International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 3 (2019) pp. 750-761
 https://dx.doi.org/10.37622/IJAER/14.3.2019.750-761

A Single Server Queue with Reneging and Working Vacation

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Abstract

This paper presents an analysis of an $M/G/1$ queue with reneging and Working Vacation. The server works with different service rate rather than completely stopping service during a vacation. Both service times in a vacation period and in a service period are generally distributed random variables. We treat reneging in this paper when the service rate is lower as the server is on working vacation. Using supplementary variable technique we derive the probability generating function for the number of customers in the system and the average number in the system. Furthermore we carryout the waiting time distribution and some particular cases of interest are discussed. Finally some numerical results are presented.

Keywords: Poisson Arrivals, Reneging, Working Vacation, Supplementary Variable Technique.

AMS Subject Classification Number: 60K25, 60K30.

1. INTRODUCTION

In real life, there are queueing situations when some customers are impatient and discouraged by a long wait in the queue. As such, customers may decide not join the queue (Balking) or leave the queue after joining without receiving any service (Reneging). We often witness such situations in real life like calls waiting in call centers, emergency patients in hospitals, programs on computer, banks etc. Balking and Reneging have attracted the attention of many authors and study of queues with behavior of impatient customers has sufficiently developed and we see an extensive amount of literature in this area. Dayley (1965) appears to be the first who studied queues with impatient customers since then queueing models with balking and reneging has been studied by many authors like Ancker et.al. (1963), Altman and Yechiali (2006), Choudhury and Medhi (2011) to quote a few. In recent years, studies related to customer's impatience has been mainly concentrated on queueing models with single server.

Recently a class of semi-vacation policies has been introduced by Servi and Finn. Such a vacation is called working vacation (WV). The server works at a lower rate rather than completely stops service during a vacation. Servi and Finn (2002) studied an M/M/1 queue with multiple working vacation and obtained the probability generating function for the number of customers in the system and the waiting time distribution. Some other notable works were done by Wu and Takagi (2006) Tian et.al. (2008) Aftab Begum (2011) Santhi and Pazhani Bala Murugan (2013) and Santhi and Pazhani Bala Murugan (2014).

2. THE MODEL DESCRIPTION

We assume the following to describe the queueing model of our study. Customers arrive at the system one by one according to a Poisson stream with arrival rate $\lambda(>0)$. The service discipline is FCFS. The service time is general distribution. Let $S_b(x), s_b(x)$ and $S_b^*(\theta)$ be the probability distribution function, the probability density function and the Laplace-Stieltjes transform (LST) of the service time S_b. Whenever the system becomes empty at a service completion instant the server starts working vacation and the duration of the vacation time follows an exponential distribution with rate η. At a vacation completion instant, if there are customers in the system the server will start a new busy period. Otherwise he continues his working vacation period. This type of vacation policy is called multiple working vacation. In the single working Vacation if there are customers in the system the server will start a new busy period. Otherwise he remains in the system until an arrival of a customer. During the working vacation period the server provides the service with the service time S_v of a typical customer follows a general distribution with the distribution function $S_v(x)[s_v(x)$ the probability density function and $S_v^*(\theta)$, the LST]. We assume that customers may renege (leave the system after joining the queue) when the server is on working vacation and reneging is assumed to follow exponential distribution with parameter β. Further, it is noted that the service interrupted at the end of a vacation is lost and it is restarted with a different distribution at the beginning of the following service period. Inter arrival times, service times and working vacation times are mutually independent of each other.

3. THE SYSTEM SIZE DISTRIBUTION

The system size distribution at an arbitrary time will be treated by the supplementary variable technique. That is, from the joint distribution of the queue length and the remaining service time of the customer in service if the server is busy, or the remaining service time of the customer if the server is on Working Vacation. Let us define the following random variables.

$N(t)$ - the system size at time t.

$S_b^0(t)$ - the remaining service time in regular service period.

$S_v^0(t)$ - the remaining service time in WV period.

$$Y(t)=\begin{cases}0 & \text{if the server is idle in WVperiod}\\ 1 & \text{if the server is idle in regular service period}\\ 2 & \text{if the server is busy in regular service period}\\ 3 & \text{if the server is busy in WV period}\end{cases}$$

so that the supplementary variables $S_b^0(t)$ and $S_v^0(t)$ are introduced in order to obtain bivariate Markov Process $\{(N(t),\partial(t));t\geq 0\}$, where

$$\partial(t)=S_b^0(t)\text{ if Y(t) = 2 ; } S_v^0(t)\text{ if Y(t) = 3}$$

We define the following limiting probabilities:

$$Q_0 = \lim_{t\to\infty}\Pr\{N(t)=0,Y(t)=0\};\quad P_0=\lim_{t\to\infty}\Pr\{N(t)=0,Y(t)=1\}$$

$$P_n(x) = \lim_{t\to\infty}\Pr\{N(t)=n,Y(t)=2,x<S_b^0(t)\leq x+dx\},\quad n\geq 1.$$

$$Q_n(x) = \lim_{t\to\infty}\Pr\{N(t)=n,Y(t)=3,x<S_v^0(t)\leq x+dx\},\quad n\geq 1.$$

We define the Laplace Stieltjes Transform and the probability generating functions as follows,

$$S_b^*(\theta) = \int_0^\infty e^{-\theta x}s_b(x)dx;\quad S_v^*(\theta)=\int_0^\infty e^{-\theta x}s_v(x)dx;\quad Q_n^*(\theta)=\int_0^\infty e^{-\theta x}Q_n(x)dx;$$

$$Q_n^*(0) = \int_0^\infty Q_n(x)dx;\quad P_n^*(\theta)=\int_0^\infty e^{-\theta x}P_n(x)dx;\quad P_n^*(0)=\int_0^\infty P_n(x)dx$$

$$Q^*(z,\theta)=\sum_{n=1}^\infty z^nQ_n^*(\theta);\quad Q^*(z,0)=\sum_{n=1}^\infty z^nQ_n^*(0);\quad P^*(z,\theta)=\sum_{n=1}^\infty z^nP_n^*(\theta);$$

$$P^*(z,0)=\sum_{n=1}^\infty z^nP_n^*(0);\quad Q(z,0)=\sum_{n=1}^\infty z^nQ_n(0);\quad P(z,0)=\sum_{n=1}^\infty z^nP_n(0)$$

4. THE SYSTEM SIZE DISTRIBUTION FOR MULTIPLE WORKING VACATION

By assuming that the system is in steady state condition, the differential difference equations governing the system is as follows:

$$\lambda Q_0 = P_1(0)+Q_1(0) \tag{1}$$

$$-\frac{d}{dx}Q_1(x) = -(\lambda+\eta)Q_1(x)+Q_2(0)s_v(x)+\lambda Q_0 s_v(x)+\beta Q_2(x) \tag{2}$$

$$-\frac{d}{dx}Q_n(x) = -(\lambda+\eta+\beta)Q_n(x)+Q_{n+1}(0)s_v(x)+\lambda Q_{n-1}(x)+\beta Q_{n+1}(x);n>1 \tag{3}$$

$$-\frac{d}{dx}P_1(x) = -\lambda P_1(x)+P_2(0)s_b(x)+\eta s_b(x)\int_0^\infty Q_1(y)dy \tag{4}$$

$$-\frac{d}{dx}P_n(x) = -\lambda P_n(x)+P_{n+1}(0)s_b(x)+\lambda p_{n-1}(x)+\eta s_b(x)\int_0^\infty Q_n(y)dy;\quad n>1 \tag{5}$$

Taking the LST of (2) to (5) we have

$$\theta Q_1^*(\theta) - Q_1(0) = (\lambda+\eta)Q_1^*(\theta) - Q_2(0)S_v^*(\theta) - \lambda Q_0 S_v^*(\theta) - \beta Q_2^*(\theta) \quad (6)$$

$$\theta Q_n^*(\theta) - Q_n(0) = (\lambda+\eta+\beta)Q_n^*(\theta) - Q_{n+1}(0)S_v^*(\theta) - \lambda Q_{n-1}^*(\theta) - \beta Q_{n+1}^*(\theta); n>1 \quad (7)$$

$$\theta P_1^*(\theta) - P_1(0) = \lambda P_1^*(\theta) - P_2(0)S_b^*(\theta) - \eta S_b^*(\theta)\int_0^\infty Q_1(y)dy \quad (8)$$

$$\theta P_n^*(\theta) - P_n(0) = \lambda P_n^*(\theta) - P_{n+1}(0)S_b^*(\theta) - \lambda P_{n-1}^*(\theta) - \eta S_b^*(\theta)\int_0^\infty Q_n(y)dy; \quad n>1 \quad (9)$$

z^n times (7) summing over n from 2 to ∞ is added up with z times (6) we get

$$[\theta-(\lambda-\lambda z+\eta+\beta-\beta/z)]Q^*(z,\theta) = Q(z,0)[\frac{z-S_v^*(\theta)}{z}] - S_v^*(\theta)[\lambda z Q_0 - Q_1(0)] \quad (10)$$

Inserting $\theta = (\lambda-\lambda z+\eta+\beta-\beta/z)$ in (10) we get

$$Q(z,0) = \frac{zS_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)[\lambda z Q_0 - Q_1(0)]}{z - S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)} \quad (11)$$

The denominator of the above equation has a unique root z_1 in $(0,1)$. Therefore

$$Q_1(0) = \lambda z_1 Q_0$$

Substituting this in (11) we have

$$Q(z,0) = \frac{\lambda z Q_0 S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)(z-z_1)}{z - S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)} \quad (12)$$

Substituting (12) in (10) and inserting $\theta = 0,$ we get

$$Q^*(z,0) = \frac{\lambda z Q_0 (z-z_1)[1-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)]}{(\lambda-\lambda z+\eta+\beta-\beta/z)(z-S_v^*(\lambda-\lambda z+\eta+\eta-\beta/z))} \quad (13)$$

z^n times (9) summing over n from 2 to ∞ is added up with z times (8), we get

$$[\theta-(\lambda-\lambda z)]P^*(z,\theta) = P(z,0)\left(\frac{z-S_b^*(\theta)}{z}\right) - S_b^*(\theta)[\eta Q^*(z,0) - P_1(0)] \quad (14)$$

Inserting $\theta = (\lambda - \lambda z)$ and $P_1(0) = \lambda Q_0(1-z_1)$ in (14), we get

$$P(z,0) = \frac{zS_b^*(\lambda-\lambda z)[\eta Q^*(z,0) - \lambda Q_0(1-z_1)]}{z - S_b^*(\lambda-\lambda z)} \quad (15)$$

Substituting (13), (15) and $P_1(0) = \lambda(1-z_1)Q_0$ in (14) and inserting $\theta = 0,$ we get

$$P^*(z,0) = \frac{\begin{bmatrix} Q_0 \lambda z(1-S_b^*(\lambda-\lambda z))[\eta z(z-z_1)(1-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)) \\ -(1-z_1)\lambda-\lambda z+\eta+\beta-\beta/z)(z-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z))] \end{bmatrix}}{D_1(z)D_2(z)} \quad (16)$$

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where $D_1(z) = (\lambda - \lambda z)(\lambda - \lambda z + \eta + \beta - \beta / z)$ (17)

$D_2(z) = (z - S_b^*(\lambda - \lambda z))(z - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z))$ (18)

We define $P_V(z) = Q^*(z,0) + Q_0$

$$P_V(z) = \frac{\begin{bmatrix} Q_0[\lambda z(z - z_1)(1 - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z)) \\ +(\lambda - \lambda z + \eta + \beta - \beta / z)(z - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z))] \end{bmatrix}}{(\lambda - \lambda z + \eta + \beta - \beta / z)(z - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z))} \quad (19)$$

as the probability generating function for the number of customers in the system when the server is on WV period and

$$P_B(z) = P^*(z,0)$$

$$\begin{aligned} P_B(z) = {} & \frac{Q_0 \lambda z(1 - S_b^*(\lambda - \lambda z))}{D_1(z) D_2(z)} \{\eta z(z - z_1)(1 - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z)) \\ & -(1 - z_1)(\lambda - \lambda z + \eta + \beta - \beta / z)(z - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z))\} \end{aligned} \quad (20)$$

where $D_1(z)$ and $D_2(z)$ are given in (17) and (18)as the probability generating function for the number of customers in the system when the server is not in regular service period then

$$P(z) = P_B(z) + P_V(z)$$

$$\begin{aligned} P(z) = {} & \frac{Q_0}{D_1(z) D_2(z)} \{\lambda z(1 - S_b^*(\lambda - \lambda z)) \{\eta z(z - z_1) \\ & \times(1 - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z)) - (1 - z_1) \\ & \times(\lambda - \lambda z + \eta + \beta - \beta / z)(z - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z))\} \\ & +\{\lambda z(z - z_1)(1 - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z)) \\ & +(\lambda - \lambda z + \eta + \beta - \beta / z)(z - S_v^*(\lambda - \lambda z + \eta + \beta - \beta / z))\} \\ & \times(\lambda - \lambda z)(z - S_b^*(\lambda - \lambda z))\} \end{aligned} \quad (21)$$

as the probability generating function for the number of customers in the system where $D_1(z)$ and $D_2(z)$ are given in (17) and (18). We shall now use the normalizing condition $P(1) = 1$ to determine the only unknown Q_0, which appears in (21). Substituting $z = 1$ in (21) and using L'hospital's rule we obtain

$$Q_0 = \frac{1 - \rho_b}{\left[\frac{(\lambda - \lambda z_1 + \eta)}{\eta} - \frac{\rho_b (1 - z_1) S_v^*(\eta)}{1 - S_v^*(\eta)} - \frac{\beta(1 - z_1)\rho_b}{\eta} \right]} \quad (22)$$

where $\rho_b = \lambda E(S_b), E(S_b)$ is the mean service time. From (22) we obtain the system stability condition,

$$\rho_b < 1. \quad (23)$$

4.1 Particular Cases

Case i: If no reneging during the server is on working vacation then on setting $\beta = 0$ in (21), we get $P(z)$ as

$$P(z) = P_B(z) + P_V(z) \quad (24)$$

where

$$P_B(z) = \frac{\begin{bmatrix} \lambda Q_0 \{z(1 - S_b^*(\lambda - \lambda z))[\eta z(z - z_1)(1 - S_v^*(\lambda - \lambda z + \eta)) \\ -(1 - z_1)(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))]\} \end{bmatrix}}{(\lambda - \lambda z)(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))(z - S_b^*(\lambda - \lambda z))}$$

$$P_V(z) = \frac{\begin{bmatrix} Q_0 \{\lambda z(z - z_1)(1 - S_v^*(\lambda - \lambda z + \eta)) \\ +(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))\} \end{bmatrix}}{(\lambda - \lambda z + \eta)(z - S_v^*(\lambda - \lambda z + \eta))}$$

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$$Q_0 = \frac{1-\rho_b}{\left[\frac{(\lambda-\lambda z_1+\eta)}{\eta} - \frac{\rho_b(1-z_1)S_v^*(\eta)}{1-S_v^*(\eta)}\right]}$$

where $\rho_b = \lambda E(S_b)$. Equation (24) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue with multiple working vacation Takagi (2006) irrespective of notations.

Case ii: If the server never takes a vacation then taking the limit as $\eta \to \infty$ in (21), we get

$$P(z) = \frac{S_b^*(\lambda-\lambda z)(1-z)(1-\lambda E(S_b))}{S_b^*(\lambda-\lambda z)-z} \quad (25)$$

Equation (25) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue Medhi (1982) irrespective of notations.

4.2 Performance Measures

Mean System Length

Let L_v and L_b denote the mean system size during the working vacation and regular service period respectively and let W_v and W_b be the mean waiting time of the customer in the system during WV period and regular service period respectively.

$L_v = \frac{d}{dz}P_V(z)$ at $z=1$

$$L_v = \frac{\lambda Q_0[D(1)N_{1'}(1) - D'(1)N_1(1)]}{(D(1))^2}$$

where

$$N_1(1) = (1-z_1)(1-S_v^*(\eta))$$

$$N_{1'}(1) = (1-S_v^*(\eta))(1+(1-z_1)) - (1-z_1)(-\lambda+\beta)S_v^{*'}(\eta)$$

$$D(1) = \eta[1-S_v^*(\eta)]$$

$$D'(1) = (-\lambda+\beta)(1-S_v^*(\eta)) + \eta(1-(-\lambda+\beta)S_v^{*'}(\eta))$$

$L_b = \frac{d}{dz}P_B(z)$ at $z=1$

$$L_b = \frac{\lambda Q_0}{2[D_{1'}(1)]^2[D_{2'}(1)]^2}[D_{1'}(1)D_{2'}(1)[N_{2''}(1)N_{3'}(1) + N_{3''}(1)N_{2'}(1)] - N_{2'}(1)N_{3'}(1)[D_{1''}(1)D_{2'}(1) + D_{1'}(1)D_{2''}(1)]].$$

where

$$\begin{aligned}
N_{2'}(1) &= -\lambda E(S_b) \\
N_{2''}(1) &= -[2\lambda E(S_b) + \lambda^2 E(S_b^2)] \\
N_{3'}(1) &= (1-S_v^*(\eta))[\eta+\lambda(1-z_1)] - \beta(1-z_1)(1-S_v^*(\eta)) - \eta(1-z_1)S_v^*(\eta) \\
N_{3''}(1) &= 2[\eta(1-S_v^*(\eta)) - \eta(-\lambda+\beta)S_v^{*'}(\eta) - \eta(1-z_1)(-\lambda+\beta)S_v^{*'}(\eta) \\
&\quad + \beta(1-z_1)(1-S_v^*(\eta)) - (1-z_1)(-\lambda+\beta)(1-(-\lambda+\beta)S_v^{*'}(\eta))] \\
D_{1'}(1) &= -\lambda\eta \\
D_{1''}(1) &= -2\lambda(-\lambda+\beta) \\
D_{2'}(1) &= (1-\lambda E(S_b))(1-S_v^*(\eta)) \\
D_{2''}(1) &= -\lambda^2 E(S_b^2)(1-S_v^*(\eta)) + 2(1-\lambda E(S_b))(1-(-\lambda+\beta)S_v^{*'}(\eta))
\end{aligned}$$

where $E(S_b)$ is the mean service time. $E(S_b^2)$ is the second moment of the service time.

4.3 Numerical Result

Fixing the values of $z_1 = 0.3, \beta = 1, \mu_v = 3, \mu_b = 6$ and ranging the values of λ from 1.7 to 2.1 insteps of 0.1 and varying the values of η from 1.80 to 2.00 insteps of 0.05, we calculated the corresponding values of L_b and W_b for multiple working vacation and tabulated in Table 1 and in Table 2 respectively. The corresponding graphs have been drawn for λ versus L_b and λ versus W_b and are shown in Figure 1 and in Figure 2 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for different values of η.

Table 1: λ versus L_b in regular service period

λ	$\eta=1.80$	$\eta=1.85$	$\eta=1.90$	$\eta=1.95$	$\eta=2.00$
1.7	0.364035	0.357720	0.352040	0.346930	0.342327
1.8	0.385628	0.379851	0.374669	0.370019	0.365844
1.9	0.409068	0.403853	0.399190	0.395020	0.391290
2.0	0.434641	0.430002	0.425871	0.422193	0.418919
2.1	0.462637	0.458582	0.454989	0.451807	0.448994

Table 2: λ versus W_b in regular service period

	$\eta=1.80$	$\eta=1.85$	$\eta=1.90$	$\eta=1.95$	$\eta=2.00$
1.7	0.214138	0.210423	0.207083	0.204076	0.201369
1.8	0.214238	0.211029	0.208150	0.205566	0.203247
1.9	0.215299	0.212554	0.210100	0.207905	0.205942
2.0	0.217320	0.215001	0.212936	0.211096	0.209459
2.1	0.220303	0.218372	0.216661	0.215146	0.213807

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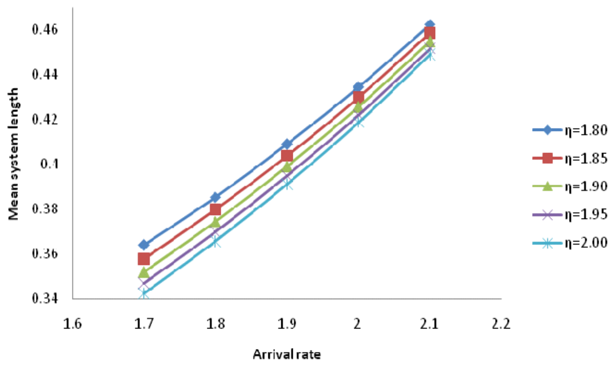

Figure 1: λ versus L_b in regular service period

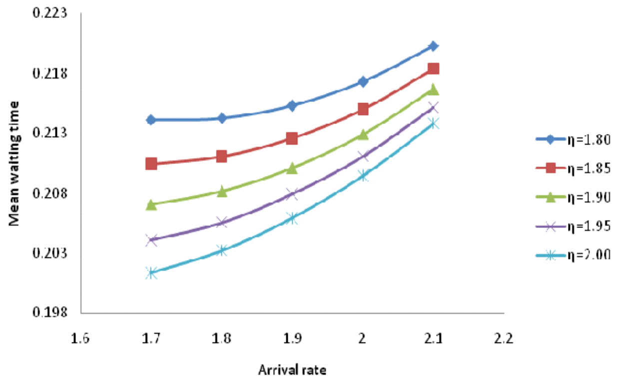

Figure 2: λ versus W_b in regular service period

Again fixing the values of $z_1 = 0.4, \beta = 1, \mu_v = 2, \mu_b = 5$ and ranging the values of λ from 2.2 to 2.6 insteps of 0.1 and varying the values of η from 1.20 to 1.28 insteps of 0.02, we calculated the corresponding values of L_v and W_v for multiple working vacation and tabulated in Table 3 and in Table 4 respectively. The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 3 and in Figure 4 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for different values of η.

Table 3: λ versus L_v in WV period

λ	$\eta = 1.20$	$\eta = 1.22$	$\eta = 1.24$	$\eta = 1.26$	$\eta = 1.28$
2.2	0.427778	0.427505	0.427091	0.426547	0.425885
2.3	0.460788	0.459608	0.458327	0.456954	0.455496
2.4	0.491892	0.489832	0.487708	0.485528	0.483297
2.5	0.520833	0.517925	0.514989	0.512033	0.509059
2.6	0.547368	0.543651	0.539942	0.536245	0.532563

Table 4: λ versus W_v in WV period

λ	$\eta = 1.20$	$\eta = 1.22$	$\eta = 1.24$	$\eta = 1.26$	$\eta = 1.28$
2.2	0.194444	0.194320	0.194132	0.193885	0.193584
2.3	0.200342	0.199830	0.199273	0.198675	0.198042
2.4	0.204955	0.204097	0.203212	0.202303	0.201374
2.5	0.208333	0.207170	0.205996	0.204813	0.203624
2.6	0.210526	0.209096	0.207670	0.206248	0.204832

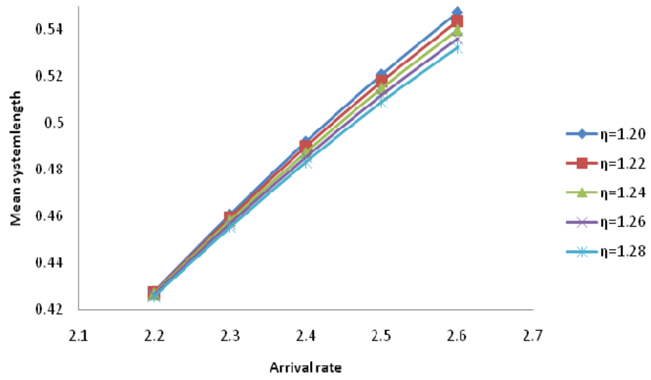

Figure 3: λ versus L_v in WV period

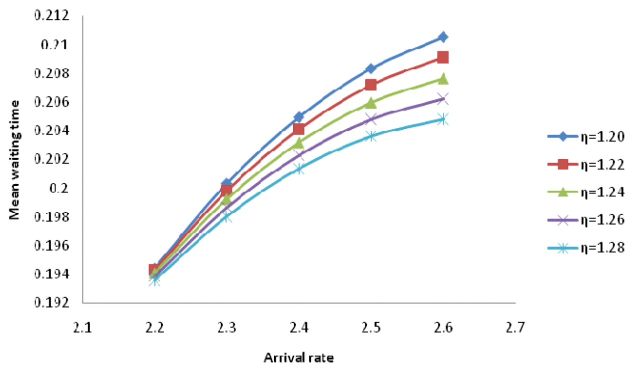

Figure 4: λ versus W_v in WV period

5. THE SYSTEM SIZE DISTRIBUTION FOR SINGLE WORKING VACATION

By considering the steady state, we have the following system of the differential difference equations.

$$(\lambda + \eta)Q_0 = P_1(0) + Q_1(0) \tag{26}$$

$$-\frac{d}{dx}Q_1(x) = -(\lambda + \eta)Q_1(x) + Q_2(0)s_v(x) + \lambda Q_0 s_v(x) + \beta Q_2(x) \tag{27}$$

$$-\frac{d}{dx}Q_n(x) = -(\lambda+\eta+\beta)Q_n(x)+Q_{n+1}(0)s_v(x)+\lambda Q_{n-1}(x)+\beta Q_{n+1}(x);\, n>1 \tag{28}$$

$$\lambda P_0 = \eta Q_0 \tag{29}$$

$$-\frac{d}{dx}P_1(x) = -\lambda P_1(x)+P_2(0)s_b(x)+\eta s_b(x)\int_0^\infty Q_1(y)dy+\lambda P_0 s_b(x) \tag{30}$$

$$-\frac{d}{dx}P_n(x) = -\lambda P_n(x)+P_{n+1}(0)s_b(x)+\lambda p_{n-1}(x)+\eta s_b(x)\int_0^\infty Q_n(y)dy;\quad n>1 \tag{31}$$

Taking the LST on both sides of the equations (27), (28), (30) and (31) we get

$$\theta Q_1^*(\theta)-Q_1(0) = (\lambda+\eta)Q_1^*(\theta)-Q_2(0)S_v^*(\theta)-\lambda Q_0 S_v^*(\theta)-\beta Q_2^*(\theta) \tag{32}$$

$$\theta Q_n^*(\theta)-Q_n(0) = (\lambda+\eta+\beta)Q_n^*(\theta)-Q_{n+1}(0)S_v^*(\theta)-\lambda Q_{n-1}^*(\theta)-\beta Q_{n+1}^*(\theta);\ n>1 \tag{33}$$

$$\theta P_1^*(\theta)-P_1(0) = \lambda P_1^*(\theta)-P_2(0)S_b^*(\theta)-\eta S_b^*(\theta)Q_1^*(0)-\lambda P_0 S_b^*(\theta) \tag{34}$$

$$\theta P_n^*(\theta)-P_n(0) = \lambda P_n^*(\theta)-P_{n+1}(0)S_b^*(\theta)-\lambda P_{n-1}^*(\theta)-\eta S_b^*(\theta)Q_n^*(0);\ n>1 \tag{35}$$

z^n times (33) taken summed with $n=2$ to ∞ and adding the resultant with z times (32), we get

$$[\theta-(\lambda-\lambda z+\eta+\beta-\frac{\beta}{z})]Q^*(z,\theta) = Q(z,0)[\frac{z-S_v^*(\theta)}{z}]-S_v^*(\theta)[\lambda z Q_0-Q_1(0)] \tag{36}$$

Inserting $\theta=(\lambda-\lambda z+\eta+\beta-\frac{\beta}{z})$ in (36) we get

$$Q(z,0)=\frac{zS_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)[\lambda z Q_0-Q_1(0)]}{z-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)} \tag{37}$$

Substituting (37) and putting $\theta=0$ in (36), we get

$$Q^*(z,0)=\frac{z(1-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z))(\lambda z Q_0-Q_1(0))}{(\lambda-\lambda z+\eta+\beta-\beta/z)(z-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z))} \tag{38}$$

Let $g(z)=(\lambda-\lambda z+\eta+\beta-\beta/z)(z-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z))$

$$=\frac{1}{z}(\lambda z-\lambda z^2+\eta z+\beta z-\beta)(z-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z))$$

Take $f(z)=\lambda z-\lambda z^2+\eta z+\beta z-\beta$. We find that $f(0)<0$ and $f(1)>0$. This implies that there exist a real root $z_1\in(0,1)$ for the equation $f(z)=0$. Hence at $z=z_1$ the equation (38) becomes $Q_1(0)=\lambda z_1 Q_0$. Substituting this in (38) we get

$$Q^*(z,0)=\frac{\lambda z(z-z_1)[1-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z)]Q_0}{(\lambda-\lambda z+\eta+\beta-\beta/z)(z-S_v^*(\lambda-\lambda z+\eta+\beta-\beta/z))} \tag{39}$$

z^n times (35) summed over n from 2 to ∞ and added up with z times (34) gives

$$\theta\sum_{n=1}^{\infty}P_n^*(\theta)z^n-\sum_{n=1}^{\infty}P_n(0)z^n=\lambda\sum_{n=1}^{\infty}P_n^*(\theta)z^n \quad -S_b^*(\theta)\sum_{n=1}^{\infty}P_{n+1}(0)z^n-\eta S_b^*(\theta)\sum_{n=1}^{\infty}Q_n^*(0)z^n$$
$$-\lambda zP_0S_b^*(\theta)-\lambda\sum_{n=2}^{\infty}P_{n-1}^*(\theta)z^n$$

$$[\theta-(\lambda-\lambda z)]P^*(z,\theta)=\left(\frac{z-S_b^*(\theta)}{z}\right)P(z,0)-S_b^*(\theta)[\eta Q^*(z,0)+\lambda zP_0-P_1(0)] \tag{40}$$

Substituting $Q_1(0)=\lambda z_1Q_0$ in (26), we get $P_1(0)=(\lambda-\lambda z_1+\eta)Q_0$. Inserting $\theta=(\lambda-\lambda z)$ and $P_1(0)=(\lambda-\lambda z_1+\eta)Q_0$ in (40) and also using (29), we get

$$P(z,0)=\frac{zS_b^*(\lambda-\lambda z)[\eta Q^*(z,0)-(\lambda(1-z_1)+\eta(1-z))Q_0]}{z-S_b^*(\lambda-\lambda z)} \tag{41}$$

Substituting (39), (41) and $P_1(0)=(\lambda-\lambda z_1+\eta)Q_0$ in (40) and inserting $\theta=0$ and also using (29) and taking $\lambda(z)=\lambda-\lambda z+\eta$ we get

$$P^*(z,0)=\frac{\left[\begin{array}{l}Q_0z(1-S_b^*(\lambda-\lambda z))\{\eta\lambda z(z-z_1)(1-S_v^*(\lambda(z)+\beta-\beta/z))-(\lambda(1-z_1)\\+\eta(1-z))(\lambda(z)+\beta-\beta/z)(z-S_v^*(\lambda(z)+\beta-\beta/z))\}\end{array}\right]}{D_1(z)D_2(z)} \tag{42}$$

where

$$D_1(z)=(\lambda-\lambda z)(\lambda(z)+\beta-\beta/z) \tag{43}$$

$$D_2(z)=(z-S_b^*(\lambda-\lambda z))(z-S_v^*(\lambda(z)+\beta-\beta/z)) \tag{44}$$

We define $P_V(z)=Q^*(z,0)+Q_0$, then it becomes

$$P_V(z)=\frac{\left[Q_0\{\lambda z(z-z_1)(1-S_v^*(\lambda(z)+\beta-\beta/z))+(\lambda(z)+\beta-\beta/z)(z-S_v^*(\lambda(z)+\beta-\beta/z))\}\right]}{(\lambda(z)+\beta-\beta/z)(z-S_v^*(\lambda(z)+\beta-\beta/z))} \tag{45}$$

as the probability generating function for the number of customers in the system when the server is on WV period and also we define $P_B(z)=P^*(z,0)+P_0$, then it becomes,

$$\begin{aligned}P_B(z)=\;&\frac{Q_0}{\lambda D_1(z)D_2(z)}\{\lambda z(1-S_b^*(\lambda-\lambda z))\{\eta\lambda z(z-z_1)(1-S_v^*(\lambda(z)+\beta-\beta/z))\\&-(\lambda(1-z_1)+\eta(1-z))(\lambda(z)+\beta-\beta/z)(z-S_v^*(\lambda(z)+\beta-\beta/z))\}\\&+\eta\{(\lambda-\lambda z)(\lambda(z)+\beta-\beta/z)\}\{(z-S_b^*(\lambda-\lambda z))(z-S_v^*(\lambda(z)+\beta-\beta/z))\}\}\end{aligned} \tag{46}$$

as the probability generating function for the number of customers in the system when the server is in regular service period, where $D_1(z)$ and $D_2(z)$ are given in (43) and (44) respectively.

Again we define

$P(z)=P_B(z)+P_V(z)$,then it becomes

$$\begin{aligned}P(z) = \ & \frac{Q_0}{\lambda D_1(z) D_2(z)}\{\lambda z(1-S_b^*(\lambda-\lambda z))\{\eta\lambda z(z-z_1)(1-S_v^*(\lambda(z)+\beta-\beta/z)) \\ & -(\lambda(1-z_1)+\eta(1-z))(\lambda(z)+\beta-\beta/z)(z-S_v^*(\lambda(z)+\beta-\beta/z))\} \\ & +\eta\{(\lambda-\lambda z)(\lambda(z)+\beta-\beta/z)\}\{(z-S_b^*(\lambda-\lambda z))(z-S_v^*(\lambda(z)+\beta-\beta/z))\} \\ & +\{\lambda z(z-z_1)(1-S_v^*(\lambda(z)+\beta-\beta/z))+(\lambda(z)+\beta-\beta/z) \\ & \times(z-S_v^*(\lambda(z)+\beta-\beta/z))\}\{\lambda(\lambda-\lambda z)(z-S_b^*(\lambda-\lambda z))\}\}\end{aligned} \tag{47}$$

as the probability generating function for the number of customers in the system,where $D_1(z)$ and $D_2(z)$ are given in (43) and (44) respectively. Now we shall use the normalizing condition $P(1)=1$ to determine the only unknown Q_0, which appears in (47). Substituting $z=1$ in (47) and using L'hospital's rule we obtain

$$Q_0 = \frac{1-\rho_b}{\left[\frac{(\lambda-\lambda z_1+\eta)}{\eta}+\frac{\eta}{\lambda}-\frac{\rho_b(1-z_1)S_v^*(\eta)}{1-S_v^*(\eta)}-\frac{\beta(1-z_1)\rho_b}{\eta}\right]} \tag{48}$$

where $\rho_b = \lambda E(S_b), E(S_b)$ is the mean service time. From (48) we obtain the system stability condition,

$$\rho_b < 1 \tag{49}$$

5.1 Particular Cases

Case i: If no reneging during the server is on working vacation then on setting $\beta = 0$ in (47), we get $P(z)$ as

$$P(z) = P_B(z) + P_V(z) \tag{50}$$

where

$$P_B(z) = \frac{\left[\begin{aligned}&Q_0\{\lambda z(1-S_b^*(\lambda-\lambda z))[\eta\lambda z(z-z_1)(1-S_v^*(\lambda(z)))-(\eta(1-z)+\lambda(1-z_1)) \\ &\times(\lambda(z))(z-S_v^*(\lambda(z)))]+\eta(\lambda-\lambda z)(z-S_v^*(\lambda(z)))(\lambda(z))(z-S_b^*(\lambda-\lambda z))\}\end{aligned}\right]}{\lambda(\lambda-\lambda z)(\lambda(z))(z-S_v^*(\lambda(z)))(z-S_b^*(\lambda-\lambda z))}$$

$$P_V(z) = \frac{Q_0[\lambda z(z-z_1)(1-S_v^*(\lambda(z)))+(\lambda(z))(z-S_v^*(\lambda(z)))]}{(\lambda(z))(z-S_v^*(\lambda(z)))}$$

$$Q_0 = \frac{1-\rho_b}{\left[\frac{(\lambda-\lambda z_1+\eta)}{\eta}+\frac{\eta}{\lambda}-\frac{\lambda(1-z_1)S_v^*(\eta)E(S_b)}{1-S_v^*(\eta)}\right]}$$

where $\rho_b = \lambda E(S_b)$. Equation (50) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue with single working vacation Julia Rose Mary (2010) irrespective of the notations.

Case ii: If the server never takes a vacation then taking the limit as $\eta \to \infty$ in (47),

we get

$$P(z) = \frac{S_b^*(\lambda-\lambda z)(1-z)(1-\lambda E(S_b))}{S_b^*(\lambda-\lambda z)-z} \tag{51}$$

Equation (51) is well known probability generating function of the steady state system length distribution of an $M/G/1$ queue Medhi (1982) irrespective of notations.

5.2 Performance Measures

Mean System Length

Let L_v and L_b denote the mean system size during the working vacation and regular service period respectively and let W_v and W_b be the mean waiting time of the customer in the system during WV period and regular service period respectively.

$$L_v = \frac{d}{dz}P_V(z)|_{z=1} = \frac{\lambda Q_0[D(1)N_{1'}(1) - D'(1)N_1(1)]}{(D(1))^2}$$

where $N_1(1) = (1-z_1)(1-S_v^*(\eta))$;

$$N_{1'}(1) = (1-S_v^*(\eta))(1+(1-z_1)) - (1-z_1)(-\lambda+\beta)S_v^{*'}(\eta)$$

$$D(1) = \eta[1-S_v^*(\eta)]$$

$$D'(1) = (-\lambda+\beta)(1-S_v^*(\eta)) + \eta(1-(-\lambda+\beta)S_v^{*'}(\eta))$$

$$L_b = \frac{d}{dz}P_B(z)|_{z=1} = \frac{d}{dz}[P^*(z,0)]|_{z=1}$$

$$L_b = \frac{Q_0}{2[D_{1'}(1)]^2[D_{2'}(1)]^2}[D_{1'}(1)D_{2'}(1)[N_{2''}(1)N_{3'}(1) + N_{3''}(1)N_{2'}(1)] - N_{2'}(1)N_{3'}(1)[D_{1''}(1)D_{2'}(1) + D_{1'}(1)D_{2''}(1)]]$$

Applying Little's formula we get $W_b = \frac{L_b}{\lambda}$

where $N_{2'}(1) = -\lambda E(S_b)$; $N_{2''}(1) = -[2\lambda E(S_b) + \lambda^2 E(S_b^2)]$

$$N_{3'}(1) = (1-S_v^*(\eta))[\eta^2 + \eta\lambda - \lambda(1-z_1)(-\lambda+\beta)] - \eta\lambda(1-z_1)S_v^*(\eta)$$

$$N_{3''}(1) = 2\eta\lambda[(1-S_v^*(\eta)) - (1-z_1)(-\lambda+\beta)S_v^{*'}(\eta) - (-\lambda+\beta)S_v^{*'}(\eta)] + 2(1-S_v^*(\eta))[\eta(-\lambda+\beta) + \lambda\beta(1-z_1)] + 2(1-(-\lambda+\beta)S_v^{*'}(\eta)) \times [\eta^2 - \lambda(1-z_1)(-\lambda+\beta)]$$

$$D_{1'}(1) = -\lambda\eta; \quad D_{1''}(1) = -2\lambda(-\lambda+\beta); \quad D_{2'}(1) = (1-\lambda E(S_b))(1-S_v^*(\eta))$$

$$D_{2''}(1) = -\lambda^2 E(S_b^2)(1-S_v^*(\eta)) + 2(1-\lambda E(S_b))(1-(-\lambda+\beta)S_v^{*'}(\eta))$$

where $E(S_b)$ is the mean service time. $E(S_b^2)$ is the second moment of the service time.

5.3 Numerical Result

Fixing the values of $z_1 = 0.5, \beta = 2, \mu_v = 5, \mu_b = 9$ and ranging the values of λ from 0.1 to 0.5 insteps of 0.1 and varying the values of η from 2.0 to 2.4 insteps of 0.1, we calculated the corresponding values of L_v and W_v for single working vacation and tabulated in Table 5 and in Table 6 respectively. The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 5 and in Figure 6 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for different values of η.

Table 5: λ versus L_b in regular service period

λ	$\eta = 2.0$	$\eta = 2.1$	$\eta = 2.2$	$\eta = 2.3$	$\eta = 2.4$
0.1	0.012763	0.012490	0.012270	0.012091	0.011944
0.2	0.029087	0.028032	0.027175	0.026473	0.025893
0.3	0.049313	0.047016	0.045137	0.043588	0.042302
0.4	0.073728	0.069775	0.066520	0.063821	0.061565

0.5	0.102578	0.096593	0.091636	0.087501	0.084026

Table 6: λ versus W_b in regular service period

λ	$\eta = 2.0$	$\eta = 2.1$	$\eta = 2.2$	$\eta = 2.3$	$\eta = 2.4$
0.1	0.127630	0.124901	0.122700	0.120910	0.119444
0.2	0.145437	0.140161	0.135873	0.132363	0.129467
0.3	0.164377	0.156721	0.150457	0.145294	0.141007
0.4	0.184321	0.174437	0.166301	0.159551	0.153913
0.5	0.205157	0.193185	0.183273	0.175002	0.168052

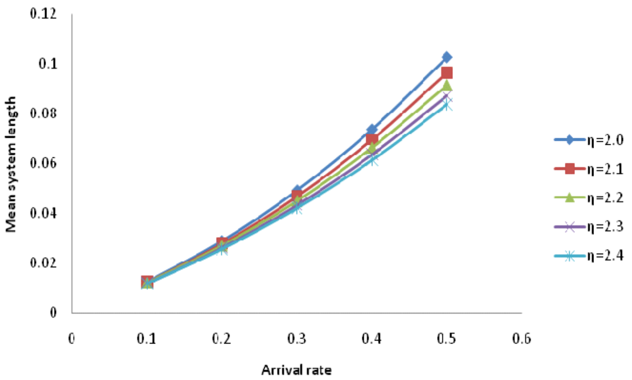

Figure 5: λ versus L_b in regular service period

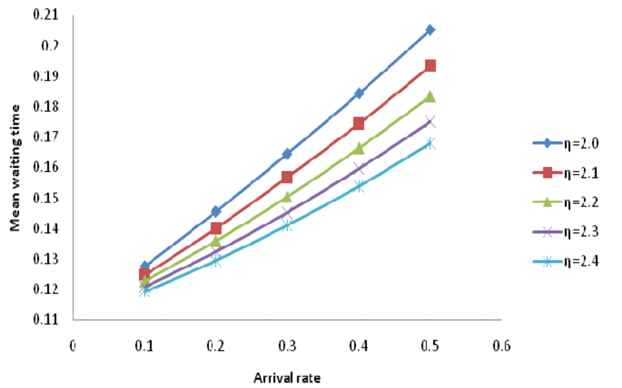

Figure 6: λ versus W_b in regular service period

Again fixing the values of $z_1 = 0.2, \beta = 1, \mu_v = 4, \mu_b = 6$ and ranging the values of λ from 3.1 to 3.5 insteps of 0.1 and varying the values of η from 3.1to 3.9 insteps of 0.2, we calculated the corresponding values of L_v and W_v for single working vacation and tabulated in Table 7 and in Table 8 respectively. The corresponding graphs have been drawn for λ versus L_v and λ versus W_v and are shown in Figure 7 and in Figure 8 respectively. From the graphs it is seen that as λ increases both L_b and W_b increases for different values of η.

Table 7: Arrival rate (λ) versus mean system length (L_v) in WV period

λ	$\eta = 3.1$	$\eta = 3.3$	$\eta = 3.5$	$\eta = 3.7$	$\eta = 3.9$
3.1	0.115474	0.111841	0.107773	0.103500	0.099174
3.2	0.122463	0.118054	0.113341	0.108527	0.103741
3.3	0.129185	0.124006	0.118656	0.113309	0.108073
3.4	0.135582	0.129647	0.123674	0.117807	0.112132
3.5	0.141597	0.134925	0.128348	0.121979	0.115883

Table 8: (λ) versus (W_v) in WV period

λ	$\eta = 3.1$	$\eta = 3.3$	$\eta = 3.5$	$\eta = 3.7$	$\eta = 3.9$
3.1	0.037250	0.036078	0.034765	0.033387	0.031992
3.2	0.038270	0.036892	0.035419	0.033915	0.032419
3.3	0.039147	0.037577	0.035956	0.034336	0.034336
3.4	0.039877	0.038131	0.036375	0.034649	0.032980
3.5	0.040456	0.038550	0.036671	0.034851	0.033109

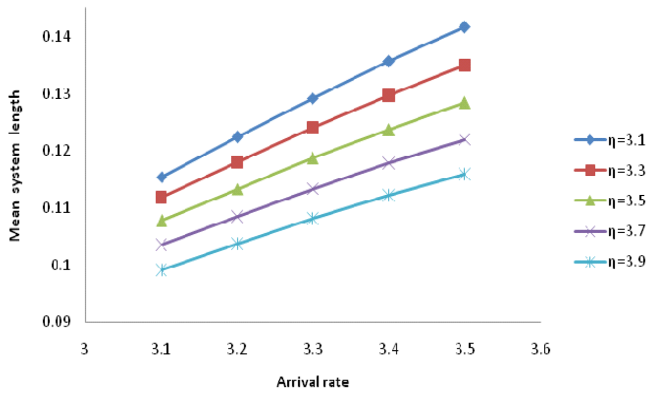

Figure 7: (λ) versus (L_v) in WV period

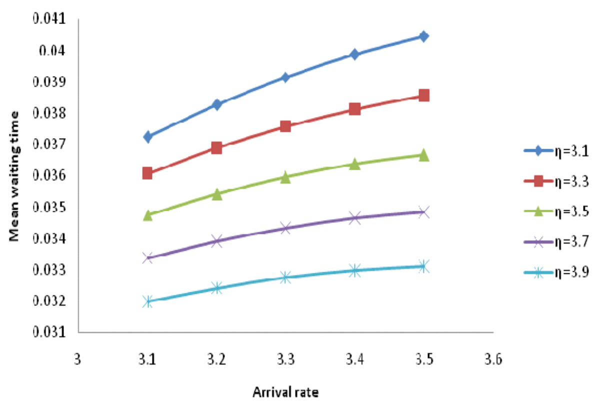

Figure 8: (λ) versus (W_v) in WV period

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