International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 4 (2019) pp. 877-880

Some Applications of Differential Transform Methods to Stiff Differential Equations

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Abstract

In this paper, Differential Transform Method (DTM) approach is discussed for solving boundary stiff differential equations. The proposed approach solves the stiff boundary problems for linear and non-linear systems so that the efficiency could be increased in such kind of problems. The method has some stages in which the first section is to use to find the series solution and after that it will be converted into exact solution which is easy and compatible using DTM. For the convergence of the method, there are necessary conditions. The recommended method is demonstrated by means of different numerical examples, mostly taken from well-known textbooks and from graphs. The results show that the accuracy in this method is more appropriate than others and it is very simple, efficient and convenient.

1. INTRODUCTION

Since last two decades, stiff differential equations have been studied broadly and a variety of methods for their solutions have been proposed.

As a rule stiffness occurs in differential equations where there are two or more different time scales of the independent variables on which the dependent variables are changing. In the case of linear problems the stiffness is caused by eigenvalues of big negative values. To measure the degree of stiffness the following stiffness ratio SR = max |λ|/ min |λ| is introduced: when SR < 20 the problem is not stiff, up to SR ≈ 1000 the problem is classified as stiff, and when SR ≥ 100000, the problem is very stiff. Stiff ODEs are called extremely stable if there is at least one eigenvalue with a large negative real part. In the case of nonlinear problems the problem is more complicated since stiffness is a global problem and cannot be reduced to the solution structure in the neighbourhood of single points.

Consider the stiff initial value problem:

$$y'(x) = f\big(x, (x)\big), y(x_0) = y_0 \qquad \textbf{(a)}$$

There are so many previous efforts to solve stiff boundary problems by many researchers such as Abasi (2014), Alt (1978), Alvarez (2002), Cash (1980), Dahlquist (1974), Ibrahim (2008), Musa (2015), Suleiman (2014), Yatim (2011) and Zawawi (2015) among others, to develop methods for stiff ODEs. Researchers like Alexander with diagonally implicit Runge-Kutta for stiff ODEs. have also been involved for getting efficient numerical approximation in terms of accuracy and computational time. The motivation of this research is to modify the method developed by Zawawi so as to improve its accuracy and stability properties.

In this paper, we first give some basic properties of one-dimensional differential transform method. Differential transform of a function $y(x)$ is defined as follows:

$$Y(k) = \frac{1}{k!}\frac{d^k y}{dx^k}\bigg|_{x=0} \qquad \textbf{(b)}$$

Where $y(x)$ is the original function and $Y(k)$ is the transformed function for k=0,1,2,3,…. The differential inverse transform of $Y(k)$ is defined as

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \qquad \textbf{(c)}$$

From Equations (b) and (c) we get

$$y(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{d^k y}{dx^k}\bigg|_{x=0} \qquad \textbf{(d)}$$

The concept of DTM is thus derived from Taylor series expansion, but this method don't evaluate the derivative symbolically. Though, iterative procedure can be used to calculate relative derivatives and original function is described by the transformed equations.

In this work, we use the lower case letters to represent the original functions and upper case letters to represent the transformed functions. In actual applications, the function $y(x)$ is expressed by a finite series Equation (e) can be written as

$$y(x) = \sum_{k=0}^{n} x^k Y(k) \qquad \textbf{(e)}$$

Here n is represented the convergence of natural frequency.

We have obtained Table of the fundamental operations of one-dimensional differential transform method from Equations (b) and (c) as given by.

Original Function	Transformed Function
$y(x) = u(x) \pm v(x)$	$Y(k) = U(k) \pm V(k)$
$y(x) = cw(x)$	$Y(k) = cW(k)$
$y(x) = \frac{dy}{dx}$	$Y(k) = (k+1)W(k+1)$
$y(x) = \frac{d'y}{dx'}$	$Y(k) = (k+1)(k+2) \dots W(k+j)W(k+j)$
$y(x) = u(x)v(x)$	$Y(k) = \sum_{r=0}^{k} U(r)V(k-r)$
$y(x) = exp(\lambda x)$	$Y(k) = \frac{\lambda^k}{k!}$

2. DIFFERENTIAL TRANSFORMATION METHOD AND ITS APPLICATION TO STIFF SYSTEM

In this section, we apply DTM to both linear and nonlinear stiff systems.

Problem 1: Consider the linear stiff system,

$$y_1' = -y_1 - 15y_2 + 15e^{-x} \quad \textbf{(f)}$$

$$y_2' = 15y_1 - y_2 - 15e^{-x} \quad \textbf{(g)}$$

With initial value $y_1(0) = 1, y_2(0) = 1$

This system has eigen values of large modulus lying closed to the imaginary axis -1±15*I*

By applying Differential Transformation, we have

$$y_1(k+1) = \frac{1}{k+1}\left[-Y_1(k) - 15y_2(k) + 15\frac{(-1)^k}{k!}\right] \quad \textbf{(h)}$$

$$y_2(k+1) = \frac{1}{k+1}\left[15Y_1(k) - y_2(k) - 15\frac{(-1)^k}{k!}\right] \quad \textbf{(i)}$$

The initial conditions of Differential Transformation are given by:

$$Y_1(0) = 1, Y_2(0) \quad \textbf{(j)}$$

For k = 0,1,2,3, … the series coefficients for $Y_1(k)$ and $Y_2(k)$ can be obtained as

$$Y_1(0) = 1, Y_1(1) = -1 \; Y_1(2) = \frac{1}{2!} \quad \textbf{(k)}$$

$$Y_1(3) = \frac{1}{3!}, Y_1(4) = \frac{1}{4!}, Y_1(5) = -\frac{1}{5!} \dots \quad \textbf{(l)}$$

$$Y_2(0) = 1, Y_1(1) = -1, Y_2(2) = \frac{1}{2!} \quad \textbf{(m)}$$

$$Y_2(3) = -\frac{1}{3!}, Y_2(4) = \frac{1}{4!}, Y_2(5) = \frac{1}{5!} \dots \quad \textbf{(n)}$$

We used MATHEMATICA to calculate the unknown coefficients $Y_1(k)$ and $y_2(k)$

Using the inverse Transform, we get

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k) \quad \textbf{(0)}$$

$$y_1(x) = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \textbf{(p)}$$

$$y_2(x) = 1 - \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \cdots \quad \textbf{(q)}$$

Equations (p) and (q) can be written in the exponential form are given by

$$y_1(x) = e^{-x} \quad \textbf{(r)}$$

and

$$y_2(x) = e^{-x} \quad \textbf{(s)}$$

Exact solution is achieved by differential transform method.

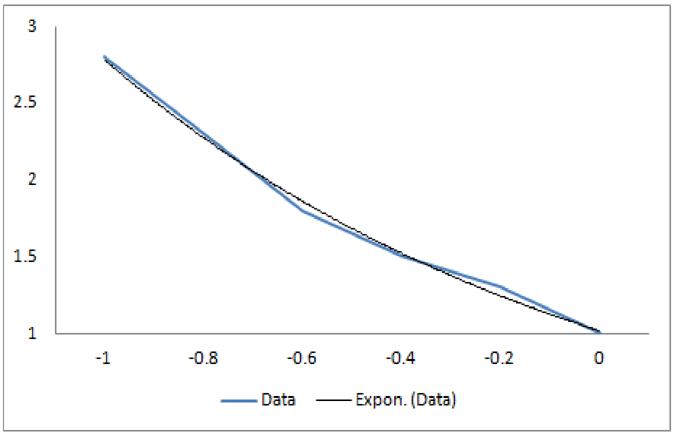

Figure 1. Graphical solution of Equation (h) via DTM

Problem 2: Consider the non-linear system in the form of initial value problems is given by:

$$Y_1 = -1002_{,Y_1} + 1000\, y^{2,}{}_2\, y_1(0) = 1, \quad \textbf{(1.11)}$$

$$Y_1 = -1002_{,Y_1} + 1000\, y^{2,}{}_2\, y_1(0) = 1, \quad \textbf{(1.12)}$$

Applying Differential Transform, we have

$$y_1(k+1) = \frac{1}{k+1}\begin{bmatrix} -1002y_{1(k)+1000\sum_{r=0}^{k} y_1}(k) \\ +1000\sum_{r=0}^{k} y_1(r)y_1(k-r) \end{bmatrix}. \quad \textbf{(1.13)}$$

$$y_1(k+1) \Big|= \frac{1}{k+1}[y_1(k) + y_2(k) - \sum_{r=0}^{k} y_1(r)y_1(k-r)]. \quad \textbf{(1.14)}$$

For k=0, 1, 2, 3….,n., the series coefficients for $Y_1\ (k)$ *and* $Y_2\ (k)$ can be obtained as

$$Y_1\ (0) = 1, Y_1(1) = -2, Y_1\ (2) = \ 2,$$

$$Y_1\ (3) = -\frac{4}{3}, Y_1(4) = -\frac{2}{3}, \ldots,$$

$$Y_2(0) = 1, Y_2(1) = -1 Y_2(2) = \frac{1}{2},$$

$$Y_2(3) = -\frac{1}{3!}, Y_2(4) = \frac{1}{4!} Y_2(5) = -\frac{1}{5!} \ldots$$

We used MATHEMATICA to find the coefficients of $Y_1(k)$ and $Y_2(k)$

Using the inverse Transform:

$$y(x) = \sum_{k=0}^{\infty} x^k Y(k), \tag{1.15}$$

$$y_1(x) = 1 - \frac{2x}{1!} + \frac{4x^2}{2!} - \frac{8x^3}{3!} + \frac{16x^4}{4!} - \cdots \tag{1.16}$$

$$y_2(x) = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} \tag{1.17}$$

This can be written as follows

$$y_1(x) = e^{-2x} \tag{1.18}$$

and

$$y_2(x) = e^{-x} \tag{1.19}$$

Problem 3: Consider the system of initial value problems:

$$y_1' = -y_1 \tag{1.20}$$

$$y_2' = -10y_2 \tag{1.21}$$

$$y_3' = -100y_3 \tag{1.22}$$

$$y_4' = -1000y_4 \tag{1.23}$$

With initial conditions $y_i(0) = 1, \mathrm{i} = 1, 2, 3, 4$. Applying Differential Transform, we have

$$Y_1(k+1) = -\frac{Y_1(k)}{(k+1)};$$

$$Y_2(k+1) = -\frac{10Y_2(k)}{(k+1)};$$

$$Y_3(k+1) = -\frac{100Y_3(k)}{(k+1)};$$

$$Y_4(k+1) = -\frac{1000Y_4(k)}{(k+1)};$$

With the transformed initial conditions is $y_i(0) = 1$, i =1,2,3,4.

For k = 0, 1, 2,3,…. the series coefficients for$Y_1(k)$, $Y_2(k)$, $Y_3(k)$ and $Y_4(k)$ can be obtained as:

$$y_1(1) = -1, Y_1(2) = \frac{1}{2}, y_1(3) = -\frac{1}{6}, Y_1(4) = \frac{1}{24}, Y_1(5) = -\frac{1}{120}, \ldots.$$

$$y_2(1) = -10, Y_2(2) = 50, y_2(3) = -\frac{500}{3}, Y_2(4) = \frac{1250}{3}, Y_2(5) = -\frac{2500}{3}, \ldots.$$

$$y_3(1) = -100, Y_3(2) = 5000, y_3(3) = -\frac{500000}{3}, Y_3(4) = -\frac{12500000}{3}, Y_3(5) = -\frac{250000000}{3}, \ldots.$$

$$y_4(1) = -1000, Y_4(2) = 500000, y_4(3) = -\frac{500000000}{3}, Y_4(4) = -\frac{125000000000}{3}, Y_4(5) = -\frac{25000000000000}{3}, \ldots.$$

Using the inverse Transform:

$$y_i(x) = \sum_{k=0}^{\infty} x^k\, Y_i(k), for\ i = 1,2,3,4 \tag{1.24}$$

We obtain

$$y_1 = e^{-x},\ y_1 = e^{-10x}, y_1 = e^{-100x},\ y_1 = e^{-1000x},$$

In this area, we have introduced three diverse linear and nonlinear hardened systems through Differential Transform Method and the arrangement of Equations (1.13) and (h) have appeared in Figures 1 and 2 individually.

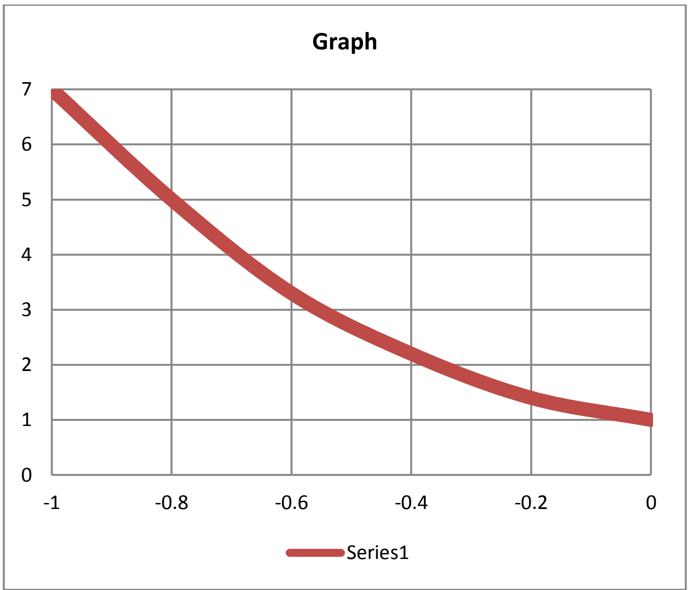

Figure 2. Graphical solution of Equation (1.13) via DTM

3. CONCLUSION

From the above equation it is concluded that the DTM method was successfully accomplished of solution for linear and non-linear stiff system. From the above graph it is clearly shown that DTM is very simple and effective for the solution of boundary value problems. We have obtained that the results of DTM is more accurate in comparison with those obtained by other methods.

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