutions of nonlinear Fisher's equations. The homotopy perturbation technique excludes the stipulation for computing the Adomian polynomials which can be difficult in some cases. In present study MATLAB software is used to study the series found from the homotopy perturbation technique and plot different effective graphs of fisher equations based problems in 2-dimensional and 3-dimensional by setting different values of parameters.

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