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On $\mathcal{I}_{\pi^* g}$-closed sets in an Ideal Topological Spaces

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ABSTRACT: In this paper, we define a new class of $\mathcal{I}_{\pi^* g}$-closed sets in an ideal topological spaces by using the notion of $\mathcal{I}_{\pi^* g}$-closed sets and also discussed relationship between $\mathcal{I}_{\pi^* g}$-closed sets and the related closed sets in an ideal topological spaces.
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1. INTRODUCTION AND PRELIMINARIES

An ideal $\mathcal{I}$ is a nonempty collection of subsets of X which satisfies the following conditions: (i) $A \in \mathcal{I}$ and $B \subseteq A$ implies $B \in \mathcal{I}$; (ii) $A \in \mathcal{I}$ and $B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ [2]. With topology τ and an ideal $\mathcal{I}$, for each subset A of X, a subset $A^*(\mathcal{I})$ or simply A^* of X is defined by $A^* = \{x \in X/U \cap A \notin \mathcal{I}$ for every $U \in \tau$ such that $x \in U\}$ [2]. A Kuratowski closure operator cl^* for a topology $\tau^*(\mathcal{I}, \tau)$, called the $*$-topology, finnier than τ is defined by $cl^*(A) = A \cup A^*$ [9]. If $\mathcal{I}$ is an ideal on X, then $(X, \tau, \mathcal{I})$ is called an ideal topological space. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $*$-closed [2] (resp. $*$-dense in itself [1]) if $A^* \subseteq A$ (resp. $A \subseteq A^*$). We denote by $cl(A)$ the intersections of all closed sets containing A; and by $int(A)$ the union of all open sets contained in A and $int^*(A)$ will denote the interior of A in (X, τ^*). A subset A of a topological space (X, τ) is α-open [7] (resp. semi-open [4], preopen [7]) if $A \subseteq int(cl(int(A)))$ (resp. $A \subseteq cl(int(A))$, $A \subseteq int(cl(A))$).

Lemma 1.1. *[2] Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If $A \subseteq A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$.*

Lemma 1.2. *A subset A of a topological space (X, τ) is said to be*

i. g-closed [6] $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is open in (X, τ)
ii. $\mathcal{I}_g$-closed [6]if $A^* \subseteq U$ and $A \subseteq U$ and U is open in (X, τ).

2. $\mathcal{I}_{\pi^* G}$-CLOSED SETS

Definition 2.1. *A subset H of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\mathcal{I}_{\pi^* g}$-closed if $cl^*(int(H)) \subseteq U$, whenever $H \subseteq U$ and U is a pre-$\mathcal{I}$-open set in $(X, \tau, \mathcal{I})$. The complement of $\mathcal{I}_{\pi^* g}$-closed is called a $\mathcal{I}_{\pi^* g}$-open set. The class of all $\mathcal{I}_{\pi^* g}$-closed sets of $(X, \tau, \mathcal{I})$ is denoted by $\mathcal{I}_{\pi^* g}\, cl\, (X, \tau, \mathcal{I})$.*

Theorem 2.2. *In an ideal topological space $(X, \tau, \mathcal{I})$, every closed set is a $\mathcal{I}_{\pi^* g}$-closed set but not conversely.*
Proof: *Obvious*

Remark 2.3. *The converse of the above Theorem 2.2 need not be as seen in the following example.*

Example 2.4. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{c\}, \{a, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}\}$. Then the $\mathcal{I}_{\pi^* g}$-closed sets in X are $\emptyset$, $\{a\}$, $\{b\}$, $\{a, b\}$, $\{b, c\}$ and X and closed sets in X are $\emptyset$, $\{b\}$, $\{a, b\}$, $\{b, c\}$ and X. It is clear that $\{a\}$ is a $\mathcal{I}_{\pi^* g}$-closed set but not closed in X.*

Theorem 2.5. *In an ideal topological space $(X, \tau, \mathcal{I})$, every $*$-closed set is a $\mathcal{I}_{\pi^* g}$-closed set but not conversely.*
Proof: *Suppose that H is a $*$-closed set in X. Let $H \subseteq U$ where U is a pre-$\mathcal{I}$-open set. Since H is a $*$-closed set, $H^* \subseteq H$. Now $cl^*(int(H)) \subseteq cl^*(H) = H^* \cup H = H \subseteq U$. Hence H is a $\mathcal{I}_{\pi^* g}$-closed set in X.*

Remark 2.6. *The converse of Theorem 2.5 need not be true in general as shown in the following example.*

Example 2.7. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then the subset $\{c\}$ of X is a $\mathcal{I}_{\pi^* g}$-closed set but not $*$-closed in X.*

Theorem 2.8. *In an ideal topological space $(X, \tau, \mathcal{I})$, every g-closed set is a $\mathcal{I}_{\pi^* g}$-closed set but not conversely.*
Proof: *Suppose that H is a g-closed set in $(X, \tau, \mathcal{I})$. Let $H \subseteq U$ where U is an open set. Since H is a g-closed set, $cl(H) \subseteq U$ and we have $cl^*(H) \subseteq cl(H)$. Now $cl^*(int(H)) \subseteq cl^*(H) \subseteq cl(H) \subseteq U$. Using the fact that every open set is a pre-$\mathcal{I}$-open set. Hence H is a $\mathcal{I}_{\pi^* g}$-closed set.*

Remark 2.9. *The converse of Theorem 2.8 need not be true in general as shown in the following example.*

Example 2.10. *In Example 2.7, the subset* $\{a\}$ *of X is a* $\mathcal{I}_{\pi^*g}$ *-closed set but not* g *-closed.*

Theorem 2.11. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $H \subseteq X$*. If* H *is both* $\mathcal{I}_g$ *-closed set and* $*$ *-dense in itself, then* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X *but not conversely.*
Proof: *Let* $H \subseteq U$ *where U is an open set in* $(X, \tau, \mathcal{I})$*. Since* H *is a* $\mathcal{I}_g$ *-closed set and* H *is* $*$ *-dense in itself, we have* $H^* \subseteq U$ *and* $H \subseteq H^*$*, respectively. Therefore* $cl^*(int(H)) = (int(H))^* \cup (int(H)) \subseteq H^* \cup H = H^* \subseteq U$*. Using the fact that every open set is a pre-*$\mathcal{I}$ *-open set. Hence* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*.*

Remark 2.12. *Example 2.13 below shows that the notions* $\mathcal{I}_g$ *-closed set and* $*$ *-dense in itself are independent to each other.*

Example 2.13. *Consider an ideal topological space* $(X, \tau, \mathcal{I})$ *where* $X = \{a, b, c, d\}$*,* $\tau = \{\emptyset, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$*. Then* $\mathcal{I}_g$ *-closed sets are* $\emptyset$*,* $\{a\}$*,* $\{b\}$*,* $\{c\}$*,* $\{a, b\}$*,* $\{a, c\}$*,* $\{b, c\}$*,* $\{c, d\}$*,* $\{a, b, c\}$*,* $\{a, c, d\}$*,* $\{b, c, d\}$*,* X *and* $*$ *-dense in itself are* $\emptyset$*,* $\{a\}$*,* $\{b\}$*,* $\{c\}$*,* $\{d\}$*,* $\{a, d\}$*,* $\{b, d\}$*,* $\{c, d\}$*,* $\{a, b, d\}$*,* $\{a, c, d\}$*,* $\{b, c, d\}$*,* X*. Hence*
i. The subset $\{a, c\}$ *is a* $\mathcal{I}_g$ *-closed set but not* $*$ *-dense in itself.*
ii. The subset $\{d\}$ *is* $*$ *-dense in itself but not* $\mathcal{I}_g$ *-closed set.*

Theorem 2.14. *In an ideal topological space* $(X, \tau, \mathcal{I})$*, every pre-*$\mathcal{I}$ *-closed set is a* $\mathcal{I}_{\pi^*g}$ *-closed set but not conversely.*
Proof: *Let* H *be a pre-*$\mathcal{I}$ *-closed set in* $(X, \tau, \mathcal{I})$*. Let* $H \subseteq U$ *where* U *is a pre-*$\mathcal{I}$ *-open set. Since* H *is a pre-*$\mathcal{I}$ *-closed set, we have* $cl^*(int(H)) \subseteq H$ *and hence* $cl^*(int(H)) \subseteq U$*. Hence* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*.*

Remark 2.15. *The converse of Theorem 2.14 need not be true in general as shown in the following example.*

Example 2.16. *In Example 2.7, the subset* $\{a\}$ *is* $\mathcal{I}_{\pi^*g}$ *-closed set in* X *but not pre-*$\mathcal{I}$ *-closed set.*

Corollary 2.17. *In an ideal topological space* $(X, \tau, \mathcal{I})$*, every pre-closed set is a* $\mathcal{I}_{\pi^*g}$ *-closed set.*
Proof: *Obvious*

Theorem 2.18. *In an ideal topological space* $(X, \tau, \mathcal{I})$*, every* α *-*$\mathcal{I}$ *-closed set is a* $\mathcal{I}_{\pi^*g}$ *-closed set but not conversely.*
Proof: *Let* H *be a* α *-*$\mathcal{I}$ *-closed set in* $(X, \tau, \mathcal{I})$*. Let* $H \subseteq U$ *where* U *is a pre-*$\mathcal{I}$ *-open set. Since* H *is a* α *-*$\mathcal{I}$ *-closed set,* $cl^*(int(cl^*(H))) \subseteq H$*. Now* $cl^*(int(H)) \subseteq cl^*(int(cl^*(H))) \subseteq U$*. Hence* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*.*

Remark 2.19. *The converse of Theorem 2.18 need not be true in general as shown in the following example.*

Example 2.20. *Let* $(X, \tau, \mathcal{I})$ *where* $X = \{a, b, c, d\}$*,* $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, b, d\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$*. Then the subset* $\{b, d\}$ *is a* $\mathcal{I}_{\pi^*g}$ *-closed set but not* α *-*$\mathcal{I}$ *-closed set.*

Corollary 2.21. *In an ideal topological space* $(X, \tau, \mathcal{I})$*, every* α *-closed set is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X *.*
Proof: *Obvious.*

Theorem 2.22. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. For any* $H \in \mathcal{I}$*,* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set but not conversely.*
Proof: *Let* $H \subseteq U$*, where* U *is a pre-*$\mathcal{I}$ *-open set. We have* $H^* = \emptyset$ *for every* $H \in \mathcal{I}$*, then* $cl^*(H) = H^* \cup H = H$*. Now* $cl^*(int(H)) \subseteq cl^*(H) = H \subseteq U$*. Hence for every* $H \in \mathcal{I}$*,* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X *.*

Theorem 2.23. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* H *and* F *be subsets of* X*. If* H *and* F *are* $\mathcal{I}_{\pi^*g}$ *-closed sets in* X*, then* $H \cap F$ *is also a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*.*
Proof: *Let* $H \cap F \subseteq U$ *where* U *is a pre-*$\mathcal{I}$ *–open set. Since* H *and* F *be* $\mathcal{I}_{\pi^*g}$ *-closed sets in* X*, we have* $cl^*(int(H)) \subseteq U$ *and* $cl^*(int(F)) \subseteq U$*. Now* $cl^*(int(H \cap F)) = cl^*(int(H) \cap int(K)) \subseteq cl^*(int(H)) \cap cl^*(int(F)) \subseteq U$*. Hence* $H \cap F$ *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*.*

Remark 2.24. *The following example shows that the union of two* $\mathcal{I}_{\pi^*g}$ *-closed sets need not be a* $\mathcal{I}_{\pi^*g}$ *-closed set.*

Example 2.25. *Let* $(X, \tau, \mathcal{I})$ *where* $X = \{a, b, c, d\}$*,* $\tau = \{\emptyset, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, X\}$ *and* $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$*. Then the subset* $\{b\}$ *and* $\{c\}$ *are* $\mathcal{I}_{\pi^*g}$ *-closed sets. But but their union* $\{b, c\}$ *is not a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*.*

Theorem 2.26. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $H \subseteq X$*. If* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*, then* $cl^*(int(H))$ *-* H *contains no non-empty pre-*$\mathcal{I}$ *-closed set in* X*.*
Proof: *Suppose that* F *is a non-empty pre-*$\mathcal{I}$ *-closed set contained in* $cl^*(int(H))$ *-* H*. Now* $F \subseteq cl^*(int(H)) - H$ *implies that* $F \subseteq cl^*(int(H)) \cap H^c$*. Consequently,* $F \subseteq cl^*(int(H))$*. Now* $F \subseteq H^c$ *implies that* $H \subseteq F^c$*. Since* F^c *is a pre-*$\mathcal{I}$ *–open set and* H *is a* $\mathcal{I}_{\pi^*g}$ *-closed set, we have* $cl^*(int(H)) \subseteq F^c$ *and* $F \subseteq (cl^*(int(H)))^c$*. Therefore* $F \subseteq (cl^*(int(H))) \cap (cl^*(int(H))^c = \emptyset$*. That is,* $F = \emptyset$*. Hence* $cl^*(int(H))$ *-* H *contains no non-empty pre-*$\mathcal{I}$ *-closed set in* X*.*

Corollary 2.27. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* H *be* $\mathcal{I}_{\pi^*g}$ *-closed subset of* X*. Then* H *is a regular-*$\mathcal{I}$ *-closed set in* X *if and only if* $cl^*(int(H))$ *-* H *is a pre-*$\mathcal{I}$ *–closed set in* X*.*
Proof: *Let* H *be a* $\mathcal{I}_{\pi^*g}$ *-closed set in* X*. If* H *is a regular-*$\mathcal{I}$ *-closed set in* X*, then we have* $cl^*(int(H))$ *-* $H = \emptyset$ *which is a pre-*$\mathcal{I}$ *-closed set in* X*. Conversely, let* $cl^*(int(H))$ *-* H *be a pre-*$\mathcal{I}$ *-closed set in* X*. Then, by Theorem 2.26,*

$cl^*(int(H))$ - H does not contains any non-empty pre-$\mathcal{I}$-closed subset of X and since $cl^*(int(H)) - H$ is a pre-$\mathcal{I}-$closed subset of itself, then $cl^*(int(H))$ - $H = \emptyset$. This implies that $H = cl^*(int(H))$. Hence H is a regular-$\mathcal{I}$-closed set in X.

Theorem 2.28. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and H and K be any two subsets of X. If H is a $\mathcal{I}_{\pi^*g}$-closed set in X and $H \subseteq K \subseteq cl^*(int(H))$, then K is also a $\mathcal{I}_{\pi^*g}$-closed set in X.*
Proof: *Let $K \subseteq U$ where U is a pre-$\mathcal{I}$-open set. Now $H \subseteq K$ implies that $H \subseteq U$ and U is a pre-$\mathcal{I}$-open set. Since H is a $\mathcal{I}_{\pi^*g}$-closed set, then $cl^*(int(H)) \subseteq U$. Using hypothesis, $cl^*(int(K)) \subseteq U$. Hence K is a $\mathcal{I}_{\pi^*g}$-closed set in X.*

Theorem 2.29. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset of X is a $\mathcal{I}_{\pi^*g}$-closed set if and only if every pre-$\mathcal{I}$-open set is a pre-$\mathcal{I}$-closed set in X.*
Proof: *Suppose every subset of X is a $\mathcal{I}_{\pi^*g}$-closed set. If $U \subseteq X$ where U is a pre-$\mathcal{I}$-open set, then U is a $\mathcal{I}_{\pi^*g}$-closed set and so $cl^*(int(H)) \subseteq U$. Hence U is a pre-$\mathcal{I}$-closed set in X.*
Conversely, suppose that every pre-$\mathcal{I}$-open set is a pre-$\mathcal{I}$-closed set. If U is a pre-$\mathcal{I}$-open set such that $H \subseteq U \subseteq X$, then $cl^*(int(H)) \subseteq H \subseteq U$ and so H is a $\mathcal{I}_{\pi^*g}$-closed set in X.

Theorem 2.30. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense and $H \subseteq X$. Then H is both regular-$\mathcal{I}$-open and $\mathcal{I}_{\pi^*g}$-closed, then it is a clopen set,*
Proof: *If H is a regular-$\mathcal{I}$-open set, then H is an open set in X and so $H = int(H)$. Since H is a $\mathcal{I}_{\pi^*g}$-closed set, then $cl^*(int(H)) \subseteq H$. Now $\mathcal{I}$ is completely codense, $cl(H) = cl^*(H) = cl^*(int(H)) \subseteq H$. Therefore $cl(H) = H$. Hence H is a clopen set.*

Theorem 2.31. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space where $\mathcal{I}$ is completely codense and $H \subseteq X$. Then the following are equivalent:*
(i) H is a clopen set.
(ii) H is a regular-I-open set and $\mathcal{I}_{\pi^*g}$-closed set.
(iii) H is a pre-I-open set and $\mathcal{I}_{\pi^*g}$-closed set.
Proof: *$(i) \Rightarrow (ii)$ If H is a clopen set, then H is both open and closed. Now $int(cl^*(H)) = int(cl(H)) = int(H) = H$. Hence H is a regular-I-open set. Let $H \subseteq U$ where U is a pre-I-open set in X. Then $cl^*(int(H)) = cl^*(H) = cl(H) \subseteq G$. Hence H is a $\mathcal{I}_{\pi^*g}$-closed set.*
$(ii) \Rightarrow (i)$ Follows from Theorem 2.30.
$(ii) \Rightarrow (iii)$ The proof follows from the fact that every regular-I-open set is pre-I-open set.

Theorem 2.32. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is a $\mathcal{I}_{\pi^*g}$-open set if and only if $F \subseteq int^*(cl(A))$ whenever F is a pre-$\mathcal{I}$-closed set and $F \subseteq A$.*

Proof: *Suppose that A is a $\mathcal{I}_{\pi^*g}$-open set. Let $F \subseteq A$ and F is a pre-$\mathcal{I}$-closed set. Then $X - A \subseteq X - F$ and $X - F$ is a pre-$\mathcal{I}$-open set. Since $X - A$ is a $\mathcal{I}_{\pi^*g}$-closed set, then $cl^*(int(X - A)) \subseteq X - F$ and $X - cl(int^*(A)) = cl^*(int(X - A)) \subseteq X - F$. Hence $F \subseteq int^*(cl(A))$. Conversely, let $X - A \subseteq U$ where U is a pre-$\mathcal{I}$-open set. Then $X - U$ is a pre-$\mathcal{I}$-closed set. By hypothesis, we have $X - U \subseteq int^*(cl(A))$ and hence $cl^*(int(X - A)) = X - int^*(cl(A)) \subseteq U$. Therefore $X - A$ is a $\mathcal{I}_{\pi^*g}$-closed set and hence A is a $\mathcal{I}_{\pi^*g}$-open set.*

Theorem 2.33. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is a $\mathcal{I}_{\pi^*g}$-open set in X and $int^*(cl(A)) \subseteq B \subseteq A$, then B is a $\mathcal{I}_{\pi^*g}$-open set in X.*
Proof: *Since A is a $\mathcal{I}_{\pi^*g}$-open set, then $X - A$ is a $\mathcal{I}_{\pi^*g}$-closed set. By Theorem 2.26, $cl^*(int(X - A)) \subseteq X - A$ contains no nonempty pre-$\mathcal{I}$-closed set. Since $int(cl^*(A) \subseteq int(cl^*(B)$, we have $X - int(cl^*(X - A) \subseteq X - int(cl^*(X - B)$, which implies that $int^*(cl(X - B) \subseteq int^*(cl(X - A)$ and so $int^*(cl(X - B) - (X - B) \subseteq int^*(cl(X - A) - (X - A))$. Hence B is a $\mathcal{I}_{\pi^*g}$-open set.*

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