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Group S_3 Cordial Prime Labeling of Some Graphs

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Abstract

Let G = (V (G), E(G)) be a graph. Consider the group S_3. For u $\in S_3$, let o(u) denote the order of u in S_3. Let G : V (G) → S_3 be a function defined in such a way that if xy $\in$ E(G) (o(g(x)), o(g(y)) = 1. Let n_j (g) denote the number of vertices of G having label j under g. Now g is called a group S_3 cordial prime labeling if $|n_i(g)-n_j(g)| \leq 1$ for every i, j$\in S_3$, i ≠ j. A graph which admits a group S_3 cordial prime labeling is called a group S_3 cordial prime graph. In this paper, we prove that the complete graph, Dumbbell graph, friendship graph and the web graph are group S_3 cordial prime.

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Keywords: Cordial labeling, prime labeling, group S_3 cordial prime labeling.

1 INTRODUCTION

Graphs considered here are finite, undirected and simple. Let A be a group. The order of a $\in$ A is the least positive integer n such that $a^n = e$. We denote the order of a by o(a).

Cahit [1] introduced the concept of cordial labeling.

Definition 1.1. Let f : V (G) → {0, 1} be any function. For each edge xy assign the label |f (x) − f (y)|. f is called a cordial labeling if the number of vertices labeled 0 and the number of vertices labeled 1 differ by at most 1. Also the number of edges labeled 0 and the number of edges labeled 1 differ by at most 1.

Entringer introduced the concept of prime labeling which was later studied by Tout et al.[4]

Definition 1.2. A prime labeling of a graph G of order n is an injective function f : V → {1, 2, ...n} such that for every pair of adjacent vertices u and v gcd{f (u), f (v)} = 1.

Motivated by these two definitions, we introduce group S_3 cordial prime labeling of graphs. Terms not defined here are used in the sense of Harary [3] and Gallian [2].

The greatest common divisor of two integers m and n is denoted by (m, n) and m and n are said to be relatively prime if (m, n) = 1. For any real number x, we denote by $\lfloor x \rfloor$, the greatest integer smaller than or equal to x and by $\lceil x \rceil$, we mean the smallest integer greater than or equal to x.

A graph G is complete, if every two of its vertices are adjacent. A complete graph on n vertices is denoted by K_n. A path is an alternating sequence of vertices and edges, $v_1, e_1, v_2, e_2, ..., v_{n-1}, e_{n-1}, v_n$ which are distinct, such that e_i is an edge joining v_i and v_{i+1} for $1 \leq I \leq n-1$.

A path on n vertices is denoted by P_n. A path $v_1, e_1, v_2, e_2, ..., e_{n-1}, v_n, e_n, v_1$ is called a cycle and a cycle on n vertices is denoted by C_n.

The graph obtained by joining two disjoint cycles $u_1u_2 \ldots u_nu_1$ and $v_1v_2 \ldots v_nv_1$ with an edge u_1v_1 is called dumbbell graph denoted by Db_n. The one-point union of t cycles of length n, $C_n^{(t)}$ is called a Friendship graph.

Given two graphs G and H , G + H is the graph with vertex set V (G) ∪ V (H) and edge set E(G) ∪ E(H) ∪ {uv|u ∈ V (G), v ∈ V (H)}. A wheel W_n is defined as $C_n + K_1$. The Helm H_n is obtained from a wheel W_n by attaching a pendent edge at each vertex of the cycle C_n. The closed Helm C H_n is a graph obtained from a Helm H_n by joining each pendent vertex to form a cycle. The Web graph W (2, n) is the graph obtained from a closed Helm CH_n by adding a single pendent edge to each vertex of the outer cycle.

2 GROUP S_3 CORDIAL PRIME GRAPHS

Definition 2.1. Let g : V (G) → S_3 be a function defined in such a way that if xy ∈E(G), then (o(g(x)), o(g(y)) = 1.Let n_j (g) denote the number of vertices of G having label j under g. Now g is called a group S_3 cordial prime labeling if $|n_i(g) - n_j(g)| \leq 1$ for every i, j ∈ S_3, i 6= j. A graph which admits a group S_3 cordial prime labeling is called a group S_3 cordial prime graph.

Definition 2.2. Consider the symmetric group S_3.

Let the elements of S_3 be {e, a, b, c, d, f } where

$$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} ; \ a= \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} ; \ b = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} ;$$

$$c = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} ; \quad d = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} ; f = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$

Now o(e) = 1;

o(a) = o(b) = o(c) = 2;

o(d) = o(f) = 3.

Example 2.3. A group S_3 cordial prime labeling of two graphs is given in Fig. 1 and Fig. 2.

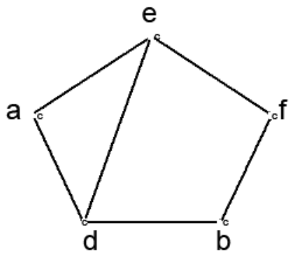

Fig. 1

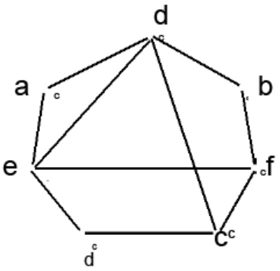

Fig. 2

Theorem 2.4. Complete graph K_n is group S_3 cordial prime i $n \leq 3$.

Proof. Fig. 3 and Fig. 4 shows that K_n is group S_3 cordial prime if $n \leq 3$.

Suppose $n > 3$. Let $v_1, v_2, \ldots, v_n$ be the n vertices. Since every vertex is adjacent to every other vertex, we can have at most one label from {a, b, c}, one from {d, f } and one from {e}. Thus we can have at most 3 distinct labels.

So K_n is not group S_3 cordial prime if $n > 3$.

Fig. 3

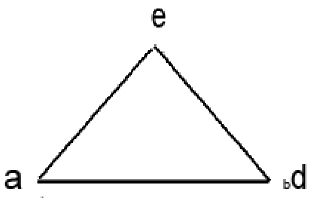

Fig. 4

Definition 2.5. The graph obtained by joining two disjoint cycles $u_1u_2 \ldots u_nu_1$ and $v_1v_2 \ldots v_nv_1$ with an edge u_1v_1 is called dumbbell graph Db_n.

Theorem 2.6. The dumbbell graph Db_n is group S_3 cordial prime $\Leftrightarrow$ n is not of the form $n = 6k+3, k \geq 0$.

Proof. Assume that Db_n is group S_3 cordial prime. Number of vertices in Db_n is 2n. Suppose $n \equiv 3 \pmod 6$. Let $n = 6k + 3 (k \geq 0, k \in Z_+)$.

Case (i): k = 0.
Now n = 3. One of $\{u_1, u_2, u_3\}$ should be labelled with a label from the set {a, b, c} and another from the set {d, f }. The third vertex should be labelled with e. Same is the case with the vertices v_1, v_2, v_3 by symmetry. But now, as e is already used once, it cannot be used again. So $n \neq 3$.

Case (ii): $k \geq 1$.

Now $n \geq 9$.

Each label should be used

$$\frac{2n}{6} = \frac{2(6k+3)}{6} = 2k+1$$

times

Label u_1 and v_1 by e. Label $\{u_2, u_3, u_4, u_5, u_6\}$ and $\{v_2, v_3, v_4, v_5, v_6\}$ in order using the labels adbf c correspondingly. Label the next six consecutive vertices in both cycles using eadbf c correspondingly. Continue this style of labelling until vertices upto u_{n-3} and v_{n-3} are labelled. As the label of u_{n-3} is c, the vertices u_{n-2}, u_{n-1} and u_n should be labelled using 2 labels from {d, f, e} and one from {a, b, c}. But now, as same is the case with v_{n-2}, v_{n-1}, v_n, it is not possible to label the vertices satisfying the group S_3 cordial prime condition. By symmetry, this is the case with any labeling. So n is not of the form $n = 6k+3, k \geq 0$.

Conversely, assume that n is not of the form $n = 6k+3, k \geq 0$.

Case (i): $n \equiv 0 \pmod 6$.

Let $n = 6, k \geq 1, k \in Z$.

Each vertex label should appear $\frac{2n}{6} = \frac{2(6k)}{6} = 2k$ times.

Label the vertices $u_i, v_i (1 \leq i \leq n)$ using the labels in the pattern eadbf c starting from u_1 as well as v_1. Clearly this is a group S_3 cordial prime labeling.

Case (ii): $n \equiv 1 \pmod 6$

Let $n = 6k + 1, k \geq 1, k \in Z$.

Total number of vertices= $2n = 2(6k + 1) = 12k + 2$.

So two labels appear 2k + 1 times and 4 others appear 2k times.

$g : V(Db_n) \to S_3$ is defined as follows:

$g(v_1) = g(v_7) = \ldots = g(v_{n-6}) = e$

$g(v_2) = g(v_8) = \ldots = g(v_{n-5}) = a$

$g(v_3) = g(v_9) = \ldots = g(v_{n-4}) = d$

$g(v_4) = g(v_{10}) = \ldots = g(v_{n-3}) = b$

$g(v_5) = g(v_{11}) = \ldots = g(v_{n-2}) = f$

$g(v_6) = g(v_{12}) = \ldots = g(v_{n-1}) = c$

$g(v_n) = f$

Define g on $\{u_1, u_2, \ldots, u_n\}$ in the same way except that $g(u_n) = d$.

Now $n_d(g) = 2k + 1$, $n_f(g) = 2k + 1$ and $n_a(g) = n_b(g) = n_c(g) = n_e(g) = 2k$.

Case (iii): $n \equiv 2 \pmod 6$.

Let $n = 6k + 2$, $k \geq 1$, $k \in Z_+$.

Now, number of vertices= $2(6k + 2) = 12k + 4$.

So, 4 labels appear 2k + 1 times and 2 labels appear 2k times.

Define $g : V(Db_n) \to S_3$ as follows:

$g(v_1) = g(v_7) = \ldots = g(v_{n-7}) = e$

$g(v_2) = g(v_8) = \ldots = g(v_{n-6}) = a$

$g(v_3) = g(v_9) = \ldots = g(v_{n-5}) = d$

$g(v_4) = g(v_{10}) = \ldots = g(v_{n-4}) = b$

$g(v_5) = g(v_{11}) = \ldots = g(v_{n-3}) = f$

$g(v_6) = g(v_{12}) = \ldots = g(v_{n-2}) = c$

$g(v_{n-1}) = f$, $g(v_n) = a$

Define g on $\{u_1, u_2, \ldots, u_n\}$ in the same way except that $g(u_{n-1}) = d$, $g(u_n) = b$. Now $n_a(g) = n_b(g) = n_d(g) = n_f(g) = 2k + 1$ and $n_c(g) = n_e(g) = 2k$.

Case (iv): $n \equiv 4 \pmod 6$

Let $n = 6k + 4$, $k \geq 1$, $k \in Z_+$.

Number of vertices= $2(6k + 4) = 12k + 8$.

Now 4 labels appear 2k + 1 times and 2 labels appear 2k + 2 times.

The labeling is same as in case (ii).

Case (v): $n \equiv 5 \pmod 6$.

Let $n = 6k + 5$, $k \geq 0$, $k \in Z_+$.

Number of vertices= $2(6k + 5) = 12k + 10$.

Here, 2 labels appear 2k + 1 times and 4 labels appear 2k + 2 times.

The labelling is same as in case (iii).

Thus, in all cases, we observe that Db_n is group S_3 cordial prime.

Definition 2.7. Let $C_n^{(t)}$ denote the one-point union of t cycles of length n.

The graph $C_3^{(t)}$ is called a Friendship graph.

Theorem 2.8. Friendship graphs $C_3^{(t)}$ are group S_3 cordial prime.

Proof. Let $C_3^{(t)}$ be the friendship graph. Let the common vertex be w and the remaining two vertices of the n cycles be denoted as $u_i, v_i (i = 1, 2, \ldots, n)$.

$$\text{Then } E(C_3^{(t)}) = \begin{cases} u_i v_i, & i = 1 \text{ to } n \\ u_i w, & i = 1 \text{ to } n \\ v_i w, & i = 1 \text{ to } n \end{cases}$$

Then w should be labelled as e. If we label w as a (or b or c), then none of the vertices of u_i, v_i can be labelled as a or b or c. Similarly if we label w as d or f , then none of the vertices of u_i, v_i can be labelled with d or f .

Hence w should be labelled as e.

Define $g : V(C_3^{(t)}) \to S_3$ as follows :

Clearly g is a group S_3 cordial prime labeling.

$g(w) = e$;

$$g(u_i) = \begin{cases} a, & i \equiv 1 (mod\ 3) \\ b, & i \equiv 2 (mod\ 3) \\ c, & i \equiv 0 (mod\ 3) \end{cases}$$

$$g(v_i) = \begin{cases} d, & i \equiv 1 (mod\ 3) \\ f, & i \equiv 2 (mod\ 3) \\ e, & i \equiv 0 (mod\ 3) \end{cases}$$

Clearly, g is a group S_3 cordial prime labelling.

Illustration of the labeling for the Friendship graph $C_3^{(2)}$ is given in Fig.

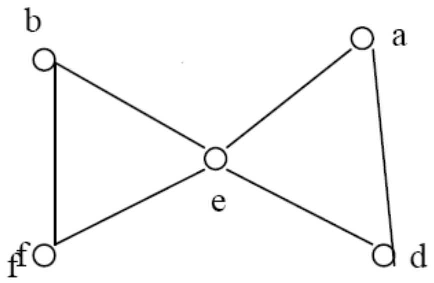

Fig. 5

Definition 2.9. The Web graph W (2, n) is the graph obtained from a closed Helm CH_n by adding a single pendent edge to each vertex of the outer cycle. **Theorem 2.10.** Web graphs W (2, n) are group S_3 cordial prime if and only if $n \neq 3$.

Proof : Let W (2, n) be the web graph. Let w be the center vertex. Let $u_1, u_2, \ldots, u_n$ be the vertices of the cycle C_n, and $v_1, v_2, \ldots, v_n$ be the vertices of the outer cycle and $w_1, w_2, \ldots, w_n$ be the pendent vertices .

Suppose n = 3. The graph has 10 vertices and contains two triangles.

We need at least 3 vertices to be labelled with the label e in order to have a group S_3 cordial prime labeling. But at most 2 e′s are available for use. So $n \neq 3$.

Conversely assume that $n \neq 3$.

Case 1: $n \equiv 0 \pmod 6$.

Let $n = 6k$, $k \geq 1$, $k \in Z_+$. Define $g : V(W(2, n)) \to S_3$ as follows:

g(w) = e

$$g(u_i) = \begin{cases} a, & i = 1,7,\ldots,6k-5 \\ d, & i = 2,8,\ldots,6k-4 \\ b, & i = 3,9,\ldots,6k-3 \\ f, & i = 4,10,\ldots,6k-2 \\ c, & i = 5,11,\ldots,6k-1 \\ e, & i = 6,12,\ldots,6k \end{cases}$$

$$g(v_i) = \begin{cases} d, & i = 1,7,\ldots,6k-5 \\ b, & i = 2,8,\ldots,6k-4 \\ f, & i = 3,9,\ldots,6k-3 \\ c, & i = 4,10,\ldots,6k-2 \\ e, & i = 5,11,\ldots,6k-1 \\ a, & i = 6,12,\ldots,6k \end{cases}$$

$$g(w_i) = \begin{cases} c, & i = 1,7,\ldots,6k-5 \\ f, & i = 2.8,\ldots,6k-4 \\ b, & i = 3,9,\ldots,6k-3 \\ d, & i = 4,10,\ldots,6k-2 \\ a, & i = 5,11,\ldots,6k-1 \\ e, & i = 6,12,\ldots,6k \end{cases}$$

Case 2: $n \equiv 1 \pmod 6$.

Let $n = 6k + 1$, $k \geq 1$, $k \in Z_+$.

Define $g : V(W(2, n)) \to S_3$ as follows.

g(w) = e.

$$g(u_i) = \begin{cases} a, & i = 1,7,\ldots,6k-5 \\ d, & i = 2,8,\ldots,6k-4 \\ b, & i = 3,9,\ldots,6k-3 \\ f, & i = 4,10,\ldots,6k-2 \\ c, & i = 5,11,\ldots,6k-1 \\ e, & i = 6,12,\ldots,6k \\ d, & i = 6k+1 \end{cases}$$

$$g(v_i) = \begin{cases} d, & i = 1,7,\ldots,6k-5 \\ b, & i = 2,8,\ldots 6k-4 \\ f, & i = 3,9,\ldots,6k-3 \\ c, & i = 4,10,\ldots,6k-2 \\ e, & i = 5,11,\ldots,6k-7 \\ a, & i = 6,12,\ldots,6k-6 \\ f, & i = 6k-1 \\ a, & i = 6k \\ e, & i = 6k+1 \end{cases}$$

$$g(w_i) = \begin{cases} c, & i = 1,7,\ldots,6k+1 \\ f, & i = 2.8,\ldots,6k-4 \\ b, & i = 3,9,\ldots,6k-3 \\ d, & i = 4,10,\ldots,6k-2 \\ a, & i = 5,11,\ldots,6k-1 \\ e, & i = 6,12,\ldots,6k \end{cases}$$

Case 3: $n \equiv 2 \pmod 6$

Let $n = 6k + 2$, $k \geq 1$, $k \in Z_+$.

Define $g : V(W(2, n)) \to S_3$ as follows:

g(w) = e

$$g(u_i) = \begin{cases} a, & i = 1,7,\ldots,6k+1 \\ d, & i = 2,8,\ldots,6k-4 \\ b, & i = 3,9,\ldots,6k-3 \\ f, & i = 4,10,\ldots,6k-2 \\ c, & i = 5,11,\ldots,6k-1 \\ e, & i = 6,12,\ldots,6k \end{cases}$$

$$g(v_i) = \begin{cases} d, & i = 1,7,\ldots,6k-5 \\ b, & i = 2,8,\ldots,6k-4 \\ f, & i = 3,9,\ldots,6k-3 \\ c, & i = 4,10,\ldots,6k-2 \\ e, & i = 5,11,\ldots,6k-1 \\ a, & i = 6,12,\ldots,6k \\ f, & i = 6k+1 \\ b, & i = 6k+2 \end{cases}$$

$$g(w_i) = \begin{cases} c, & i = 1,7,\dots,6k+1 \\ f, & i = 2.8,\dots,6k+2 \\ b, & i = 3,9,\dots,6k-3 \\ d, & i = 4,10,\dots,6k-2 \\ a, & i = 5,11,\dots,6k-1 \\ e, & i = 6,12,\dots,6k \end{cases}$$

Case 4: n ≡ 3(mod 6)

Let n = 6k + 3, k ≥ 1, k ∈ Z_+.

Define g : V (W (2, n)) → S_3 as follows:

g(w) = e

$$g(u_i) = \begin{cases} a, & i = 1,7,\dots 6k-5 \\ d, & i = 2,8,\dots,6k-4 \\ b, & i = 3,9,\dots,6k-3 \\ f, & i = 4,10,\dots,6k-2 \\ c, & i = 5,11,\dots,6k-1 \\ e, & i = 6,12,\dots,6k-6 \\ d & i = 6k \\ a, & i = 6k+1 \\ f, & i = 6k+2 \\ e, & i = 6k+3 \end{cases}$$

$$g(v_i) = \begin{cases} d, & i = 1,7,\dots,6k-5 \\ b, & i = 2,8,\dots 6k-4 \\ f, & i = 3,9,\dots,6k-3 \\ c, & i = 4,10,\dots,6k-2 \\ e, & i = 5,11,\dots,6k-1 \\ a, & i = 6,12,\dots,6k \\ d, & i = 6k+1 \\ b, & i = 6k+2 \\ e, & i = 6k+3 \end{cases}$$

$$g(w_i) = \begin{cases} c, & i = 1,7,\dots,6k+1 \\ f, & i = 2.8,\dots,6k+2 \\ b, & i = 3,9,\dots,6k+3 \\ d, & i = 4,10,\dots,6k-2 \\ a, & i = 5,11,\dots,6k-1 \\ e, & i = 6,12,\dots,6k \end{cases}$$

Case 5: n ≡ 4(mod 6)

Let n = 6k + 4, k ≥ 0, k ∈ Z_+.

Define g : V (W (2, n)) → S_3 as follows:

g(w) = e

$$g(u_i) = \begin{cases} a, & i = 1,7,\dots,6k-5 \\ d, & i = 2,8,\dots,6k-4 \\ b, & i = 3,9,\dots,6k-3 \\ f, & i = 4,10,\dots,6k-2 \\ c, & i = 5,11,\dots,6k-1 \\ e, & i = 6,12,\dots,6k \end{cases}$$

$$g(v_i) = \begin{cases} d, & i = 1,7,\dots,6k+1 \\ c, & i = 2,8,\dots,6k+2 \\ f, & i = 3,9,\dots,6k+3 \\ b, & i = 4,10,\dots,6k-2 \\ e, & i = 5,11,\dots,6k-1 \\ a, & i = 6,12,\dots,6k \\ a, & i = 6k+4 \end{cases}$$

$$g(w_i) = \begin{cases} c, & i = 1,7,\dots,6k+1 \\ f, & i = 2,8,\dots,6k+2 \\ b, & i = 3,9,\dots,6k+3 \\ d, & i = 4,10,\dots,6k-2 \\ a, & i = 5,11,\dots,6k-1 \\ e, & i = 6,12,\dots,6k \\ e, & i = 6k+4 \end{cases}$$

Case 6: n ≡ 5(mod 6)

Let n = 6k + 5, k ≥ 0, k ∈ Z_+.

Define g : V (W (2, n)) → S_3 as follows:

g(w) = e

$$g(u_i) = \begin{cases} a, & i = 1,7,\dots,6k+1 \\ d, & i = 2,8,\dots,6k+2 \\ b, & i = 3,9,\dots,6k+3 \\ f, & i = 4,10,\dots,6k+4 \\ c, & i = 5,11,\dots,6k-1 \\ e, & i = 6,12,\dots,6k \\ d, & i = 6k+5 \end{cases}$$

$$g(v_i) = \begin{cases} d, & i = 1,7,\dots,6k+1 \\ b, & i = 2,8,\dots,6k+2 \\ f, & i = 3,9,\dots,6k+3 \\ c, & i = 4,10,\dots,6k+4 \\ e, & i = 5,11,\dots,6k+5 \\ a, & i = 6,12,\dots,6k \end{cases}$$

$$g(w_i) = \begin{cases} c, & i = 1,7,\dots,6k+1 \\ f, & i = 2.8,\dots,6k+2 \\ b, & i = 3,9,\dots,6k+3 \\ d, & i = 4,10,\dots,6k+4 \\ a, & i = 5,11,\dots,6k+5 \\ e, & i = 6,12,\dots,6k \end{cases}$$

Table 1 shows that g is a group S_3 cordial prime labeling.

Table 1

Nature of n	$n_a(g)$	$n_b(g)$	$n_c(g)$	$n_d(g)$	$n_e(g)$	$n_f(g)$
$n = 6k(k \geq 1)$	3k	3k	3k	3k	3k + 1	3k
$n = 6k + 1(k \geq 1)$	3k	3k	3k + 1	3k + 1	3k + 1	3k + 1
$n = 6k + 2(k \geq 1)$	3k + 1	3k + 1	3k + 1	3k + 1	3k + 1	3k + 2
$n = 6k + 3(k \geq 1)$	3k + 1	3k + 2	3k + 1	3k + 2	3k + 2	3k + 2
$n = 6k + 4(k \geq 0)$	3k + 2	3k + 2	3k + 2	3k + 2	3k + 2	3k + 3
$n = 6k + 5(k \geq 0)$	3k + 2	3k + 3	3k + 2	3k + 3	3k + 3	3k + 3

REFERENCES

[1] Cahit I, Cordial graphs: a weaker version of graceful and harmonious graphs, Ars Combin. 23(1987) 201-207

[2] Gallian J A, A Dynamic survey of Graph Labeling, The Electronic Journal of Combinatories DEC7(2015), N O.D56.

[3] Harary F, Graph Theory, Addison Wesley, Reading Mass, 1972.

[4] Tout , A, Dabboucy, A N and Howalla K, Prime labeling of graphs, Nat. Acad. Sci letters, 11, pp 365-368, 1982.