International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 4 (2019) pp. 936-941

Probabilistic Linear Programming in Project Management

[1]K. Srinivasa Rao, [2]S.Adilakshmi , [1]U.V. Adinarayana Rao, [3]N.Ravi Shankar

[1]*Dept.of Operations, Institute of Management, GITAM (Deemed to be University), Visakhapatnam, India.*

[2]*Dept. of Basic Science & Engineering, Vignan's Institute of Information Technology, Visakhapatnam, India.*

[3]*Dept.of Applied Mathematics, Institute of Science, GITAM (Deemed to be University), Visakhapatnam, India.*

Abstract

The aim of this paper is to develop a probabilistic linear programming that can help the Project Manager (PM) with solving the project network problems. In this paper, we first revisit the related papers and focus on how to develop a probabilistic linear programming model to find critical activities, critical path and total duration of a project. This model is applied on four different types of project networks with eight different activity times for each activity in the project network of each type. The mean estimate of each activity is calculated using existed time estimates which are obtained from various continuous distribution methods with continuous random variable as activity times and calculated critical path with total duration for 32 project networks. The proposed model is an alternative to traditional Program Evaluation Review Technique (PERT) method to find critical activities, critical path and total duration of a project.

Mathematics Subject Classification: 90C05, 60E05

Keywords: PERT, Critical path, activity times, project network

INTRODUCTION

A project is classically defined as a set of activities which must be performed according to some precedence constraints requiring that some activities cannot start before the completion of some others. When duration of the activities are well known, critical path method (CPM) [1] (Kelly and Walker, 1959), provide the minimal project duration and identify the critical paths. Since the late 1950s, Critical Path Method (CPM) techniques have become widely recognized as valuable tools for the planning and scheduling large scale projects. In many situations, projects can be complicated and challenging to manage. When the activity times in the project are deterministic and known, CPM has been demonstrated to be a useful tool in managing projects in an efficient manner to meet this challenge. However, there are many cases where the activity times may not be presented in a precise manner. In real world, the durations of particular project activities cannot be precisely defined. This is the way the original Program Evaluation Review Technique (PERT) has been developed [2,3](Malcolm et al.,1959, Clark,1962). PERT is the most widely used management technique for planning and coordinating large scale projects. Since estimation of operation times of activities in a project network is difficult, therefore it is important to compute the variance of the project completion time in a network

The creators of PERT considered beta distribution

$$f_Y(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \frac{(y-a)^{\alpha-1}(b-y)^{\beta-1}}{(b-a)^{\alpha+\beta-1}}, a < y < b, \alpha, \beta > 0.$$

as an adequate distribution of the activity duration y where α and β are parameters of the beta distribution and the interval (a,b) is the domain of y.

They suggested the estimates of the mean and variance values

$$\mu = \frac{1}{6}(a+4m+b),$$

$$\sigma^2 = \frac{1}{36}(b-a)^2,$$

where a,m and b are the optimistic, most likely and pessimistic activity duration estimates respectively. In PERT, when a little sample information is available to 'fit' the distribution a, m and b are subjectively determined. PERT approximations are developed using some probability continuous random variable distributions.

In Section 2 , we briefly present PERT approximations using beta, normal, lognormal,and biparabolic distributions [2-8]. In Section 3, we develop a new linear programming model using PERT approximations to find critical activities, critical path and total duration of a project network.. In Section 4,the proposed probabilistic linear programming model is applied on thirty two project networks of four different types (i) Network –I,(ii) Network –II, (iii) Wheatstone Bridge,and (iv) Double Wheatstone Bridge as represented in Fig. 1 to obtain critical activities, critical path and total duration of a project network.

2. PERT APPROXIMATIONS USING BETA, NORMAL, LOGNORMAL, AND BIPARABOLIC DISTRIBUTIONS

In this section, some PERT approximations (mean and variance) using beta, normal, lognormal, and biparabolic distributions are reviewed and presented in Table I. Here, the activity times for the project network are considered to be continuous random variables with three parameters a , m and b .

Table I: PERT approximations

Traditional PERT (Malcolm et al.,1959)[2] using Beta distribution	$\mu = \frac{1}{6}(a+4m+b)$ $\sigma^2 = \frac{1}{36}(b-a)^2$
Ginzburg PERT approximations (Ginzburg, 1988) [4] using Beta distribution	$\mu = \frac{2a+9m+2b}{13}$ $\sigma^2 = \frac{(b-a)^2}{1268}\left[22+81\frac{m-a}{b-a}-81\left(\frac{m-a}{b-a}\right)^2\right].$
Shankar and Sireesha (S-S) PERT approximations [5] (Shankar and Sireesha, 2009) using Beta distribution	$\mu = \frac{5a+17m+5b}{27}$ $\sigma^2 = \frac{(b-a)^2}{35}$
PERT approximations using Normal distribution [6] (Cottrell, 1999)	PERT approximations using time estimates (*b*,m) are $\left.\begin{matrix}\mu = m \\ \sigma = \frac{b-m}{z}\end{matrix}\right\}$ where $z = 3.44$.
PERT approximations using Lognormal distribution(Mohan et al., 2007) [7]	PERT approximations using activity time estimates (a,m) are $\left.\begin{matrix}\sigma^* = \frac{z}{2} - \left[\frac{z^2}{4} + \ln(a/m)\right]^{1/2} \\ \mu^* = \ln[a] + z\sigma^*\end{matrix}\right\}$ PERT approximations using activity time estimates (b,m) are $\left.\begin{matrix}\sigma^* = -\frac{z}{2} + \left[\frac{z^2}{4} + \ln(b/m)\right]^{1/2} \\ \mu^* = \ln[b] - z\sigma^*\end{matrix}\right\}$ where $\sigma^* and\ \mu^*$ are the mean and standard deviation of the normal distribution and z=3.
PERT approximations using biparabolic distribution (Garcia et al., 2010) [8]	$\mu = \frac{3a+2m+3b}{8}$ $\sigma^2 = \frac{12(m-a)^2 - 12(m-a)(b-a) + 19(b-a)^2}{320}$

3. CRITICAL PATH METHOD USING PROBABILISTIC LINEAR PROGRAMMING PROBLEM

Consider a project network S = <V, A, t> consisting of a finite set V of nodes and a set $A \subset V \times V$ of arcs with random numbers as activity times, which are determined by a function t : A→ X where X is a random number and attached to the arcs. Denote $t_{(i,j)}$ as the probabilistic time period of activity (i, j) ∈ A. M($t_{(i,j)}$) is the mean of the time period calculated using probability density function. Let $x_{(i,j)}$ be the decision variable denoting the amount of flow in (i, j)∈A. Since only one unit of flow could be in any arc at any one time, the variable $x_{(i,j)}$ must assume binary values (0 or 1) only. The probabilistic linear programming problem with n nodes is formulated as

$$T = \max \sum_{i=1}^{n} \sum_{j=1}^{n} M(t_{(i,j)}) x_{(i,j)}$$

Subject to $\sum_{j=1}^{n} x_{(1,j)} = 1$

$$\sum_{j=1}^{n} x_{(i,j)} = \sum_{j=1}^{n} x_{(j,i)}, i = 2,....,n-1$$

$$\sum_{k=1}^{n} x_{(k,n)} = 1$$

$x_{(i,j)}$ = 0 or 1, (i, j)∈A.

The objective is to maximize the total duration time of the project network from node 1 to node n. The constraints are called the flow conservation equations and indicate the flow may be neither created nor destroyed in the project network. The critical path for this PERT network consists of a set of activities (i, j)∈A from the start to the finish in which each activity in the path corresponds to the optimal decision variable $x^*_{(i,j)} = 1$ in the optimal solution to probabilistic linear programming problem. The total duration time needed to complete the project is given as the maximal objective value T of probabilistic linear programming problem.

4. APPLICATION OF PROPOSED PROBABILISTIC LINEAR PROGRAM ON PERT NETWORKS

Thirty two project networks of four different types (i) Network –I,(ii) Network –II, (iii) Wheatstone Bridge,and (iv) Double Wheatstone Bridge as represented in Fig. 1 are analyzed using probabilistic Linear programming problem.

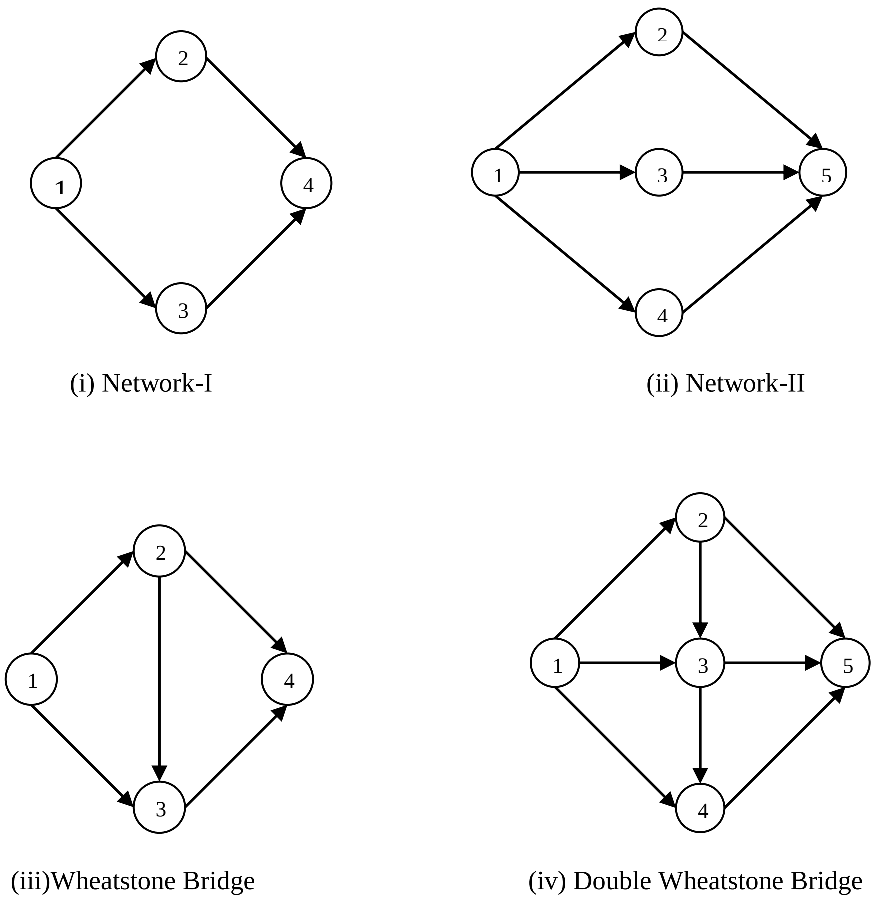

Fig.1 Fuzzy Network Structures

The activity times of 32 project networks of 4 types of networks are presented in Table II, Table III, Table IV ,and Table V.

Table II: Eight project networks of Network –I type diagram in Fig.1 with activities and their corresponding activity times

Activity	**Activity times of project networks 1 to 8**							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
(1,2)	(2.5,3,3.5)	(2.5,3,3.5)	(2.5,3,3.1)	(2.5,3,3.1)	(0.6,3,5.4)	(0.6,3,5.4)	(0.5,3,6)	(0.5,3,6)
(1,3)	(4,5,6)	(4,5,6)	(4,5,5.5)	(4,5,5.5)	(1,5,9)	(1,5,9)	(0.1,5,8)	(0.1,5,8)
(2,4)	(3.5,4.1,4.7)	(7,8,9)	(3.5,4.1,4.2)	(7,8,8.5)	(0.8,4.1,7.2)	(2,8,14)	(0.4,4.1,9.1)	(1,8,14)
(3,4)	(1.6,2,2.4)	(1.6,2,2.4)	(1.8,2,2.4)	(1.8,2,2.4)	(0.4,2,3.6	(0.4,2,3.6	(0.2,2,4)	(0.2,2,4)

Table III: Eight project networks of Network –II type diagram in Fig.1 with activities and their corresponding activity times

Activity	**Activity times of project networks 9 to 16**							
	9	10	11	12	13	14	15	16
(1,2)	(4,5,6)	(4,5,6)	(4,5,5.5)	(4,5,5.5)	(1,5,9)	(1,5,9)	(1,5,12)	(1,5,12)
(1,3)	(3.7,1,4.5)	(7,8,9)	(3.5,4.1,4.2)	(7,8,8.5)	(0.8,4.1,7.4)	(2,8,14)	(0.5,4.1,7.2)	(1,8,14)
(1,4)	(1.6,2,2.4)	(1.6,2,2.4)	(1.8,2,2.4)	(1.8,2,2.4)	(0.4,2,3.6)	(0.4,2,3.6)	(0.2,2,4)	(0.2,2,4)
(2,5)	(2.5,3,3.5)	(2.5,3,3.5)	(2.5,3,3.1)	(2.5,3,3.1)	(0.6,3,5.4)	(0.6,3,5.4)	(0.6,3,7)	(0.6,3,7)
(3,5)	(3.5,4,4.5)	(3.5,4,4.5)	(3.9,4,4.5)	((3.9,4,4.5)	(0.8,4,7.2)	(0.8,4,7.2)	(0.5,4,7.2)	(0.5,4,7.2)
(4,5)	(4.8,5.9,7)	(1.6,2,2.4)	(5.5,5.9,7)	(1.6,2,2.2)	(0.8,5.9,11)	(0.4,2,3.6)	(0.5,5.9,12)	(0.4,2,5)

Table IV: Eight project networks of Wheatstone Bridge type diagram in Fig.1 with activities and their corresponding activity times

Activity	**Activity times of project networks 17 to 24**							
	17	**18**	**19**	**20**	**21**	**22**	**23**	**24**
(1,2)	(3.3,4.1,4.9)	(3.2,4,4.8)	(3.8,4.1,4.2)	(3.2,4,4.1)	(0.8,4.1,7.4)	(0.5,4,7.5)	(0.8,4.1,10)	(0.8,4,10)
(1,3)	(5.8,6.9,9.8)	(1.6,2,2.4)	(6.8,6.9,8)	(1.9,2,2.4)	(0.8,6.9,13)	(0.4,2,3.6)	(1,6.9,14)	(0.1,2,5)
(2,3)	(2.5,3,3.5)	(6.4,8,9.6)	(2.5,3,3.1)	(7,8,10)	(0.6,3,5.4)	(1.6,8,14.4)	(0.5,3,5.4)	(1,8,14)
(2,4)	(5,6,7)	(5,6,7)	(5.9,6,7)	(5.9,6,7)	(1,6,11)	(1,6,11)	(0.5,6,11)	(0.5,6,11)
(34)	(2.5,3.1,3.7)	(2.4,3,3.6)	(2.6,3.1,3.2)	(2.9,3,3.5)	(0.6,3.1,5.6)	(0.6,3,5.4)	(0.5,3.1,7)	(0.5,3,7)

Table V: Eight project networks of Double Wheatstone Bridge type diagram in Fig.1 with activities and their corresponding activity times

Activity	**Activity times of project networks 25 to 32**							
	25	**26**	**27**	**28**	**29**	**30**	**31**	**32**
(1,2)	(2.5,3,3.5)	(2.5,3,3.5)	(2.9,3,3.4)	(2.9,3,3.4)	(0.6,3,5.4)	(0.6,3,5.4)	(0.5,3,6)	(0.5,3,6)
(1,3)	(4,4.9,5.8)	(0.8,1,1.2)	(4.5,4.9,5.1)	(0.9,1,1.2)	(1,4.9,8.8)	(0.2,1,1.8)	(0.6,4.9,10)	(0.1,1,4)
(1,4)	(1.6,2,2.4)	(1.6,2,2.4)	(1.9,2,2.4)	(1.9,2,2.4)	(0.4,2,3.6)	(0.4,2,3.6)	(0.3,2,4)	(0.3,2,4)
(2,3)	(1.7,2.1,2.5)	(6.4,8,9.6)	(1.9,2.1,2.4)	(7,8,10)	(0.4,2.1,3.8)	(1.6,8,14.4)	(0.4,2.1,5)	(1,8,14)
(2,5)	(2.6,3.2,3.8)	(2.5,3,3.5)	(2.6,3.2,3.3)	(2.5,3,3.1)	(0.6,3.2,5.8)	(0.6,3,5.4)	(0.5,3.2,7)	(0.6,3,7)
(3,4)	(8,9,10)	(8,9,10)	(8.5,9,9.1)	(8.5,9,9.1)	(1,9,17)	(1,9,17)	(0.5,9,17)	(0.5,9,17)
(3,5)	(6,7,8)	(6,7,8)	(6.8,7,7.9)	(6.8,7,7.9)	(1,7,13)	(1,7,13)	(0.5,7,15)	(0.5,7,15)
(4,5)	(7.9,9.8,11.7)	(3.5,4,4.5)	(8.8,9.8,11)	(3.9,4,4.5)	(1.6,9.8,18)	(0.8,4,7.2)	(1,9.8,20)	(0.5,4,7.2)

Probabilistic linear programming model for Network –I in Fig.1 is

$$Maximize\, z = M(t_{(1,2)})x_{(1,2)} + M(t_{(1,3)})x_{(1,3)} + M(t_{(2,4)})x_{(2,4)} + M(t_{(3,4)})x_{(3,4)}$$

subject to constrains

$$x_{(1,2)} + x_{(1,3)} = 1$$

$$x_{(1,2)} - x_{(2,4)} = 0$$

$$x_{(1,3)} - x_{(3,4)} = 0$$

$$x_{(2,4)} + x_{(3,4)} = 1$$

$$x_{(1,2)}, x_{(1,3)}, x_{(2,4)}, x_{(3,4)} \geq 0 \; or\, 1.$$

Total duration network and critical path of network I are calculated using TORA system and presented in Table VI.

Table VI: Total duration and critical path of a network I

Method	Critical path	Total duration
Beta distribution	$1 \to 2 \to 4$	7.1
Ginzburg time	$1 \to 2 \to 4$	7.1
S-S time	$1 \to 2 \to 4$	7.1
Normal distribution	$1 \to 2 \to 4$	7.1
Lognormal distribution	$1 \to 2 \to 4$	7.10
Biparabolic distribution	$1 \to 2 \to 4$	6.97

Probabilistic linear programming model for Network –II in Fig.1 is

$$\text{Maximize } Z = M\left(t_{(1,2)}\right)x_{(1,2)} + M\left(t_{(1,3)}\right)x_{(1,3)} + M\left(t_{(1,4)}\right)x_{(1,4)} + M\left(t_{(2,5)}\right)x_{(2,5)} + M\left(t_{(4,5)}\right)x_{(4,5)} + M\left(t_{(3,5)}\right)x_{(3,5)}$$

Subject to constrains

$$x_{(1,2)} + x_{(1,3)} + x_{(1,4)} = 1$$
$$x_{(1,2)} - x_{(2,5)} = 0$$
$$x_{(1,3)} - x_{(3,5)} = 0$$
$$x_{(1,4)} - x_{(4,5)} = 0$$
$$x_{(2,5)} + x_{(3,5)} + x_{(4,5)} = 1$$

$$x_{(1,2)}, x_{(1,3)}, x_{(1,4)}, x_{(2,5)}, x_{(3,5)}, x_{(4,5)} \geq 0\, or\, 1.$$

Total duration network and critical path of network II are calculated using TORA system and presented in Table VII.

Table VII : Total duration and critical path of a network II

Method	Critical path	Total duration
Beta distribution	$1 \to 3 \to 5$	8.10
Ginzburg time	$1 \to 3 \to 5$	8.1
S-S time	$1 \to 3 \to 5$	8.1
Normal distribution	$1 \to 3 \to 5$	8.1
Lognormal distribution	$1 \to 3 \to 5$	8.1
Biparabolic distribution	$1 \to 3 \to 5$	8.1

Probabilistic linear programming model for Double of wheatstone Bridge network III in Fig.1 is

$$\text{Maximize } Z = M\left(t_{(1,2)}\right)x_{(1,2)} + M\left(t_{(1,3)}\right)x_{(1,3)} + M\left(t_{(2,4)}\right)x_{(2,4)} + M\left(t_{(3,4)}\right)x_{(3,4)}$$

Subject to constrains

$$x_{(1,2)} + x_{(1,3)} = 1$$
$$x_{(1,2)} - x_{(2,3)} - x_{(2,4)} = 0$$
$$x_{(1,3)} - x_{(3,4)} = 0$$
$$x_{(2,3)} - x_{(3,4)} = 0$$
$$x_{(2,4)} + x_{(3,4)} = 1$$

$$x_{(1,2)}, x_{(1,3)}, x_{(2,3)}, x_{(2,4)}, x_{(3,4)} \geq 0\, or\, 1.$$

Total duration network and critical path of wheatstone Bridge network III are calculated using TORA system and presented in Table VIII .

Table VIII : Total duration and critical path of a network III

Method	Critical path	Total duration
Beta distribution	$1 \to 3, 2 \to 3, 3 \to 4$	13
Ginzburg time	$1 \to 3, 2 \to 3, 3 \to 4$	12.9
S-S time	$1 \to 3, 2 \to 3, 3 \to 4$	13
Normal distribution	$1 \to 3, 2 \to 3, 3 \to 4$	13
Lognormal distribution	$1 \to 3, 2 \to 3, 3 \to 4$	12.25
Biparabolic distribution	$1 \to 3, 2 \to 3, 3 \to 4$	13

Probabilistic linear programming model for Double Wheatstone Bridge Network – IV in Fig.1 is

Maximize Z = $M\left(t_{(1,2)}\right)x_{(1,2)} + M\left(t_{(1,3)}\right)x_{(1,3)} + M\left(t_{(1,4)}\right)x_{(1,4)}$
$+ M\left(t_{(2,3)}\right)x_{(2,3)} + M\left(t_{(2,5)}\right)x_{(2,5)} + M\left(t_{(3,4)}\right)x_{(3,4)}$
$+ M\left(t_{(4,5)}\right)x_{(4,5)} + M\left(t_{(3,5)}\right)x_{(3,5)}$

Subject to constrains

$$
\begin{aligned}
x_{(1,2)} + x_{(1,3)} + x_{(1,4)} &= 1\\
x_{(1,2)} - x_{(2,3)} - x_{(2,5)} &= 0\\
x_{(1,3)} - x_{(3,4)} - x_{(3,5)} &= 0\\
x_{(1,4)} - x_{(4,5)} &= 0\\
x_{(3,4)} - x_{(4,5,)} &= 0\\
x_{(2,5)} + x_{(3,5)} + x_{(4,5)} &= 1
\end{aligned}
$$

$$x_{(1,2)}, x_{(1,3)}, x_{(1,4)}, x_{(2,3)}, x_{(2,5)}, x_{(1,3)}, x_{(3,4)}, x_{(3,5)}, x_{(4,5)} \geq 0\, or\, 1.$$

Total duration network and critical path of Double wheatstone Bridge network IV are calculated using TORA system and presented in Table IX .

Table IX: Total duration and critical path of a network IV

Method	**Critical path**	**Total duration**
Beta distribution	$1\to3, 2\to3, 3\to4, 4\to5$	25.8
Ginzburg time	$1\to3, 2\to3, 3\to4, 4\to5$	25.8
S-S time	$1\to3, 2\to3, 3\to4, 4\to5$	25.8
Normal distribution	$1\to3, 2\to3, 3\to4, 4\to5$	25.8
Lognormal distribution	$1\to3, 2\to3, 3\to4, 4\to5$	25.8
Biparabolic distribution	$1\to3, 2\to3, 3\to4, 4\to5$	25.8

CONCLUSION

The proposed probabilistic linear programming model to obtain critical activities, critical path and total duration of a project network is illustrated through 32 projects on four different types of project networks with eight different activity times. The proposed model is an alternative to traditional Program Evaluation Review Technique (PERT) method to find critical activities, critical path and total duration of a project. Total duration and critical path of a network with four different activity times are calculated using TORA system and compared with some methods using continuous probability distribution. From comparison, it is observed that the total duration and the critical paths obtained by all methods are same. Beta distribution time estimates are performed well in calculating project duration time compared to other time estimates.

REFERENCES

[1] Cottrell W., Simplified program evaluation and review technique (PERT), Journal of construction Engineering and Management 125 (1), 16-22, 1999.

[2] Kelly J.E., Critical path planning and scheduling mathematical basis. Operations Research, 9,296-320.doi:10.1287/opre.9.3.296,1961.

[3] Malcom D.G.,Roseboom J.H.,Clark C,E., and Fazar W., Applications of a Technique for Research and Development Program Evaluation , Operations Research, vol. 7 no .5 646-669, 1959.

[4] Mohan S., Gopalakrishna M., Balasubramanian H. and Chandrashekar A., A lognormal approximation of activity duration in PERT using two time estimates ,Journal of the Operational Research Society 58, 827-831, 2007 .

[5] Garcia C.B., Garcia P.J . and Cruz S.R., proposal of a new distribution in PERT methodology , Annals of operation research, DOI 10.1007/S 1047-010-0783-1,2010.

[6] Sireesha V., and Shankar , Ravi Shankar N., A new approach to find total float time and critical path in a fuzzy project network ,International Journal of Engineering Science and Technology , Vol.2, pp.600-609, 2010.

[7] Ginzburg G.D., on the distribution of activity time in PERT ,Journal of Operational research society, 39,767 – 771,1988.