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On the Stabilization of Jensen Equations through Fixed and Direct Methods

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Abstract

In last few decades, the stability of functional equations has found a celebrated place in functional analysis. This article deals with the stabilization of Jensen type functional equation in multi-Banach spaces through fixed point and direct methods.

Keywords and Phrases: - Fixed Point Method, Direct Methrod, Jensen equations, Multi-Normed space

1. INTRODUCTION

D. H. Hyer [1] was first who gave the first affirmative answer of the stability of functional equations in Banach space. In 1978, Th. M. Rassias [11] presented the more efficient version of D. H. Hyer's result and proved the stability for approximately linear mappings. P. Gavruta [6] a great mathematician further extended the work of P. Gavruta [11] in 1994 and examined the stability results for the bound $\varepsilon(\|x\|^p + \|y\|^p)$. In last few decades, the stability of various functional equations has been introduced on various spaces such as Random Normed space, Banach space, Normed space, IRN space, etc [3, 4, 5, 7, 8, 9, 10].

In this article the stability of following Jensen functional equations are studied in multi-normed space using fixed point and direct approach:

$$Z(g) = 2g\left(\frac{(u \pm v)}{2}\right) - g(u) - g(v) \qquad (1)$$

and $Z^1(g) = g(mu \pm mv) - 2m^2 g(u) - 2m^2 g(v)$ (2)

This article is divided into five sections. Section 2 deals with the basic terminologies of multi – Banach spaces which are used in further sections. In section 3, the stability of Jensen functional equations is studied using direct method and in Section 4 the stability is proved using fixed approach.

2. PRELIMINARIES

This section presents the basic terminologies of multi-Banach space. Throughout this section we consider F^t as the linear space $F \oplus F \oplus \ldots \oplus F$ consisting of t-tuples, zero is taken as zeroth element for F and F^t, $\mathbf{M}_t = \{1, 2, \ldots, t\}$ and P_t denotes the permutation group with t symbols.

Definition 1 [2]. A sequence $\{\|\cdot\|_t : t \in M\}$, defined on $\{F^t : t \in M\}$ is said to be a multi-norm on F^t, where $\|x\|_1 = \|x\|$ and satisfies the following conditions for $t \geq 2$:

(M1) $\|(u_{\partial(1)}, \ldots, u_{\partial(t)})\|_t = \|(u_1, \ldots, u_t)\|_t$, where $\partial \in P_t$, and $u_1, \ldots, u_t \in F$;

(M2) $\|(\gamma_1 u_1, \ldots, \gamma_t u_t)\|_t \leq (\max_{i \in M_t} |\gamma_i|)\ \|(u_1, \ldots, u_t)\|_t$, where $\gamma_1, \ldots, \gamma_t \in C$, and $u_1, \ldots, u_t \in F$;

(M3) $\|(u_1, \ldots, u_{t-1}, 0)\|_t = \|(u_1, \ldots, u_{t-1})\|_{t-1}$, where $u_1, \ldots, u_{t-1} \in F$;

(M4) $\|(u_1, \ldots, u_{t-1})\|_t = \|(u_1, \ldots, u_{t-1})\|_{t-1}$, where $u_1, \ldots, u_{t-1} \in F$

Then, the pair $(F^t, \|\cdot\|_t)$ is named as multi-normed space.

Also, we have the following properties of multi norm space [2].

(a) $\|(u, \ldots, u)\|_t = \|u\|$, when $r \in F$,

(b) $\max_{i \in M_t} \|u_i\| \leq \|(u_1, \ldots, u_t)\|_t \leq \sum_{i=1}^{t} \| u_i \| \leq t \max_{i \in M_t} \|u_i\|$, when $u_1, \ldots, u_t \in F$.

Thus, we have if the F is a complete normed space, then the pair $(F^t, \|\cdot\|_t)$ is also a complete normed space, hence the pair $((F^t, \|\cdot\|_t) : t \in \mathbf{M})$ is called a multi-Banach space.

Definition 2. [2]. For a multi-normed space $(F^t, \|\cdot\|_t)$, a sequence $<u_n>$ in F is said to be a multi-null sequence if for a positive real number $\varepsilon > 0$, $\exists$ positive integer *m* such that

$$\sup_{t \in N} \|(u_n, \ldots, u_{n+t-1})\|_t \leq \varepsilon, \quad n \geq m.$$

Also the sequence $<u_n>$ in F is said to be multi convergent to a point u in F if

$$\lim_{n \to \infty} u_n = u.$$

Lemma 3. [2] Let us consider $(u_1, u_2, \ldots, u_t) \in F^t$ and let $< u_n^j >_{n \in N}$ be a real sequence in F such that $\lim_{n \to \infty} u_n^j = u_j$, then the relation

$$\lim_{n \to \infty} < u_n^1 - v_1, \ldots, u_n^k - v_t > = < u_1 - v_1, \ldots, u_t - v_t > \quad \text{holds.}$$

for each tuple $(v_1, v_2, \ldots, v_t) \in F^t$.

3. STABILIZATION OF JENSEN EQUATIONS VIA DIRECT APPROACH

This section deals with the stabilization of equation (1) and (2).

Theorem 1. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If g is a mapping from G into F with g(0) = 0 such that

$$\sup_{t\in N} \| Z(g(u_1, v_1),\dots, g(u_t, v_t))\|_t \le \eta \qquad (3)$$

where $u_1,\dots,u_t$, and $v_1,\dots,v_t \in G$. Then, $Q : G\to F$ is a unique function which satisfies

$$\sup_{t\in N} \|(g(u_1) - Q(u_1),\ g(u_2) - Q(u_2),\dots\dots, g(u_t) - Q(u_t))\|_t \le \frac{\eta}{3} \qquad (4)$$

Proof: Taking $v_1, v_2,\dots,v_t = 0$ and also changing $u_1, u_1,\dots, u_t$ with $2u_1, 2u_2, \dots, 2u_t$ in inequality (3), we have

$$\sup_{t\in N}\left\|\left(\frac{g(2u_1)}{4}-g(u_1),\frac{g(2u_2)}{4}-g(u_2),..,\frac{g(2u_t)}{4}-g(u_t)\right)\right\|_t \le \frac{\eta}{4} \qquad (5)$$

Again changing $u_1, u_1,\dots, u_t$ with $2u_1, 2u_2, \dots, 2u_t$ in inequality (5), and then dividing with 4 throughout (5), we can write

$$\sup_{t\in N}\left\|\left(\frac{g(2^2u_1)}{4^2}-g(u_1),\frac{g(2^2u_2)}{4^2}-g(u_2),...,\frac{g(2^2u_t)}{4^2}-g(u_t)\right)\right\|_t \le \frac{\eta}{4^2}+\frac{\eta}{4}$$

Proceeding in this way n-times, we obtain

$$\sup_{t\in N}\left\|\left(\frac{g(2^nu_1)}{4^n}-g(u_1),\frac{g(2^nu_2)}{4^n}-g(u_2),...,\frac{g(2^nu_t)}{4^n}-g(u_t)\right)\right\|_k \le \sum_{i=0}^{n-1}\frac{\eta}{4^{i+1}}$$

$$\le \sum_{i=0}^{\infty}\frac{\eta}{4^{i+1}} \qquad (6)$$

Now, we prove $< g(2^n u)/4^n >$ is convergent, therefore, changing $u_1, u_2,\dots, u_t$ with $u, 2u,\dots, 2^{t-1}u$ and fix $u\in G$ we get

$$\sup_{t\in N}\left\|\left(\frac{g(2^n u)}{4^n}-\frac{g(2^m u)}{4^m},\dots,\frac{g(2^{n+t-1}u)}{4^{n+t-1}}-\frac{g(2^{m+t-1}u)}{4^{m+t-1}}\right)\right\|_t$$

$$\le \sup_{t\in N}\left\|\left(\frac{g(2^n u)}{4^n}-\frac{g(2^m u)}{4^m},\dots,\frac{1}{4^{t-1}}\left(\frac{g(2^n(2^{t-1}u)}{4^n}-\frac{g(2^m(2^{t-1}u))}{4^m}\right)\right)\right\|_t$$

Then, from (M3), we obtain

$$\le \sup_{t\in N}\left\|\left(\frac{g(2^n u)}{4^n}-\frac{g(2^m u)}{4^m},\dots,\frac{g(2^n(2^{t-1}u)}{4^m}-\frac{g(2^m(2^{t-1}u)}{4^n}\right)\right\|_t$$

$$\le \sum_{i=m}^{n-1}\frac{\eta}{4^{i+1}} \qquad (7)$$

As n tends to infinity the inequality (7) gives $< g(2^n u)/4^n >$ is a Cauchy sequence. Further, as F is a Banach space, it is also convergent in F, such that

$$\lim_{n\to\infty}\frac{g(2^n u)}{4^n} = Q(u) \qquad (8)$$

Also, Equation (6) implies (4) as $n\to\infty$,

$$\sup_{t\in N}\|(Q(u_1) - g(u_1), Q(u_2) - g(u_2),\dots, Q(u_t) - g(u_t))\|_t$$

$$\le \sum_{i=0}^{\infty}\frac{\eta}{4^{i+1}} \le \frac{\eta}{3}$$

To show $Q : G\to F$ is quadratic, therefore, substituting $u_1 = u_2 = ,\dots, = u_t = 2^n u$ and $v_1, = , \dots , = v_k = 2^n v$ in (3) and then dividing throughout with 4^n, we obtain

$$\left\|\frac{1}{4^n}\, g\left(\frac{2^n(u-v)}{2}\right)+\frac{1}{4^n}\, g\left(\frac{2^n(u+v)}{2}\right)-\frac{g(2^n u)+g(2^n v)}{4^n}\right\| \le \frac{\eta}{4^n}$$

As n tends to infinity, we obtain

$$Q\left(\frac{u+v}{2}\right)+Q\left(\frac{u-v}{2}\right)-\frac{Q(u)+Q(v)}{2} = 0$$

Thus, the mapping $Q : G\to F$ is quadratic.

Now, to examine Q is unique. Let $Q^1 : G\to F$ be second mapping satisfying (3), such that

$$\| Q'(u)-Q(u) \| \le \frac{1}{4^n}\|Q'(2^n u) - Q(2^n u)\|$$

$$\le \frac{1}{4^n}\|g(2^n u) - Q(2^n u)\| + \frac{1}{4^n}\|Q'(2^n u) - g(2^n u)\|$$

$$\le \frac{2\eta}{3.4^n}$$

Thus, we have $Q = Q'$. This completes the proof.

Corollary 2. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $g : G\to F$ is a function with g(0) = 0 such that

$$\sup_{t\in N}\|Zg(u_1, v_1),\dots, Zg(u_t, v_t)\|_t \le \psi(u_1, v_1,\dots, u_t, v_t)$$

where $u_1,\dots,u_t$, $v_1,\dots,v_t\in G$, and $\psi : G^{2t} \to [0, \infty)$. Then, $Q : G\to F$ is a unique function which satisfies

$$\sup_{t\in N}\|g(u_1) - Q(u_1),\dots, g(u_t) - Q(u_t)\|_t$$

$$\le \sum_{i=0}^{\infty}\frac{1}{4^{i+1}}\psi(2^i u_1, 0 ,\dots, 2^i u_t,0)$$

Proof: Similar to Theorem 1.

Corollary 3. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $g : G \to F$ is a function with $g(0) = 0$ and also $0<p < 2$, $\omega \geq 0$ such that

$$\sup_{t\in N} \|Zg(u_1, v_1), \dots, Zg(u_t, v_t)\|_t \leq \omega(\|u_1\|^p + \|v_1\|^p, \dots, \|u_t\|^p + \|v_t\|^p) \quad (9)$$

where $u_1,\dots,u_t$, and $v_1,\dots,v_t \in G$. Then, $Q: G\to F$ is a unique function which satisfies

$$\sup_{t\in N} \|(g(u_1) - Q(u_1), \dots, g(u_t) - Q(u_t))\|_t \leq \frac{\omega}{4-2^p}(\|u_1\|^p, \dots, \|v_t\|^p) \quad (10)$$

for all $u_1, \dots, u_t \in G$.

Proof: Similar to Theorem 1.

Theorem 4. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $g : G\to F$ is a function with $g(0) = 0$ such that

$$\sup_{t\in N} \|Z^1g(u_1, v_1), Z^1g(u_2, v_2), \dots, Z^1g(u_t, v_t)\|_t \leq \eta \quad (11)$$

where $u_1,\dots,u_t$, and $v_1,\dots,v_t \in G$. Then, $Q : G\to F$ is a unique function which satisfies

$$\sup_{t\in N} \|(g(u_1) - Q(u_1), g(u_1) - Q(u_1), \dots, g(u_t) - Q(u_t))\|_t \leq \frac{\eta}{2(m^2-1)} \quad (12)$$

for all $u_1, \dots, u_t \in G$.

Proof: Taking $v_1,\dots, v_t = 0$ in the inequality (11), we have

$$\sup_{k\in N} \left\| \left(\frac{g(mu_1)}{m^2} - g(u_1), \frac{g(mu_2)}{m^2} - g(u_2), \dots, \frac{g(mu_t)}{m^2} - g(u_t) \right) \right\|_t \leq \frac{\varepsilon}{2m^2} \quad (13)$$

And letting $u_1,\dots, u_t$ with $mu_1, mu_2, \dots, mu_t$ and then dividing throughout with m^2 in (13), we obtain

$$\sup_{t\in N} \left\| \left(\frac{g(m^2u_1)}{m^4} - g(u_1), \frac{g(m^2u_2)}{m^4} - g(u_2), \dots, \frac{g(m^2u_t)}{m^4} - g(u_t) \right) \right\|_t \leq \frac{\varepsilon}{2m^4} + \frac{\varepsilon}{2m^2}$$

Then, from induction on n, we get

$$\sup_{t\in N} \left\| \left(\frac{g(m^nu_1)}{m^{2n}} - g(u_1), \frac{g(m^nu_2)}{m^{2n}} - g(u_2), \dots, \frac{g(m^nu_t)}{m^{2n}} - g(u_t) \right) \right\|_t \leq \frac{1}{2}\sum_{i=1}^{n} \frac{\varepsilon}{m^{2i}} \quad (14)$$

Now, we prove $< g(m^n u)/m^{2n} >$ is convergent, therefore, changing $u_1, u_2, \dots, u_t$ with $u, mu, \dots, m^{2(t-1)}u$ and fix $u\in G$ we get

$$\sup_{t\in N} \left\| \left(\frac{g(m^nu)}{m^{2n}} - \frac{g(m^mu)}{m^{2m}}, \dots, \frac{g(m^{n+t-1}u)}{m^{2(n+t-1)}} - \frac{g(m^{m+t-1}u)}{m^{2(m+t-1)}} \right) \right\|_t$$

$$\leq \sup_{t\in N} \left\| \left(\frac{g(m^nu)}{m^{2n}} - \frac{g(m^mu)}{m^{2m}}, \dots, \frac{1}{m^{2(t-1)}}\left(\frac{g(m^{n+t-1}u)}{m^{2n}} - \frac{+(m^m(m^{t-1}u)}{m^{2m}} \right) \right) \right\|_t$$

Then, from condition (M3), we obtain

$$\leq \frac{1}{2}\sum_{i=m+1}^{n} \frac{\eta}{m^{2i}} \quad (15)$$

As n tends to infinity the inequality (15) gives $< g(m^n u)/m^{2n} >$ is a Cauchy sequence. Further, as F is a Banach space, it is also convergent in F, such that

$$\lim_{n\to\infty} \frac{g(m^nu)}{m^{2n}} = Q(u) \quad (16)$$

Also, Equation (14) implies (12) as $n\to\infty$,

$$\sup_{t\in N} \|(Q(u_1) - g(v_1), \dots, Q(u_t) - g(v_t))\|_t \leq \frac{\eta}{2(m^2-1)}$$

Remaining part is similar to Theorem 1.

Corollary 5. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. Consider $\psi : G^{2n} \to [0, \infty)$ be a map with some $0 < \beta < m^2$ defined as

$$\psi(mu_1, 0, \dots, mu_t, 0) \leq \beta\psi(u_1, 0, \dots, u_k, 0)$$

If $g : G\to F$ is a function with $g(0) = 0$ such that

$$\sup_{t\in N} \|Z^1(g(u_1, v_1), \dots, Z^1(g(u_t, v_t))\|_t \leq \psi(u_1, v_1, \dots, u_t, v_t)$$

Then, $Q : G\to F$ is a unique function which satisfies

$$\sup_{t\in N} \|(g(u_1) - Q(u_1), \dots, g(u_t) - Q(u_t))\| \leq \frac{\psi(u_1, 0, \dots, u_t, 0)}{2(m^2-\beta)}$$

for all $u_1, \dots, u_t \in G$.

Corollary 6. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $g : G\to F$ is a function with $g(0) = 0$ and also $0<p < 2$, $\omega \geq 0$ such that

$$\sup_{t\in N} \|Z^1(g(u_1, v_1), \dots, Z^1(g(u_t, v_t))\|_t \leq \omega(\|u_1\|^p + \|v_1\|^p, \dots, \|u_t\|^p + \|v_t\|^p)$$

where $u_1,\dots,u_t$, and $v_1,\dots,v_t \in G$. Then, $Q : G\to F$ is a unique function which satisfies

$$\sup_{t\in N} \|(g(u_1) - Q(u_1), \dots, g(u_t) - Q(u_t))\| \leq \frac{\omega}{2(m^2-m^p)}(\|u_1\|^p, \dots, \|u_t\|^p)$$

for all $u_1,\dots, u_t \in G$.

4 STABILIZATION OF JENSEN EQUATIONS VIA FIXED POINT METHOD

Theorem 7 (Fixed Point Alternative). Let us consider T: P→P be a strictly contractive function and (P, d) be a complete generalized metric space, such that

$$d(Tu, Tv) \leq £d(u, v), \quad \text{for all } u, v\in P,$$

for some number $£ \leq 1$. Then, for $u\in P$, we have either

$$d(T^n u, T^{n+1}u) = \infty, \quad \text{for all } n \geq 0,$$

or $\quad d(T^n u, T^{n+1}u) < \infty, \quad \text{for all } n \geq n_0,$

where $n_0 \geq 0$.

Furthermore, if $d(T^n u, T^{n+1}u) < \infty$ holds, then we can write

(i) There exists a sequence $<T^n u>$ which will be convergent to $z\in T$;

(ii) Where the fixed point z is unique T, such that

$$W = \{z\in P \mid d(T^{n_0} u, v) < \infty\}$$

(iii) $d(u, z) \leq 1/(1 - £)\, d(u, Tu)$, for all $u\in W$.

Lemma 8. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $\psi:F^n \to [0, \infty)$ and $0 < \beta < m^2$ then

$$\varphi(mu_1, mu_2, \ldots, mu_t) \leq \beta\psi(u_1, u_2, \ldots, u_t)$$

for each $u_1, u_2, \ldots, u_t\in G$. Let us define a generalized metric space on a set $R=\{f : G\to F : f(0)=0\}$, such that

$$d(g, f) = \inf\{ \sup_{t\in N} \|(g(u_1)-f(u_1),\ldots, g(u_t)-f(u_t))\|_t \leq \gamma\psi(u_1, u_2, \ldots, u_t)\ ;\ \gamma\in(0, \infty)\}$$

for all $u_1,\ldots, u_t\in G$. Then, (R, d) is a complete generalized metric space. Further, let us define $T_0 : R\to R$ as

$$\frac{f(m^n u)}{m^{2n}} = T_0 f(u)$$

for all $f\in R$ which is always a strictly contractive mapping.

Proof: Let g, $f\in R$, be two mappings and $\gamma\in(0, \infty)$ be a constant which satisfies $d(g, f) \leq \gamma$. Then, from definition of metric space d, we get

$$\sup_{t\in N} \|(g(u_1) - f(u_1), g(u_2) - f(u_2),\ldots, g(u_t) - f(u_t))\| \leq \gamma\psi(u_1, \ldots, u_t),$$

Thus, we have

$$\sup_{t\in N} \|(T_0 g(u_1) - T_0 f(u_1), T_0 g(u_2) - T_0 f(u_2), \ldots, T_0 f(u_t) - T_0 f(u_t))\|$$

$$\leq \left\|\left(\frac{g(m^n u_1)}{m^{2n}} - \frac{f(m^n u_1)}{m^{2n}}, \frac{g(m^n u_2)}{m^{2n}} - \frac{f(m^n u_2)}{m^{2n}}, \ldots, \frac{g(m^n u_t)}{m^{2n}} - \frac{f(m^n u_t)}{m^{2n}}\right)\right\|$$

$$\leq \frac{1}{m^{2n}} \|(g(m^n u_1)-f(m^n u_1), g(m^n u_1)-f(m^n u_1),\ldots, g(m^n u_t)-f(m^n u_t))\|$$

$$\leq \frac{\beta^n}{m^{2n}} \gamma\psi(u_1,\ldots, u_t)\ \forall\ u_1, u_2,\ldots, u_t\in G.$$

Then, it gives

$$d(T_0 g, T_0 f) \leq \frac{\beta^n}{m^{2n}} d(g, f),\ \forall\ g, f\in R.$$

Thus, T_0 is a strictly contractive mapping.

Theorem 9. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $g : G\to F$ is a function with $g(0) = 0$, such that

$$\sup_{t\in N} \|Z^1 g(u_1, v_1), Z^1 g(u_2, v_2),\ldots, Z^1 g(u_t, v_t)\|_t \leq \eta \tag{17}$$

Then, $Q : G\to F$ is a unique function which satisfies

$$\sup_{t\in N} \|(g(u_1) - Q(u_1),\ldots, g(u_t)-Q(u_t))\|_t \leq \frac{m^{2n}\eta}{(2m^2-2)(m^{2n}-1)} \tag{18}$$

for all $u_1,\ldots, u_t, v_1,\ldots, v_t\in G$.

Proof: Taking $v_1,\ldots, v_t = 0$ in the inequality (17), we have

$$\sup_{t\in N}\left\|\left(\frac{g(mu_1)}{m^2} - g(u_1), \frac{g(mu_2)}{m^2} - g(u_2), \ldots, \frac{g(mu_t)}{m^2} - g(u_t)\right)\right\|_t \leq \frac{\eta}{2m^2} \tag{19}$$

And letting $u_1,\ldots, u_t$ with $mu_1, mu_2, \ldots, mu_t$ and then dividing throughout with m^2 in (19), we obtain

$$\sup_{t\in N}\left\|\left(\frac{g(m^2 u_1)}{m^4} - g(u_1), \frac{g(m^2 u_2)}{m^4} - g(u_2), \ldots, \frac{g(m^2 u_t)}{m^4} - g(u_t)\right)\right\|_t \leq \frac{\eta}{2m^4} + \frac{\eta}{2m^2} \tag{20}$$

Taking induction on 'n', we have

$$\sup_{t\in N}\left\|\left(\frac{g(m^n u_1)}{m^{2n}} - g(u_1), \frac{g(m^n u_2)}{m^{2n}} - g(u_2), \ldots, \frac{g(m^n u_t)}{m^{2n}} - g(u_t)\right)\right\|_t$$

$$\leq \frac{\eta}{2}\sum_{i=1}^{n}\frac{1}{m^{2i}}$$

$$\leq \frac{\eta}{2}\sum_{i=1}^{\infty}\frac{1}{m^{2i}} \tag{21}$$

Let us define a generalized metric space on a set $R=\{f : G\to F : f(0)=0\}$, such that

$$d(g, f) = \inf\{ \sup_{t\in N} \|(g(u_1)-f(u_1),\ldots, g(u_t)-f(u_t))\|_t \leq \gamma\varphi(u_1, u_2, \ldots, u_t)\ ;\ \gamma\in(0, \infty)\}$$

Then, (R, d) is a complete generalized metric space. Further, let us define $T_0 : R\to R$ as $\frac{f(m^n u)}{m^{2n}} = T_0 f(u)$ for all $f\in R$. To prove T_0 is a strictly contractive mapping, let us consider Let g, $f\in R$, be two mappings and $\gamma\in(0, \infty)$ be a constant which satisfies $d(g, f) \le \gamma$. Then, from definition of metric space d, we get

$$\sup_{t\in N} \|(g(u_1) - f(u_1), g(u_2) - f(u_2),\ldots, g(u_t) - f(u_t))\| \le \gamma.$$

Then, we get

$$\sup_{t\in N} \|(T_0g(u_1) - T_0f(u_1),\ T_0g(u_2) - T_0f(u_2),\ \ldots,\ T_0f(u_t) - T_0f(u_t))\|$$

$$\le \left\|\left(\frac{g(m^n u_1)}{m^{2n}} - \frac{f(m^n u_1)}{m^{2n}}, \frac{g(m^n u_2)}{m^{2n}} - \frac{f(m^n u_2)}{m^{2n}}, \ldots, \frac{g(m^n u_t)}{m^{2n}} - \frac{f(m^n u_t)}{m^{2n}}\right)\right\|$$

$$\le \frac{\gamma}{m^{2n}}$$

Thus, we examine that

$$d(T_0g, T_0f) \le \frac{\gamma}{m^{2n}} \le \frac{1}{m^{2n}} d(g, f) \tag{22}$$

Now, from inequality (21), we get

$$d(T_0g, g) \le \frac{\eta}{2(m^2-1)}.$$

Thus, T_0 is a strictly contractive.

Now we prove the existence of quadratic mapping Q: $G\to F$ through fixed point alternative as follows:

(i) $m^{2n}Q(u)=Q(m^n u)$, $\forall\ u\in G$.

(ii) $d(T_0^n g, Q)$ tends to 0, for $u\in G$ implies

$$\lim_{n\to\infty} \frac{g(m^n u)}{m^{2n}} = Q(u) \qquad \forall\ u\in G. \tag{23}$$

(iii) $\frac{1}{1-\gamma} d(T_0g, g) \ge d(g, f)$ implies

$$\frac{m^{2n}\varepsilon}{(2m^2-2)(m^{2n}-1)} \ge \frac{1}{1-\frac{1}{m^{2n}}} d(T_0g, g) \ge d(g, f) \tag{24}$$

To show that the function Q : $G\to F$ is quadratic, therefore, substituting $u_1 = u_2 = ,\ldots, = u_t = m^n u$ and $v_1, = , \ldots , = v_t = m^n v$ in (17) and then dividing throughout with m^{2n}, we obtain

$$\frac{1}{m^{2n}} \sup_{t\in N} \|Zg(m^n u, m^n v),\ldots, Zg(m^n u, m^n u)\|_t$$

$$\le \lim_{n\to\infty} \frac{1}{m^{2n}} \|Zg(m^n u, m^n v)\|$$

$$\le \lim_{n\to\infty} \frac{\eta}{m^{2n}} = 0$$

Thus, Q is a quadratic. The uniqueness of the theorem can be proved similar to above theorem. Hence proved.

Theorem 10. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If $\psi : F^{2t} \to[0, \infty)$ and $0 < \beta < m^2$, then

$$\varphi(mu_1, mv_1,\ldots, mu_t, mv_t) \le \beta\psi(u_1, v_1,\ldots, u_t, v_t) \tag{25}$$

Let g : $G\to F$ be a function with g(0) = 0 such that

$$\|Zg(u_1, v_1) ,\ldots, Zg(u_t, v_t)\|_t \le \psi(u_1, v_1,\ldots, u_t, v_t) \tag{26}$$

for all $u_1,\ldots,u_t$, $v_1,\ldots,v_t\in G$. Then, Q: $G\to F$ is a unique quadratic system, such that

$$\|(g(u_1)-Q(u_1),\ldots, g(u_t)-Q(u_t))\|_t \le \frac{m^{2n}}{\beta(m^{2n}-\beta^n)(m^2-\beta)} \psi(u_1, 0,\ldots, u_t, 0) \tag{27}$$

Proof: Taking $v_1, v_2,\ldots,v_t = 0$ in the inequality (25), then we have

$$\|(g(mu_1)-m^2g(u_1),\ldots, g(mu_t)-m^2g(u_t))\|_t \le \frac{1}{2} \psi(u_1, 0 ,\ldots, u_t, 0) \tag{28}$$

Substituting u = mu, we get

$$\|(g(m^2u) - m^4g(u),\ldots, g(m^2u_t) - m^2g(u_t))\|_t$$

$$\le \frac{1}{2} \psi(u_1, 0 ,\ldots, u_t, 0)+ \frac{m^2}{2} \psi(mu_1, 0 ,\ldots, mu_t, 0)$$

Applying induction on n, we found

$$\left\|\frac{g(m^n u_1)}{m^{2n}} - g(u_1),\ldots\ldots, \frac{g(m^n u_t)}{m^{2n}} - g(u_t)\right\|_t \le \frac{1}{2}\sum_{i=0}^{n-1} \frac{1}{m^{2(i+1)}} \psi(m^i u_1, 0 ,\ldots, m^i x_t, 0)$$

$$\le \frac{1}{2}\sum_{i=0}^{n-1} \frac{\beta^i}{m^{2(i+1)}} \psi(u_1, 0 ,\ldots., u_t, 0) \qquad \text{(using (25))}$$

$$\le \frac{1}{2(m^2-\beta)} \psi(u_1, 0,\ldots, u_t, 0) \tag{29}$$

Let d be a generalized metric space on a set $R=\{f : G\to F : f(0)=0\}$, then, we define $T_0 : R\to R$ as

$$\frac{f(m^n u)}{u^{2n}} = T_0 f(u)$$

where T_0 is a strictly contractive mapping, let us consider Let g, $f\in R$, be two mappings and $\gamma\in(0, \infty)$ be a constant which satisfies $d(g, f) \le \gamma$. Then, from definition of metric space d, we get

$$\sup_{t\in N} \|(g(u_1)-f(u_1),\dots, g(u_t)-f(u_t))\|_t \le \gamma\psi(u_1,\dots,u_t)$$

From (29), we have

$$d(T_0 g, g) \le \frac{1}{2(m^2-\beta)}\psi(u_1, 0,\dots, u_t, 0)$$

Now from fixed point theorem, the unique fixed point for the system T_0 is studied, i. e., there exists a unique system Q: G→F such that $m^{2n}Q(u) = Q(m^n u)$. Further, as $d(T_0^n g, Q)$ tends to 0, gives

$$\lim_{n\to\infty}\frac{g(m^n u)}{u^{2n}} = Q(u) \text{ for all } u\in G$$

Hence, $$d(g, Q) \le \frac{1}{1-\lambda} d(T_0 g, g)$$

Which implies

$$d(g, Q) \le \frac{m^{2n}}{2(m^{2n}-\beta^n)(m^2-\beta)}\psi(u_1, 0,\dots, u_t, 0)$$

Replacing $u_1, u_2,\dots,u_t$ with $m^n u$ and $v_1, v_2,\dots, v_t$ with $m^n v$ in the inequality (26) and then dividing throughout by m^{2n}, then we get

$$\left\|\frac{g(m^n(mu+mv))}{m^{2n}}+\frac{g(m^n(mu-mv))}{m^{2n}}-\frac{2m^2 g(m^n u)}{m^{2n}}-\frac{2m^2 g(m^n v)}{m^{2n}}\right\|_t \le \frac{1}{m^{2n}}\psi(m^n u, m^n v,\dots, m^n u, m^n u).$$

Limiting as n→∞, we get

$$Q(mu+mv)+Q(mu-mv) = 2m^2 Q(mu)+2m^2 Q(mv)$$

Hence Q is a unique fixed point for T_0. Hence proved.

Corollary 11. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If g: G→F is a function with g(0)=0 and $\gamma \ge 0$ such that

$$\|Zg(u_1, v_1),\dots, Zg(u_t, v_t)\|_t \le \gamma(\|u_1\|+\|v_1\|,\dots, \|u_t\|+\|v_t\|) \quad (30)$$

for all $u_1,\dots,u_t$, $v_1,\dots,v_t\in G$. Then, Q: G→F is a unique quadratic system, such that

$$\|(g(u_1)-Q(u_1),\dots,g(u_t)-Q(u_t))\|_t \le \frac{m^n\gamma}{2(m^{2n}-1)(m^2-m)}(\|u_1\|,\dots,\|u_t\|) \quad (31)$$

Theorem 12. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If g: G→F is a function with g(0)=0 such that

$$\sup_{t\in N}\|Zg(u_1, v_1),\dots, Zg(u_t, v_t)\|_t \le \eta \quad (32)$$

for all $u_1,\dots,u_t$, $v_1,\dots,v_t\in G$. Then, Q: G→F is a unique quadratic system such that

$$\sup_{k\in N}\|(g(u_1)-Q(u_1),\dots, g(u_t)-Q(u_t))\|_t \le \frac{\eta}{3} \quad (33)$$

for all $u_1, u_2,\dots, u_t\in G$.

Proof: Taking $v_1, v_2,\dots,v_t = 0$ and substituting $u_1,\dots,u_t$ with $2u_1,\dots,2u_t$ in the inequality (32), then we have

$$\sup_{t\in N}\|(g(2u_1)-4g(u_1),\dots, g(2u_t)-4(g(u_t))\|_t \le \eta \quad (34)$$

Let d be a generalized metric space defined on a set R={f : G→F : f(0)=0} by

$$d(f, g) = \inf\{\sup_{t\in N}\|(f(u_1)-g(u_1),\dots, f(u_t)-g(u_t))\|_t \le \gamma\} \quad (35)$$

Let T_0 : R→R defined by

$$T_0 g(u) = \frac{1}{4}g(2u), \ \forall\, u\in G. \quad (36)$$

With Lipschitz constant ¼ and also strictly contractive on R. Let g, f∈R, be two mappings and γ∈(0, ∞) be a constant which satisfies d(g, f) ≤ γ. Then, from definition of metric space d, we get

$$\sup_{t\in N}\|(g(u_1)-f(u_1),\dots, g(u_t)-f(u_t))\|_t \le \gamma \quad (37)$$

Thus, we have

$$\sup_{t\in N}\|(T_0 g(u_1)-T_0 f(u_1),\dots, T_0 g(u_t) - T_0 f(u_t))\|_t$$

$$= \sup_{t\in N}\left\|\left(\frac{1}{4}g(2u_1)-\frac{1}{4}f(2u_1),\dots,\frac{1}{4}g(2u_t)-\frac{1}{4}f(2u_t)\right)\right\|_t \le \frac{\gamma}{4}$$

Hence,

$$d(T_0 g, T_0 f) \le \frac{\gamma}{4} \text{ implies } d(T_0 g, T_0 f) \le \frac{1}{4}d(g, f) \quad \forall\, g, f\in R.$$

Then from (34), we get

$$d(T_0 g, g) \le \frac{\eta}{4}$$

Now from fixed point theorem, the unique fixed point for the system T_0 exists, i. e., there exists a unique system Q: G→F satisfying

(i) $Q(2u) = 4Q(u), \ \forall\, u\in G$.

(ii) Further, $d(T_0^n g, Q)$ tend to zero, implies

$$Q(u) = \lim_{n\to\infty}(T_0^n g)(u) = \lim_{n\to\infty}\frac{g(2^n u)}{4^n}, \ \forall\, u\in G.$$

(iii) And $d(g, Q) \le \frac{d(T_0 g, g)}{(1-\lambda)}$ which implies

$$d(g, Q) \leq \frac{1}{1-\frac{1}{4}}\, d(T_0 g, g) \leq \frac{\varepsilon}{3}$$

Thus, (33) holds. Now, replacing $u_1, u_2, \ldots, u_t$ with $2^n u$ and $v_1, v_2, \ldots, v_t$ with $2^n v$ in the inequality (32) and then dividing throughout by 4^n, then we get

$$\sup_{t\in N} \left\| \left(\frac{Zg(2^n u, 2^n v)}{4^n}, \ldots, \frac{Zg(2^n u, 2^n v)}{4^n} \right) \right\|_t \leq \frac{\eta}{4^n}$$

Limiting as $n \to \infty$, we have

$$\sup_{t\in N} \| Z g(u, v)\| = 0$$

Thus, Q satisfies the given inequality and also it is unique as Q is a unique fixed point of T_0. Hence proved.

Corollary 13. Let $(F^t, \|\cdot\|_t)$ be a multi-Banach space and G be a linear space. If g: $G \to F$ is a function with $g(0)=0$ such that

$$\sup_{t\in N} \|(Zg(u_1, v_1), \ldots, Zg(u_t, v_t))\| \leq \psi(u_1, v_1, \ldots, u_t, v_t)$$

for each $u_1, \ldots, u_t, v_1, \ldots, v_t \in G$, where $\psi : G^{2t} \to [0, \infty)$. Then, Q: $G \to F$ is considered as unique quadratic system such that

$$\sup_{t\in N} \|(g(u_1)-Q(u_1), \ldots, g(u_t)-Q(u_t))\|_t \leq \frac{1}{3} \psi(u_1, 0, \ldots, u_t, 0)$$

$\forall u_1, \ldots, u_t \in G$.

Proof: Proof can be proved on the same lines of Theorem 12.

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