International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 19 (2019) pp. 3756-3762

Fuzzy Anti Inner Product Space

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Abstract

In this paper we have defined fuzzy anti inner product space and studied some of its properties. The relative fuzzy anti norm with respect to fuzzy anti inner product function has also been defined. We have proved parallelogram law and polarization identity for fuzzy anti inner product space.

Keywords: fuzzy anti inner product space, α-anti norm, parallelogram law, polarization identity.

1. INTRODUCTION

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The concept of fuzzy set was first introduced by Zadeh [1] in 1965 and thereafter several authors applied it to different branches of pure and applied mathematics. Katsaras was first to introduce the concept of fuzzy norm in 1984, while in 1992 Felbin [3] defined the concept of fuzzy norm in a different way.

Jebril and Samanta [4] introduced the concept fuzzy anti norm on a linear space. The motivation for introducing fuzzy anti norm is to study fuzzy set theory with respect to the non membership function. Several other workers [] have also contributed significantly to the study of fuzzy normed linear space.

Majumdar and Samanta [5] introduced the definition of fuzzy inner product function on a linear space and studied some properties of fuzzy inner product space. A fuzzy norm was induced from the fuzzy inner product and the result involving parallelogram law was derived.

In this paper we have defined fuzzy anti inner product space and some theorems on fuzzy anti inner product have been established. We have also proved parallelogram law and polarization identity in fuzzy anti inner product space.

2. PRELIMINARIES

In this section some definitions and preliminary results are given which will be used in this paper.

Definition 2.1 [4]: Let X be a linear space over a real field F. A fuzzy subset N^* of $X \times R$ is called a fuzzy anti norm on X if and only if it satisfies

(FaN1) for all $t \in R$ with $t \leq 0$, $N^*(x, t) = 1$

(FaN2) for all $t \in R$ with $t > 0$, $N^*(x, t) = 0$ if and only if $x = 0$

(FaN3). for all $t \in R$ with $t > 0$, $N^*(cx, t) = N^*(x, \frac{t}{|c|})$ if $c \neq 0, c \in F$

(FaN4) for all $s,t \in R$ $N^*(x+u, s+t) \leq \max\{N^*(x,s), N^*(u,t)\}$

(FaN5) $N^*(x,t)$ is a non-increasing function of $t \in R$ and $\lim N^*(x,t) = 0$ as $t \to \infty$.

Then N^* is said to be a fuzzy anti-norm on a linear space X and the pair (X, N^*) is called a fuzzy anti- normed linear space or in short Fa-NLS. $N^*(x, t)$ indicates the truth value of the statement, the real number t is less or equal to the norm of x.

Example: Let (X, N) be a normed linear space. Define

$$N^*(x,t) = \begin{cases} \dfrac{\|x\|}{t+\|x\|}, & when \quad t > 0, t \in R, x \in X \\ 1 & when \quad t \leq 0, t \in R, x \in X \end{cases}$$

then (X, N^*) *is an* $Fa-NLS$.

Theorem 2.1 [4]: Let (X, N^*) be a fuzzy anti-normed linear space, assume that for all $t > 0$, $N^*(x, t) < 1$ implies $x = 0$. Define, $\|x\|^*_\alpha = \inf\{t : N^*(x, t) < \alpha, \alpha \in (0,1)\}$ is called α anti-norm. Then $\{\| \|^*_\alpha : \alpha \in (0, 1)\}$ is an decreasing family of $\alpha -$ anti norms on linear space X.

Definition 2.2 [7]: An inner product space is a vector space X along with a function $\langle .,. \rangle$ called an inner product which associates each pair of vectors x, y with a scalar $\langle x, y \rangle$ and which satisfies:

(1) $\langle x, x \rangle = 0$ with equality if and only if x = 0

(2) $\langle x, y \rangle = \langle y, x \rangle$ and

(3) $\langle \alpha x + y, z \rangle = \alpha \langle x, z \rangle + \langle y, z \rangle$

Combining (2) and (3), we also have $\langle x, \alpha y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$. Condition (1) is called positive definite, condition (2) is called symmetric and condition (3) with the note above is called bilinear. Thus an inner product is an example of a positive definite, symmetric bilinear function or form on the vector space X.

Definition 2.3. [7]: Let X be a linear space over the field C of complex number.

Let $\mu: X \times X \times C \to [0,1]$ be a mapping such that the following holds:

(FaIP1) For s,t ∈C , $\mu(x+y, z, |t|+|s|) \geq \min\{\mu(x, z, |t|), \mu(y, z, |s|\}$

(FaIP2) For s,t ∈C , $\mu(x, y, |st|) \geq \min\{\mu(x, x, |s|^2), \mu(y, y, |t|^2)\}$

(FaIP3) For t ∈C, $\mu(x, y, t) = \mu(y, x, \bar{t})$

(FaIP4) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|})$ $\alpha (\neq 0) \in C$, $t \in C$

(FaIP5) $\mu(x, x, t) = 0$, $\forall t \in C \setminus R^+$

(FaIP6) $\mu(x, x, t) = 1$, $\forall t > 0$ iff $x = 0$

(FaIP7) $\mu(x, x, \bullet): R \to [0,1]$ is a monotonic non-decreasing function of R and

$$\lim_{t\to\infty} \mu(x, x, t) = 1 \text{ as } t \to \infty$$

Then μ is said to be fuzzy inner product (FIP in short) function on X and the pair (X, μ) is called a fuzzy inner product space.

3. FUZZY ANTI INNER PRODUCT SPACE

In this section we introduce the definition of fuzzy anti inner product space and its properties.

Definition 3.1: Let X be a linear space over the field C of complex number. Let $\mu^*: X \times X \times C \to [0,1]$ be a mapping such that the following holds:

(FaIP1) For s,t ∈C , $\mu^*(x+y, z, |t|+|s|) \leq \max\{\mu^*(x, z, |t|)\ \mu^*(y, z, |s|\}$

(FaIP2) For s,t ∈C , $\mu^*(x, y, |st|) \leq \max\{\mu^*(x, x, |s|^2), \mu^*(y, y, |t|^2)\}$

(FaIP3) For t ∈C, $\mu^*(x, y, t) = \mu^*(y, x, \bar{t})$

(FaIP4) $\mu^*(\alpha x, y, t) = \mu^*(x, y, \frac{t}{|\alpha|})$ $\alpha (\neq 0) \in C$, $t \in C$

(FaIP5) $\mu^*(x, x, t) = 1$, $\forall t \in C \setminus R^+$

(FaIP6) $\mu^*(x, x, t) = 0$, $\forall t > 0$ iff $x = 0$

(FaIP7) $\mu^*(x, x, \bullet): R \to [0,1]$ is a monotonic non-increasing function of R and

$$\lim_{t\to\infty} \mu^*(x, x, t) = 0 \text{ as } t \to \infty$$

Then μ^* is said to be fuzzy anti inner product (FaIP in short) function on X and the pair (X, μ^*) is called a fuzzy anti inner product space.

Example 3.1: Let (X, $\langle , \rangle$) be a inner product space. Define a function $\mu^*: X \times X \times C \to [0,1]$ by

$$\mu^*(x,y,t) = \frac{|\langle x,y \rangle|}{|t| + |\langle x,y \rangle|} \qquad \forall\, t \in C \setminus \{0\}$$

and ∀ linearly independent x, y ∈X

$= 1$ for t=0 and ∀ linearly independent x, y ∈X

$= 1$ $\forall\, t \in (C \setminus R^+) \cup \{0\}$ ∀ linearly dependent x, y ∈X

$= \dfrac{|\langle x,y \rangle|}{|t| + |\langle x,y \rangle|}$, $\forall\, t \in (R^+ \setminus \{0\})$ ∀ linearly dependent x, y ∈X

Then μ^* is a fuzzy anti inner product function.

Proof : (FaIP1): If at least one of |t| and |s| is zero, then the result is obvious. Let none of |t| and |s| is zero.

Let us assume without loss of generality that,

$$\mu^*(x, z, |t|) \geq \mu^*(y, z, |s|)$$

$$\Rightarrow \frac{|\langle x,z \rangle|}{|t| + |\langle x,z \rangle|} \geq \frac{|\langle y,z \rangle|}{|s| + |\langle y,z \rangle|}$$

$$\Rightarrow \frac{|t| + |\langle x,z \rangle|}{|\langle x,z \rangle|} \leq \frac{|s| + |\langle y,z \rangle|}{|\langle y,z \rangle|}$$

$$\Rightarrow \frac{|t|}{|\langle x,z \rangle|} + 1 \leq \frac{|s|}{|\langle y,z \rangle|} + 1$$

$$\Rightarrow \frac{|t|}{|\langle x,z \rangle|} \leq \frac{|s|}{|\langle y,z \rangle|}$$

$$\Rightarrow |\langle y,z \rangle| \leq \frac{|s|}{|t|} . |\langle x,z \rangle|$$

$$\Rightarrow |\langle \mathrm{x,z} \rangle| + |\langle y,z \rangle| \leq |\langle \mathrm{x,z} \rangle| + \frac{|s|}{|t|} . |\langle x,z \rangle|$$

$$\Rightarrow |\langle \mathrm{x,z} \rangle| + |\langle y,z \rangle| \leq \left(1 + \frac{|s|}{|t|}\right) |\langle x,z \rangle|$$

$$\Rightarrow |\langle x+y,z \rangle| \leq |\langle \mathrm{x,z} \rangle| + |\langle y,z \rangle| \leq \left(1 + \frac{|s|}{|t|}\right) |\langle x,z \rangle|$$

$$\Rightarrow |\langle x+y,z \rangle| \leq \left(\frac{|t| + |s|}{|t|}\right) |\langle x,z \rangle|$$

$$\Rightarrow \frac{|\langle x+y,z \rangle|}{|t| + |s|} \leq \frac{|\langle x,z \rangle|}{|t|}$$

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 19 (2019) pp. 3756-3762
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$$\Rightarrow \frac{|t|}{|\langle x,z\rangle|} \le \frac{|t|+|s|}{|\langle x+y,z\rangle|}$$

$$\Rightarrow \frac{|t|}{|\langle x,z\rangle|}+1 \le \frac{|t|+|s|}{|\langle x+y,z\rangle|}+1$$

$$\Rightarrow \frac{|t|+|\langle x,z\rangle|}{|\langle x,z\rangle|} \le \frac{|t|+|s|+|\langle x+y,z\rangle|}{|\langle x+y,z\rangle|}$$

$$\Rightarrow \frac{|\langle x,z\rangle|}{|t|+|\langle x,z\rangle|} \ge \frac{|\langle x+y,z\rangle|}{|t|+|s|+|\langle x+y,z\rangle|}$$

$\Rightarrow \mu^*(x+y, z, |t|+|s|) \le \max\{\mu^*(x, z, |t|), \mu^*(y, z, |s|)\}$

(FaIP2): If at least one of $|t|$ and $|s|$ is zero, then the result is obvious. Let none of $|t|$ and $|s|$ zero. Let us assume without loss of generality that,

$\mu^*(x, x, |s|^2) \ge \mu^*(y, y, |t|^2)$

$$\Rightarrow \frac{|\langle x,x\rangle|}{|s|^2+|\langle x,x\rangle|} \ge \frac{|\langle y,y\rangle|}{|t|^2+|\langle y,y\rangle|}$$

$$\Rightarrow \frac{|s|^2+|\langle x,x\rangle|}{|\langle x,x\rangle|} \le \frac{|t|^2+|\langle y,y\rangle|}{|\langle y,y\rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|}+1 \le \frac{|t|^2}{|\langle y,y\rangle|}+1$$

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|} \le \frac{|t|^2}{|\langle y,y\rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|\,|\langle x,x\rangle|} \le \frac{|t|^2}{|\langle x,x\rangle|\,|\langle y,y\rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|^2} \le \frac{|t|^2}{\|x\|^2\|y\|^2}$$

$$\Rightarrow \frac{|s|^4}{|\langle x,x\rangle|^2} \le \frac{|s|^2|t|^2}{\|x\|^2\|y\|^2}$$

Taking square root on both side,

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|} \le \frac{|s|\,|t|}{\|x\|\,\|y\|} \le \frac{|st|}{|\langle x,y\rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|} \le \frac{|st|}{|\langle x,y\rangle|}$$

$$\Rightarrow \frac{|s|^2}{|\langle x,x\rangle|}+1 \le \frac{|st|}{|\langle x,y\rangle|}+1$$

$$\Rightarrow \frac{|s|^2+|\langle x,x\rangle|}{|\langle x,x\rangle|} \le \frac{|st|+|\langle x,y\rangle|}{|\langle x,y\rangle|}$$

$$\Rightarrow \frac{|\langle x,x\rangle|}{|s|^2+|\langle x,x\rangle|} \ge \frac{|\langle x,y\rangle|}{|st|+|\langle x,y\rangle|}$$

$\Rightarrow \mu^*(x, x, |s|^2) \ge \mu^*(x, y, |st|)$

$\Longrightarrow \mu^*(x, z, |st|) \le \max\{\mu^*(x, x, |s|^2), \mu^*(y, y, |t|^2)\}$

(FaIP3) for $t \in \mathbb{C}$, $|t| = |\bar{t}|$,

$$\Rightarrow \frac{|\langle y,x\rangle|}{|\bar{t}|+|\langle y,x\rangle|} = \frac{|\overline{\langle x,y\rangle}|}{|t|+|\overline{\langle x,y\rangle}|} = \frac{|\langle x,y\rangle|}{|t|+|\langle x,y\rangle|}$$

(FaIP4): $\alpha(\ne 0) \in \mathbb{C}$, we consider the following cases:

Case (i) If $t(\ne 0) \in \mathbb{C}$ and x,y are linearly independent. Then

(a) If $\langle x,y\rangle = 0$, then

$$\mu^*(\alpha x,y,t) = \frac{|\langle \alpha x,y\rangle|}{|t|+|\langle \alpha x,y\rangle|} = 0$$

$$= \frac{|\langle x,y\rangle|}{\frac{|t|}{|\alpha|}+|\langle x,y\rangle|} = \mu^*\left(x,y,\frac{t}{|\alpha|}\right)$$

(b) If $\langle x,y\rangle \ne 0$ then $\langle \alpha x,y\rangle = \alpha\langle x,y\rangle \ne 0$ *if* $\alpha \ne 0$

$$\mu^*(\alpha x,y,t) = \frac{|\langle \alpha x,y\rangle|}{|t|+|\langle \alpha x,y\rangle|} = \frac{|\alpha|.|\langle x,y\rangle|}{|t|+|\alpha|.|\langle x,y\rangle|}$$

$$= \frac{|\langle x,y\rangle|}{\frac{|t|}{|\alpha|}+|\langle x,y\rangle|}$$

$$= \mu^*\left(x,y,\frac{t}{|\alpha|}\right)$$

Case (ii) If t =0 and x, y are linearly independent, then

$$\mu^*(\alpha x,y,t) = \mu^*\left(x,y,\frac{t}{|\alpha|}\right) = 1$$

case (iii) If $t \in (\mathbb{C} \setminus \mathbb{R}^+) \cup \{0\}$ $\forall$ linearly dependent x, y $\in$X, then

$$\mu^*(\alpha x,y,t) = \mu^*\left(x,y,\frac{t}{|\alpha|}\right) = 1$$

Case (iv) Let $t \in R^+ \setminus \{0\}$. If x, y are linearly dependent then also the result hold as :

$$\mu^*(\alpha x, y, t) = \frac{|\langle \alpha x, y\rangle|}{|t| + |\langle \alpha x, y\rangle|} = \frac{|\alpha|.|\langle x, y\rangle|}{|t| + |\alpha|.|\langle x, y\rangle|}$$

$$= \frac{|\langle x, y\rangle|}{\frac{|t|}{|\alpha|} + |\langle x, y\rangle|}$$

$$= \mu^*\left(x, y, \frac{t}{|\alpha|}\right)$$

(FaIP5): $\mu^*(x,x,t)\dfrac{|\langle x,x\rangle|}{|t| + |\langle x,x\rangle|} = 1 \quad \forall\, t \in c \setminus R^+$.

(FaIP6): Let $x = 0 \Leftrightarrow \langle x,x\rangle = \|x\|^2 = 0$

$$\Leftrightarrow \forall\; t > 0, \mu^*(x,x,t) = \frac{|\langle x,x\rangle|}{|t| + |\langle x,x\rangle|} = 0$$

(FaIP7): $\forall t \in R$ and $t > 0$, $\mu^*(x,x,t) = \dfrac{|\langle x,x\rangle|}{|t| + |\langle x,x\rangle|}$

$$= \frac{1}{\frac{|t|}{|\langle x,x\rangle|} + 1} \to 0 \quad as, \quad t \to \infty$$

And it is obviously monotonic non- increasing.

Theorem 3.1: Let μ^* be a fuzzy anti inner product on X. Then the function $N^*: X \times R \to [0,1]$ defined by

$$N^*(x,t) = \mu^*(x,x,t^2) \;\; \forall\, t \in R \text{ and } t > 0$$
$$= 1 \;\; , \forall\, t \in R \text{ and } t \leq 0$$

is a fuzzy anti norm on X.

Proof:

(FaN1): For $t \in R, t \leq 0$, from definition it follows, $N^*(x,t) = 1$,

(FaN2): $\forall\, t > 0$, $N^*(x,t) = \mu^*(x,x,t^2) = 0$, $\Leftrightarrow x = 0$ (by (FaIP6))

(FaN3): For $t \in R$, with $t > 0$ and $c \neq 0$,

$$N^*(cx,t) = \mu^*(cx,cx,t^2) = \mu^*\left(x, cx, \frac{t^2}{|c|}\right)$$

$$= \mu^*\left(cx, x, \frac{t^2}{|c|}\right) = \mu^*\left(x, x, \frac{t^2}{|c|^2}\right) = \mu^*\left(x, x, \left(\frac{t}{|c|}\right)^2\right)$$

$$= N^*\left(x, \frac{t}{|c|}\right)$$

(FaN4): We have to prove that, $\forall\, s,t \in R$ and $x, u \in X$,

$N^*(x+u, s+t) \leq \max\{N^*(x,s), N^*(u,t)\}$, we consider the following cases:

(a) $s+t < 0$ (b) $s = t = 0$ (c) $s+t > 0$, $s>0$, $t<0$, or $s < 0$, $t>0$, (d) $s>0$, $t>0$ $s+t > 0$

For (a), (b) and (c) the results is obvious. For (d),

$$N^*(x+y, s+t) = \mu^*(x+y, x+y, (s+t)^2)$$
$$= \mu^*(x+y, x+y, s^2 + t^2 + st + st)$$
$$\leq \mu^*(x,x,s^2) \vee \mu^*(y,y,t^2) \vee \mu^*(x,y,st)$$
$$By\ (FaIP1)\ and\ (FaIP2)$$
$$\leq \mu^*(x,x,s^2) \vee \mu^*(y,y,t^2) \quad By\ (FaIP2)$$
$$\leq N^*(x,s) \vee N^*(y,t)$$

(FaN5): From (FaIP7) it follows that $N^*(x,\bullet)$ is monotonic non increasing and tends to 0 as $t \to \infty$.

Theorem 3.2: If μ_1^* and μ_2^* are two fuzzy anti-inner product on X, then $\mu^* = \max\{\mu_1^*, \mu_2^*\}$ is also a fuzzy anti-inner product on X.

Proof: (FaIP1) $\mu^*(x+z, y, |t|+|s|) = \max\{\mu_1^*(x+z, y, |t|+|s|), \mu_2^*(x+z, y, |t|+|s|)\}$

$\leq \max\{\max[\mu_1^*(x, y, |t|), \mu_1^*(z, y, |s|)], \max[\mu_2^*(x, y, |t|), \mu_2^*(z, y, |s|)]\}$

$= \max\{\max[\mu_1^*(x, y, |t|), \mu_2^*(x, y, |t|)], \max[\mu_1^*(z, y, |s|), \mu_2^*(z, y, |s|)]\}$

$= \max[\mu^*(x\ y, |t|), \mu^*(z, y, |s|)]$

$\Longrightarrow \mu^*(x+z, y, |t|+|s|) \leq \max[\mu^*(x, y, |t|), \mu^*(z, y, |s|)]$

(FaIP2) For $s,t \in C$,

$\mu^*(x\ y, |st|) = \max[\mu_1^*(x, y, |st|), \mu_2^*(x, y, |st|)]$

$\leq \max\{\max[\mu_1^*(x, x, |s|^2), \mu_1^*(y, y, |t|^2)], \max[\mu_2^*(x, x, |s|^2), \mu_2^*(y, y, |t|^2)]\}$

$= \max[\mu^*((x, x, |s|^2), \mu^*(x, y, |t|^2)]$

(FaIP3): $\mu^*(x\,y,t) = \max[\mu_1^*(x,y,t), \mu_2^*(x,y,t)]$

$$= \max[\mu_1^*(y,x,\bar{t}), \mu_2^*(y,x,\bar{t})]$$
$$= \mu^*(y,x,\bar{t})$$

(FaIP4): *For* $\alpha \neq 0 \quad \mu^*(\alpha x, y, t) = \max\{\mu_1^*(\alpha x,y,t), \mu_2^*(\alpha x,y,t)\}$

$$= \max\left\{\mu_1^*(x,y,\frac{t}{|\alpha|}), \mu_2^*(x,y,\frac{t}{|\alpha|})\right\}$$
$$= \mu^*(x,y,\frac{t}{|\alpha|})$$

(FaIP5): $\mu^*(x,x,t) = \max\{\mu_1^*(x,x,t), \mu_2^*(x,x,t)\}$

$$= 1, \quad \forall t \in C/R^+$$

(FaIP6): Follows from the definition.

(FaIP7): We know that if $\mu_1^*(x, x,\bullet)$ and $\mu_2^*(x, x,\bullet)$ are monotonic non-increasing in R and tends to 0 as $t\to\infty$, so also is max $\{\mu_1^*, \mu_2^*\}$. Hence the result follows.

Theorem 3.3:

(i) For x, y, z ∈ X and t,s ∈ C

$\mu^*(x, y+z, |t|+|s|) \leq \max\{\mu^*(x,y,|t|), \mu^*(x,z,|s|)\}$

(ii) For $\lambda \in C$ and $\lambda \neq 0$, $\mu^*(x,\lambda y,t) = \mu^*(\lambda x, y, t)$, where $t \in R$

(iii) $\forall t \in R$ and $t > 0$, $\mu^*(0,0,t) \leq \mu^*(x,y,t)$, $\forall\, x,y \in X$

Proof : (i) From (FaIP1) and (FaIP3):

$\mu^*(x, y+z, |t|+|s|) \leq \max\{\mu^*(x,y,|t|), \mu^*(x,z,|s|)\}$

(ii) For $\lambda \in C$ and $\lambda \neq 0$, $\mu^*(x,\lambda y,t) = \mu^*(\lambda y,x,t) = \mu^*\left(y,x,\frac{t}{|\lambda|}\right) = \mu^*\left(x,y,\frac{t}{|\lambda|}\right)$

$$= \mu^*(\lambda x, y, t)$$

(iii) ∀ t∈R and t > 0, $\mu^*(0,0,t) = 0$ …… (1)

and $\mu^*(x,y,t) \leq 1$ ……. (2)

So from (1) and (2) $\mu^*(0,0,t) = 0 \leq \mu^*(x,y,t)$, $\forall$ x, y ∈ X.

Definition 3.2: Let (X, μ^*) be a fuzzy anti-inner product space satisfying

(FaIP8) $\mu^*(x,x,t^2) < 1 \;\; \forall t > 0 \Rightarrow x = 0$

And $\forall\alpha \in (0,1)$, $\|x\|_\alpha^* = \wedge\{t > 0: \mu^*(x,x,t^2) < \alpha, \alpha \in (0,1)\}$ is called the α- anti norm on X generated from μ^*.

(FaIP9) For all x, y ∈ X and p, q ∈ R

$\inf\{\mu^*(x+y, \quad x+y, 2q^2), \mu^*(x-y, \quad x-y, 2p^2)\} \leq \inf\{\mu^*(x,x,p^2), \mu^*(y,y,q^2)\}$

Theorem 3.4 (Parallelogram Law) : Let μ^* be a fuzzy anti-inner product on X satisfying (FaIP8) and (FaIP9). Let α ∈ (0,1) and $\|\;\|_\alpha^*$ be the α-anti norm, generated form the fuzzy anti –inner product μ^*on X. Then, $\|x-y\|_\alpha^{*2} + \|x+y\|_\alpha^{*2} = 2(\|x\|_\alpha^{*2} + \|y\|_\alpha^{*2})$.

Proof : We have,

$\|x-y\|_\alpha^{*2} + \|x+y\|_\alpha^{*2} = \inf\{t^2 : t\in R^+$ and $N^*(x-y,t) < \alpha\} + \inf\{s^2 : s\in R^+$ and $N^*(x+y,t) < \alpha\}$, where N^* is the fuzzy anti norm induced from μ^*.

$= \inf\{t^2 + s^2 : t,s \in R^+$ and $N^*(x-y,t) < \alpha, N^*(x+y,t) < \alpha\}$

$= \inf\{t^2 + s^2 : t,s \in R^+$ and $N^*(x-y,t) \wedge N^*(x+y,t) < \alpha\}$ ………. (A)

Also,

$2(\|x\|_\alpha^{*2} + \|y\|_\alpha^{*2}) = 2\inf\{p^2 : p \in R^+$ and $N^*(x,p) < \alpha\} + 2\inf\{q^2 : q\in R^+$ and $N^*(y,q) < \alpha\}$

$= 2\inf\{p^2 + q^2 : p,q \in R^+$ and $N^*(x,p) \wedge N^*(y,q) < \alpha\}$ ………. (B)

Now by (FaIP9), $N^*(x-y, \sqrt{2}p) \wedge N^*(x+y, \sqrt{2}q) \leq N^*(x,p) \wedge N^*(y,q)$

From (A) and (B) we get,

$\|x-y\|_\alpha^{*2} + \|x+y\|_\alpha^{*2} \leq 2(\|x\|_\alpha^{*2} + \|y\|_\alpha^{*2})$ …….. (C)

Again,

$$2(\|x\|_\alpha^{*2} + \|y\|_\alpha^{*2}) = 2\left(\left\|\frac{x+y}{2} + \frac{x-y}{2}\right\|_\alpha^{*2} + \left\|\frac{x+y}{2} - \frac{x-y}{2}\right\|_\alpha^{*2}\right)$$
$$= \frac{1}{2}(\|(x+y)+(x-y)\|_\alpha^{*2} + \|(x+y)-(x-y)\|_\alpha^{*2})$$
$$\geq \frac{1}{2}[\|x+y\|_\alpha^{*2} + \|x-y\|_\alpha^{*2} + 2\|x+y\|_\alpha^*\|x-y\|_\alpha^* + \|y+x\|_\alpha^{*2} + \|x-y\|_\alpha^{*2} - 2\|y+x\|_\alpha^*\|x-y\|_\alpha^*]$$
$$\geq \frac{1}{2} \times 2[\|x+y\|_\alpha^{*2} + \|x-y\|_\alpha^{*2}]$$
$$= \|x+y\|_\alpha^{*2} + \|x-y\|_\alpha^{*2}$$

Therefore,

$$\|x-y\|_\alpha^{*2} + \|x+y\|_\alpha^{*2} = 2(\|x\|_\alpha^{*2} + \|y\|_\alpha^{*2}).$$

Theorem 3.5 : Let (X, μ^*) be a fuzzy anti-inner product space. If $\mu^*(x,y,t) = \mu^*(z,y,t)$ for all y ∈ X then x and y are dependent.

Proof: By definition,

$\mu^*(x-z, y, t) \leq \max\{\mu^*(x,y,t), \mu^*(-z,y,t)\}$

$$= \max\{\mu^*(x,y,t), \mu^*\left(z,y,\frac{t}{|-1|}\right)$$
$$= \max\{\mu^*(x,y,t), \mu^*(z,y,t)\}$$
$$= \max\{\mu^*(x,y,t)\}$$

So x and y are linearly dependent.

Theorem (3.6): Polarization identity : If x,y,z are the elements of the fuzzy anti inner product space (X, μ^*), then

$$4\mu^*(x,y,st) = \|x+y\|_\alpha^{*2} - \|x-y\|_\alpha^{*2} + i\|x+iy\|_\alpha^{*2} - i\|x-iy\|_\alpha^{*2}.$$

Proof : We know that,

$$\|x+y\|_{\alpha}^{*2} - \|x-y\|_{\alpha}^{*2} + i\|x+iy\|_{\alpha}^{*2} - i\|x-iy\|_{\alpha}^{*2}$$

$$= \mu^{*}\left((x+y),(x+y),(s+t)^{2}\right) - \mu^{*}\left((x-y),(x-y),(s-t)^{2}\right)$$
$$+ i\mu^{*}\left((x+iy),(x+iy),(s+t)^{2}\right) - i\mu^{*}\left((x-iy),(x-iy),(s-t)^{2}\right)$$

$$= \mu^{*}\left((x+y),(x+y),s^{2}+st+st+t^{2}\right) - \mu^{*}\left((x-y),(x-y),s^{2}-st-st+t^{2}\right)$$
$$+ i\mu^{*}\left((x+iy),(x+iy),s^{2}+st+st+t^{2}\right) - i\mu^{*}\left((x-iy),(x-iy),s^{2}-st-st+t^{2}\right)$$

$$= \mu^{*}(x,x,s^{2}) + \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + \mu^{*}(y,y,t^{2}) - \mu^{*}(x,x,s^{2}) - \mu^{*}(x,-y,-st)$$
$$- \mu^{*}(-y,x,-st) - \mu^{*}(-y,-y,t^{2}) + i\mu^{*}(x,x,s^{2}) + i\mu^{*}(x,iy,st) + i\mu^{*}(iy,x,st)$$
$$+ i\mu^{*}(iy,iy,t^{2}) - i\mu^{*}(x,x,s^{2}) - i\mu^{*}(x,-iy,-st) - i\mu^{*}(-iy,x,-st) - i\mu^{*}(-iy,-iy,t^{2})$$

$$= \mu^{*}(x,x,s^{2}) + \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + \mu^{*}(y,y,t^{2}) - \mu^{*}(x,x,s^{2}) - \mu^{*}\left(x,y,\frac{-st}{|-1|}\right)$$
$$- \mu^{*}\left(y,x,\frac{st}{|-1|}\right) - \mu^{*}(y,y,t^{2}) + i\mu^{*}(x,x,s^{2}) + i\mu^{*}(x,iy,st) + i\mu^{*}(iy,x,st)$$
$$+ i\mu^{*}(iy,iy,t^{2}) - i\mu^{*}(x,x,s^{2}) - i\mu^{*}\left(x,iy,\frac{-st}{|-1|}\right) - i\mu^{*}\left(iy,x,\frac{-st}{|-1|}\right) - i\mu^{*}(iy,iy,t^{2})$$

$$= \mu^{*}(x,y,st) + \mu^{*}(y,x,st) - \mu^{*}(x,y,-st) - \mu^{*}(y,x,-st) + i\mu^{*}(x,iy,st) + i\mu^{*}(iy,x,st)$$
$$+ i\mu^{*}(x,iy,st) + i\mu^{*}(iy,x,st)$$

$$= \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + i\mu^{*}(x,iy,st) + i\mu^{*}(iy,x,st)$$
$$+ i\mu^{*}(x,iy,st) + i\mu^{*}(iy,x,st)$$

$$= \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + i(-i)\mu^{*}(x,y,st) + i(i)\mu^{*}(y,x,st)$$
$$+ i(-i)\mu^{*}(x,y,st) + i(i)\mu^{*}(y,x,st)$$

$$= \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + \mu^{*}(x,y,st) + \mu^{*}(y,x,st) + \mu^{*}(x,y,st) - \mu^{*}(y,x,st)$$
$$+ \mu^{*}(x,y,st) - \mu^{*}(y,x,st)$$

$$= 4\mu^{*}(x,y,st)$$

Hence the result.

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