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Study of M/G/1 Feedback Queue with Deterministic Server Vacations

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Abstract

We study a single server queue with Poisson input,$k$stages of general hetrogeneous services and deterministic vacations of constant duration. The server provides service to customers one by one on a first come first served basis. After complicating k^{th} stage of service, a customer may leave the system or may opt to repeat the service in which case this customer rejoins the queue. Further just after completion of a customer's k^{th} stage of service, the server may take a vacation of random length or may opt to continue staying in the system to serve next customer. We find the time probability generating function in terms of Laplace transforms and derive explicitly the corresponding steady state results.

AMS Subject classification: 60K25, 60K30

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1. Introduction

Queueing systems that allow servers to take vacation have wide range of applications in many engineering systems such as flexible manufacturing envoirnment, production, computer, communication systems and telecommunicatin networks. In the analysis of queueing models, sever's vacations are useful for the systems in which the servers want to use their idle itmes for different purposes.

Single server queueing systems have been studied by various authors due to their wide applications in the analysis of processor schedules in computer and switching systems, the analysis of polling systems used in data communication networks, the analysis of manufacturing systems etc. Several excellent surveys on single server queueing models with vacations have been done by Doshi, Tegham, Takagi, Levy and Yechiali, shantikumar and many others.

An M/G/1 queue with vacation model is often referred as a tool of understanding congestion phenomena in local networks. Since the past two or three decades, it has emerged as an important area of study in real life problems such as telecommunication engineering, naufacturing, production, computer and communication networks etc. Several contributions have been made by dealing queueing systems of M/G/1 type which include Doshi, Bertsimas, Madan, choudhury and several others.

The single server queue with phases of service with vacations has been paid attention recently by several researchers. Presently such type of models have been the subject matter of current research mainly due to its applications in computer and communication systems. Several interesting works on single server queuimg systems with phases of services and vacations have been studied by Doshi, Madan, Choi and Kim, Choudhury and Paul, Thangaraj and Vanitha and many others.

In the present work, we consider a single server feedback queue with k stages of heterogeneous services and deterministic server vacations. This paper is organised as follows. The mathematical description of our model is given in Section 2. Definitions and governing the system are given in Section 3. The time dependent solution have been obtained in Section 4 and the corresponding steady state results have been explicitly in Section 5 and particular cases are discussed in Section 6.

2. Assumptions underlying the model

1. Customers arrive at the system one by one in according to a Poisson stream with arrival rate ($\lambda > 0$).
2. Each customer undergoes k stages of heterogeneous service provided by a single server on a first come first served basis. The service time of k stages follow different general (arbitrary) distributions with distribution function $B_j(v)$ and the density function $b_j(v)$,j=1,2,3,... k.
3. After completion of k^{th} stage of service, if the customer is dissatisfied with its service for certain reason or if it received unseccessful service, the customer may immediately join the tail of the original queue with probability p $(0 < p < 1)$. Otherwise the customer may depart forever from the system with probability $q = 1 - p$.
4. Let $\mu_i(x)\mathrm{d}x$ be the conditional probability of completion of the i^{th} stage of service during the interval $(x, x + \mathrm{d}x)$ given that elapsed time is x, so that
$$\mu_i(x) = \frac{b_i(x)}{1 - B_i(x)}, i = 1, 2 \dots k, \tag{2.1}$$
and therefore
$$b_i(v) = \mu_i(v)e^{-\int_0^v \mu_i(x)\mathrm{d}x}, i = 1, 2 \dots k. \tag{2.2}$$
5. As soon as the service of k^{th} stage of a customer is complete, then with probability θ the server decides to take a vacation and with probability $1 - \theta$, he continues to be available for the next service.
6. We assume that whenever the server takes a vacation, it is of constant duration $d(> 0)$.
7. Various stochastic processes involved in the system are independent of each other.

3. Definitions, Notations and the Time - Dependant equations governing the system

The system has then the following set of differential - difference equations

$$\frac{\partial}{\partial x}P_n^{(1)}(x,t)+\frac{\partial}{\partial t}P_n^{(1)}(x,t)+(\lambda+\mu_1(x))P_n^{(1)}(x,t) = \lambda P_{n-1}^{(1)}(x,t),\quad n=1,2,\ldots \tag{3.1}$$

$$\frac{\partial}{\partial x}P_0^{(1)}(x,t)+\frac{\partial}{\partial t}P_0^{(1)}(x,t)+(\lambda+\mu(x))P_0^{(1)}(x,t) = 0, \tag{3.2}$$

$$\frac{\partial}{\partial x}P_n^{(j)}(x,t)+\frac{\partial}{\partial t}P_n^{(j)}(x,t)+(\lambda+\mu_j(x))P_n^{(j)}(x,t) = \lambda P_{n-1}^{(j)}(x,t),\quad n=1,2,\ldots,j=2,\ldots k \tag{3.3}$$

$$\frac{\partial}{\partial x}P_0^{(j)}(x,t)+\frac{\partial}{\partial t}P_0^{(j)}(x,t)+(\lambda+\mu_j(x))P_0^{(j)}(x,t) = 0, \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_0(t) = \theta q\int_0^\infty P_0^{(k)}(x,t)\mu_k(x)\mathrm{d}x+V_0(t)[-K_0-K_1-K_2\cdots], \tag{3.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}V_n(t) = \theta q\int_0^\infty P_n^{(k)}(x,t)\mu_j(x)\mathrm{d}x+\theta p\int_0^\infty P_{n-1}^{(k)}(x,t)\mu_j(x)\mathrm{d}x +V_n(t)[-K_0-K_1-K_2\ldots],\ n=1,2,\ldots, \tag{3.6}$$

$$\frac{d}{dt}Q(t) = -\lambda Q(t)+V_0(t)K_0+(1-\theta)q\int_0^\infty P_0^{(k)}(x,t)\mu_k(x)\mathrm{d}x. \tag{3.7}$$

Equations (3.2) - (3.7) are to be solved subject to the following boundary conditions

$$P_0^{(1)}(0,t) = Q(t)\lambda+V_0(t)K_1+V_1(t)K_0+(1-\theta)p\int_0^\infty P_0^{(k)}(x,t)\mu_k(x)\mathrm{d}x, +(1-\theta)q\int_0^\infty P_1^{(k)}(x,t)\mu_k(x)\mathrm{d}x, \tag{3.8}$$

$$P_n^{(1)}(0,t) = V_0(t)K_{n+1}+V_1(t)K_n+\cdots+V_n(t)K_1+V_{n+1}(t)K_0+ +(1-\theta)p\int_0^\infty P_n^{(k)}(x,t)\mu_k(x)\mathrm{d}x,+(1-\theta)q\int_0^\infty P_{n+1}^{(k)}(x,t)\mu_k(x)\mathrm{d}x,\quad n=1,2,\ldots, \tag{3.9}$$

$$P_n^{(j)}(0,t) = \int_0^\infty P_n^{(j-1)}(x,t)\mu_{j-1}(x)\mathrm{d}x,\ n=0,1,\ldots,j=2,3,\ldots k. \tag{3.10}$$

We assume that initially there is no customer in the system and the server is idle so that the initial conditions are

$$Q(0)=1,\ P_n^{(j)}(0)=0,\ V_0(0)=0=V_n(0),n\geq 0,j=2,3\ldots k,. \tag{3.11}$$

4. Generating functions of the queue length: The time-dependent solution

We define the probability generating functions,

$$\left.\begin{aligned} P^{(j)}(x,z,t) &= \sum_{n=0}^{\infty} z^n P^{(j)}(x,t),\\ P^{(j)}(z,t) &= \sum_{n=0}^{\infty} z^n P^{(j)}(t), j=2,3\ldots k,\\ V(z,t) &= \sum_{n=0}^{\infty} z^n V_n(t). \end{aligned}\right\} \tag{4.1}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function $f(t)$ as

$$\overline{f}(s) = \int_0^\infty e^{-st} f(t)\mathrm{d}t, \ \Re(s) > 0. \tag{4.2}$$

Taking the Laplace transforms of equations (3.1) to (3.10) and using (3.11), we obtain

$$\frac{\partial}{\partial x}\overline{P}_n^{(1)}(x,s) + (s+\lambda+\mu(x))\overline{P}_n^{(1)}(x,s) = \lambda\overline{P}_{n-1}^{(1)}(x,s), \quad n = 1,2,\ldots \tag{4.3}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(1)}(x,s) + (s+\lambda+\mu(x))\overline{P}_0^{(1)}(x,s) = 0, \tag{4.4}$$

$$\frac{\partial}{\partial x}\overline{P}_n^{(j)}(x,s) + (s+\lambda+\mu_j(x))\overline{P}_n^{(j)}(x,s) = \lambda\overline{P}_{n-1}^{(j)}(x,s), \quad n = 1,2,\ldots,.j = 2,3,\ldots k \tag{4.5}$$

$$\frac{\partial}{\partial x}\overline{P}_0^{(j)}(x,s) + (s+\lambda+\mu_j(x))\overline{P}_0^{(j)}(x,s) = 0, \tag{4.6}$$

$$s\overline{V}_0(s) = \alpha\overline{Q}(s) + \overline{V}_0(s)[-K_0 - K_1 - K_2\ldots] + \theta q\int_0^\infty \overline{P}_0^{(k)}(x,s)\mu_k(x)\mathrm{d}x, \tag{4.7}$$

$$s\overline{V}_n(s) = \theta q\int_0^\infty \overline{P}_n^{(k)}(x,s)\mu_k(x)\mathrm{d}x + \theta p\int_0^\infty \overline{P}_{n-1}^{(k)}(x,s)\mu_k(x)\mathrm{d}x + \overline{V}_n(s)[\,-K_0 - K_1 - K_2\ldots], \ n = 1,2,\ldots, \tag{4.8}$$

$$(s+\lambda)\overline{Q}(s) = 1 + \overline{V}_0(s)K_0 + (1-\theta)q\int_0^\infty \overline{P}_0^{(k)}(x,s)\mu_k(x)\mathrm{d}x, \tag{4.9}$$

$$\overline{P}_0^{(1)}(0,s) = \overline{Q}(s)\lambda + \overline{V}_0(s)K_1 + \overline{V}_1(s)K_0 + (1-\theta)p\int_0^\infty \overline{P}_0^{(k)}(x,s)\mu_k(x)\mathrm{d}x + (1-\theta)q\int_0^\infty \overline{P}_1^{(k)}(x,s)\mu_k(x)\mathrm{d}x\ , \tag{4.10}$$

$$\overline{P}_n^{(1)}(0,s) = \overline{V}_0(s)K_{n+1} + \overline{V}_1(s)K_n + \cdots + \overline{V}_n(s)K_1 + \overline{V}_{n+1}(s)K_0 + +(1-\theta)p\int_0^\infty \overline{P}_n^{(k)}(x,s)\mu_k(x)\mathrm{d}x, +(1-\theta)q\int_0^\infty \overline{P}_{n+1}^{(k)}(x,s)\mu_k(x)\mathrm{d}x, \quad n = 1,2,\ldots, \tag{4.11}$$

$$\overline{P}_n^{(j)}(0,s) = \int_0^\infty \overline{P}_n^{(j-1)}(x,s)\mu_1(x)\mathrm{d}x, \ n = 0,1,\ldots,j = 2,3,\ldots k. \tag{4.12}$$

Now multiplying equation (4.3) by z^n and summing over n from 1 to ∞, adding to equation (4.4) and using the generating functions defined in (4.1), we get

$$\frac{\partial}{\partial x}\overline{P}^{(1)}(x,z,s) + (s+\lambda-\lambda z+\mu_1(x))\overline{P}(x,z,s) = 0, \tag{4.13}$$

Performing similar operations on equations (4.5) to (4.8) we obtain

$$\frac{\partial}{\partial x}\overline{P}^{(j)}(x,z,s) + (s+\lambda-\lambda z+\mu_j(x))\overline{P}^{(j)}(x,z,s) = 0, j = 2,3,\ldots k. \tag{4.14}$$

$$(s+1)\overline{V}(z,s) = \theta(q+pz)\int_0^\infty \overline{P}^{(k)}(x,z,s)\mu_k(x)\mathrm{d}x, \tag{4.15}$$

For the boundary conditions, we multiply both sides of equation (4.10) by z, multiply both sides of equation (4.11) by z^{n+1}, sum over n from 1 to ∞, add the two results and use equation (4.1) to get

$$\begin{aligned} z\overline{P}^{(1)}(0,z,s) &= \lambda z\overline{Q}(s) + \overline{V}(z,s)e^{-\lambda d[1-z]} + (1-\theta)(q+pz)\int_0^\infty \overline{P}^{(k)}(x,z,s)\mu_k(x)\mathrm{d}x \\ &\quad -(1-\theta)q\int_0^\infty \overline{P}_0^{(k)}(x,s)\mu_k(x)\mathrm{d}x - \overline{V}_0(s)K_0. \end{aligned} \quad . \tag{4.16}$$

Performing similar operation on equation (4.12),we have

$$\overline{P}^{(j)}(0,z,s) = \int_0^\infty \overline{P}^{(j-1)}(x,z,s)\mu_{j-1}(x)\mathrm{d}x, j = 2,3,\dots k. \tag{4.17}$$

Using equation (4.9), equation (4.16) become

$$\begin{aligned} z\overline{P}(0,z,s) &= \lambda z\overline{Q}(s) + \overline{V}(z,s)e^{-\lambda d[1-z]} + (1-\theta)(q+pz)\int_0^\infty \overline{P}^{(k)}(x,z,s)\mu_k(k)\mathrm{d}x \\ &\quad +1-(s+\lambda)\overline{Q}(s). \end{aligned} \tag{4.18}$$

Integrating equation (4.13) from 0 to x yields

$$\overline{P}^{(1)}(x,z,s) = \overline{P}^{(1)}(0,z,s)\, e^{-(s+\lambda-\lambda z)x - \int_0^x \mu_1(t)dt}, \tag{4.19}$$

where $\overline{P}^{(1)}(0,z,s)$ is given by equation (4.18). Again integrating equation (4.19) by parts with respect to x yields

$$\overline{P}^{(1)}(z,s) = \overline{P}^{(1)}(0,z,s)\left[\frac{1-\overline{B}_1(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \tag{4.20}$$

where

$$\overline{B}_1(s+\lambda-\lambda z) = \int_0^\infty e^{-(s+\lambda-\lambda z)x}dB_1(x) \tag{4.21}$$

is the Laplace-Stieltjes transform of the essential service time $B_1(x)$. Now multiplying both sides of equation (4.19) by $\mu_1(x)$ and integrating over x, we obtain

$$\int_0^\infty \overline{P}^{(1)}(x,z,s)\mu_(x)\mathrm{d}x = \overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z). \tag{4.22}$$

Similarly, on integrating equation (4.14) from 0 to x, we get

$$\overline{P}^{(j)}(x,z,s) = \overline{P}^{(j)}(0,z,s)\, e^{-(s+\lambda-\lambda z)x - \int_0^x \mu_j(t)dt}, j = 2,3,\dots k, \tag{4.23}$$

where $\overline{P}^{(j)}(0,z,s)$ is given by equation (4.17). Again integrating equation (4.23) by parts with respect to x yields

$$\overline{P}^{(j)}(z,s) = \overline{P}^{(j)}(0,z,s)\left[\frac{1-\overline{B}_j(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], j = 2,3,\dots k, \tag{4.24}$$

where

$$\overline{B}_j(s+\lambda-\lambda z+\alpha) = \int_0^\infty e^{-(s+\lambda-\lambda z)x}dB_j(x), j = 2,3,\dots k, \tag{4.25}$$

is the Laplace-Stieltjes transform of k-1 service times $B_j(x)$. We see that by virtue of equation (4.23), we have

$$\int_0^\infty \overline{P}^{(j)}(x,z,s)\mu_j(x)\mathrm{d}x = \overline{P}^{(j)}(0,z,s)\overline{B}_j(s+\lambda-\lambda z), j = 2,3,\dots k. \tag{4.26}$$

By using equation (4.22), equation (4.17) reduces to

$$\overline{P}^{(j)}(0,z,s)=\overline{P}^{(j-1)}(0,z,s)\overline{B}_{j-1}(s+\lambda-\lambda z), j=2,3,\dots k. \tag{4.27}$$

Using equation (4.27), equation (4.26) becomes

$$\int_0^\infty \overline{P}^{(j)}(x,z,s)\mu_j(x)\mathrm{d}x=\overline{P}^{(1)}(0,z,s)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_j(s+\lambda-\lambda z), j=2,3,\dots k. \tag{4.28}$$

By using above equation (4.18) reduces to

$$\overline{P}^{(1)}(0,z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_k(s+\lambda-\lambda z)}. \tag{4.29}$$

Substituting the value of $\overline{P}^{(1)}(0,z,s)$ into equation (4.20), we get

$$\overline{P}^{(1)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\overline{B}_k(s+\lambda-\lambda z)} \left[\frac{1-\overline{b}(s+\lambda-\lambda z}{s+\lambda-\lambda z}\right]. \tag{4.30}$$

Now using equations (4.27) and (4.29), equation (4.24) become

$$\overline{P}^{(j)}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\overline{B}_k(s+\lambda-\lambda z)} \overline{B}_1(s+\lambda-\lambda z)\overline{B}_1(s+\lambda-\lambda z)\dots\overline{B}_{j-1}(s+\lambda-\lambda z)\left[\frac{1-\overline{B}_j(s+\lambda-\lambda z)}{s+\lambda-\lambda z}\right], \quad j=2,3,\dots k. \tag{4.31}$$

From equation (4.15)

$$(s+1)\overline{V}(z,s) = \frac{\overline{V}(z,s)e^{-\lambda d[1-z]}+[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{z-(q+pz)(1-\theta)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_k(s+\lambda-\lambda z)} \theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_k(s+\lambda-\lambda z).$$

which further reduces to

$$\overline{V}(z,s) = \theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_k(s+\lambda-\lambda z)\left[\frac{[1-s\overline{Q}(s)]+\lambda\overline{Q}(s)[z-1]}{DR}\right]. \tag{4.32}$$

where

$$DR = (s+1)\left[z-(1-\theta)(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_k(s+\lambda-\lambda z) +\theta(q+pz)\overline{B}_1(s+\lambda-\lambda z)\overline{B}_2(s+\lambda-\lambda z)\dots\overline{B}_k(s+\lambda-\lambda z)[1-e^{-\lambda d[1-z]}]\right] \tag{4.33}$$

Let $\overline{P}(z,s)=\overline{P}^{(1)}(z,s)+\overline{P}^{(2)}(z,s)+\dots\overline{P}^{(j)}(z,s), j=2,3\dots k$ denote the probability generating function of the number in the queue irrespective of the type of service being provided. If we let $z=1$ in equation (4.34), we can easily verify that

$$\overline{Q}(s)+\overline{V}(z,s)+\overline{P}(z,s)=\frac{1}{s}, \tag{4.34}$$

as it should be.

Thus $\overline{V}(z,s), \overline{P}^{(1)}(z,s),\dots\overline{P}^{(j)}(z,s)$ are completely determined from equations (4.32), (4.33), (4.30) and (4.31) .

5. Steady state solution

In this section, we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, we suppress the argument t wherever it appears in the time-dependent analysis. This can be obtained by applying the well-known Tauberian property,

$$\lim_{s\to 0}\overline{f}(s) = \lim_{t\to\infty} f(t). \tag{5.1}$$

In order to determine $\overline{P}^{(1)}(z,s), \overline{P}^{(2)}(z,s)\dots\overline{P}^{(j)}(z,s)$ and $\overline{V}(z,s)$ completely, we have yet to determine the unknown Q which appears in the numerators of the right hand sides of equations $(4.30), (4.31)$ and (4.32) by using initial conditions (4.29) and (4.27) . For that purpose, we shall use the normalizing condition

$$P^{(1)}(1) + P^{(2)}(1) + \dots + P^{(j)}(1) + V(1) + Q = 1. \tag{5.2}$$

Thus multiplying both sides of equations (4.30), (4.31) and (4.32) by s, taking limit as $s \to 0$, applying property (5.1) and simplifying we have

$$P^{(1)}(z) = \left[\frac{V(z)e^{-\lambda d[1-z]} + \lambda Q(z-1)}{z-(q+pz)(1-\theta)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_k(\lambda-\lambda z)}\right]\left[\frac{1-B_1(\lambda-\lambda z)}{(\lambda-\lambda z)}\right] \tag{5.3}$$

$$\begin{aligned} P^{(j)}(z) &= \left[\frac{V(z)e^{-\lambda d[1-z]} + \lambda Q(z-1)}{z-(q+pz)(1-\theta)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_k(\lambda-\lambda z)}\right] \\ &\quad B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_{j-1}(\lambda-\lambda z)\left[\frac{1-B_j(\lambda-\lambda z)}{(\lambda-\lambda z)}\right], j = 2,3,\dots k \end{aligned} \tag{5.4}$$

and

$$V(z) = \frac{\lambda\theta(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_k(\lambda-\lambda z)Q(z-1)}{D(z)}, \tag{5.5}$$

where

$$\begin{aligned} D(z) &= z-(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots + B_k(\lambda-\lambda z) \\ &\quad +\theta(q+pz)B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_k(\lambda-\lambda z)[1-e^{-\lambda d[1-z]}] \end{aligned} \tag{5.6}$$

Then substituting for $V(z)$ from equation (5.6) into equations (5.3) and (5.4)

$$P^{(1)}(z) = \frac{[B_1(\lambda-\lambda z)-1]}{DR} \tag{5.7}$$

$$P^{(j)}(z) = \frac{B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_{j-1}(\lambda-\lambda z)\left[B_j(\lambda-\lambda z)-1\right]Q}{DR} \qquad \text{j} = 2,3,\dots k \tag{5.8}$$

where $D(z)$ is given by equation (5.6).

Let $P(z) = P^{(1)}(z) + P^{(2)}(z) + \cdots + P^{(j)}(z)$.

Now from equations (5.7) and (5.8)

$$P(z) = \frac{\left[B_1(\lambda-\lambda z)B_2(\lambda-\lambda z)\dots B_k(\lambda-\lambda z)-1\right]Q}{DR} \tag{5.9}$$

where $D(z)$ is given by equation (5.6).

6. Particular case

In this case we assume that there are three stages of hetrogeneous service and therefore we take k=3.

We see that for $z = 1$, the right hand side of both equations (5.3), (5.4) and (5.5) are indeterminate of the form $\frac{0}{0}$. Therefore, applying L'Hopital's rule we obtain

$$V(1) = \frac{\theta\lambda Q\mu_1\mu_2\mu_3}{q\mu_1\mu_2\mu_3 - \lambda\mu_1\mu_2 - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \theta\lambda d\mu_1\mu_2\mu_3} \quad (6.1)$$

$$P^{(1)}(1) = \frac{\lambda Q\mu_1\mu_3}{q\mu_1\mu_2\mu_3 - \lambda\mu_1\mu_2 - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \theta\lambda d\mu_1\mu_2\mu_3} \quad (6.2)$$

$$P^{(2)}(1) = \frac{\lambda Q\mu_1\mu_2}{q\mu_1\mu_2\mu_3 - \lambda\mu_1\mu_2 - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \theta\lambda d\mu_1\mu_2\mu_3} \quad (6.3)$$

$$P^{(3)}(1) = \frac{\lambda Q\mu_1\mu_2}{q\mu_1\mu_2\mu_3 - \lambda\mu_1\mu_2 - \lambda\mu_2\mu_3 - \lambda\mu_1\mu_3 - \theta\lambda d\mu_1\mu_2\mu_3} \quad (6.4)$$

wherein we have used the facts that $\overline{B}_1(0) = \overline{B}_2(0) = \overline{B}_3(0) = 1, -\overline{B}_1^{'}(0) = \frac{1}{\mu_1}, -\overline{B}_2^{'}(0) = \frac{1}{\mu_2}$ and $-\overline{B}_3^{'}(0) = \frac{1}{\mu_3}$.

Now to determine the only unknown Q, we use (6.1) and (6.4) in the normalizing condition $Q + P^{(1)}(1) + P^{(2)}(1) + P^{(3)}(1) + V(1) = 1$ and have

$$Q = \frac{\mu_1\mu_2\mu_3[q - \theta\lambda d] - \lambda(\mu_1\mu_2 + \mu_2\mu_3 + \mu_1\mu_3)}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3} \quad (6.5)$$

$$= 1 - \frac{\lambda[\theta\mu_1\mu_2\mu_3 + \mu_1\mu_2 + \mu_1\mu_2 + \mu_1\mu_2]}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3} \quad (6.6)$$

where $\lambda < \mu_1\mu_2\mu_3[q - \theta\lambda d]$

Equation (6.6) gives the stready state probability that there is no customer in the system and the server is idle.

Also from equation (6.6), we obtain ρ, the utilisation of the factor of the system as

$$\rho = 1 - Q = \frac{\lambda[\mu_1\mu_2 + \mu_1\mu_3 + \mu_2\mu_3 + \theta\mu_1\mu_2\mu_3]}{\mu_1\mu_2\mu_3[q - \theta\lambda d] + \theta\lambda\mu_1\mu_2\mu_3} < 1 \quad (6.7)$$

Equation (6.7) gives the stready state probability that there is no customer in the system and the server is idle.

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