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On Optimal Solution for Traveling Salesman Problem: Direct Approach

Susanta Kumar Mohanta

Department of Mathematics, Govt. College Sundargarh, Sundargarh-770002, Odisha, India

Abstract

The main objective of this article is to propose a direct approach to find an optimal solution for the traveling salesman problems in a single attempt with reasonable short time from the network of a complete graph, complete digraph or connected graph.

Keywords: Absolute favorable cost, Complete graph, Complete digraph, Connected graph, Traveling salesman problem.

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1. INTRODUCTION

The traveling salesman problem consists of a salesman and a set of cities. The salesman has to visit each one of the cities starting from a certain one (e.g. hometown) and returning to the same city. The main objective of the problem is that; the salesman wishes to minimize the total length or cost of the trip. It is the particular case of transportation problem in optimization, the related basic information regarding traveling salesman problem can be obtained from any standard text book.

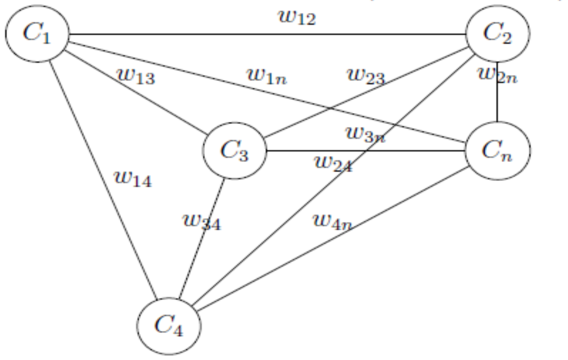
Generally network of a salesman (complete graph, complete digraph or connected graph) is useful to develop an assignment matrix (mathematical model). Assuming a salesman has to visit n cities ($c_i,\ i = 1, 2, \dots n$). Salesman wishes to start from a particular city, visit each city once and then return to his starting point. The following network representation is useful to develop the assignment cost matrix. Let w_{ij} be the distance or time or cost of going from city i to city j. Let the decision variable x_{ij} be 1; if the salesman has to travels from city i to city j; otherwise let it be zero. Mathematically,

$$x_{ij} = \begin{cases} 1, & \text{city i to city j} \\ 0, & \text{otherwise} \end{cases}$$

and subject to additional constraints that x_{ij} is so chosen that, no city is visited twice before all the cities are visited. In particular, the salesman has no permission to visit city i to city i. Mathematically,

$$w_{ij} = \begin{cases} c_{ij}, & \text{city i to city j} \\ \infty, & \text{city i to city i} \end{cases}$$

Figure 1: Network of a salesman (complete graph)



Now the traveling salesman problem can be stated in the form of $n \times n$ symmetric cost matrix $[c_{ij}]$ of real numbers for complete graph as stated below:

$$\begin{array}{c} \\ C_1 \\ C_2 \\ \vdots \\ C_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{bmatrix} \infty & c_{12} & \dots & c_{1n} \\ c_{21} & \infty & \dots & c_{2n} \\ & \vdots & & \vdots \\ c_{n1} & c_{n2} & \dots & \infty \end{bmatrix} \end{array}$$

In case of complete digraph from a network representation the obtained matrix will be non symmetric because $c_{ij} \neq c_{ji}$ and for connected graph with an arbitrarily long edge with large value assign to c_{ij} ; if there is no route between city i to j the obtained matrix will be symmetric matrix. It can be represented as an undirected weighted graph, such that cities are the graph's vertices, paths are the graph's edges, and a path's distance is the edge's weight. It is a minimization problem starting and finishing at a specified vertex after having visited each other vertex exactly once. The main objective of this model is selecting a sequence of cities that to minimize the total traveling distance or cost or time satisfying the restrictions of the problem.

In this article the basic information about the traveling salesman problems has been recited in section 1; The new proposed method and some necessary definition and results has been newly define in section 2; A good number of numerical examples have been illustrated in section 3; The algorithmic complexity of the proposed method has been studied in the worst cases in section 4; The results of numerical examples has been compare in section 5; The article end with conclusion in section 6.

2. PROPOSED METHOD

Definition 2.1. Let N be a network of n cities(C_i, $i = 1, 2, \ldots n$) obtained from a complete graph such that $|V(N)| = n$. Define $\xi : |V(N)| \rightarrow Z$ such that

$$\xi(C_i) = \left| \max_j c_{ij} - (\min_j c_{ij} + nextmin_j c_{ij}) \right|, \quad \text{where } i = 1, 2, \ \ldots \ n.$$

is the absolute favorable cost for each cities of the network. i.e., by selecting the minimum and next minimum assignment cost, subtracting their sum from the largest assignment cost of a city from the edges incident on it.

Definition 2.2. The set of (from) cities is the favorable set in the network, if $\sum f.\xi_F \geq \sum f.\xi_T$; $f \geq 2$ and for non-symmetric cost matrix, where f is the frequency of absolute favorable cost ξ_F ((from) cities) or ξ_T ((to) cities). Otherwise set of (to) cities is the favorable set in the network.

Definition 2.3. Depending upon the nature of the absolute favorable costs from the favorable set of cities; one of the following three assignment preference table will have to choose to construct a sequence:

Table 1: Assignment preference table.

Non zero ξ :	ascending order
Frequency (f) :	one
Assignment Preference:	larger to smaller of ξ

Table 2: Assignment preference table.

Non zero ξ :	ascending order
Frequency (f) :	one or more
Assignment Preference:	larger to smaller f and ξ

Table 3: Assignment preference table.

ξ :	ascending order followed by zero
Frequency (f) :	one or more
A. Pref:	start with $\xi = 0$; then larger f and ξ

Definition 2.4. A favorable sequence of cities of network is said to be optimal; if the favorable sequence contains all the cities of the network with a particular city as its source and destination.

The proposed method has been developed on the basis of favorable cost for each (from and to) cities. The following steps will describe the process of calculations.

Methodology :

Basis step:

Step-1 Develop a cost matrix from the network of a salesman (complete graph, complete digraph or connected graph) of the traveling salesman problem.

- The cost matrix will be symmetric for the network (complete graph) and calculate absolute favorable cost of from cities or to cities using the function in the definition (2.1).
- The cost matrix will be non-symmetric for the network (complete digraph) and calculate absolute favorable cost of both (from) cities and (to) cities using the function in the definition (2.1).

Step-2 Choose a suitable assignment preference table from the definition (2.3). If there is a tie between larger frequencies, then start with smaller to larger ξ_F for non-symmetric matrix, else start with larger to smaller ξ_F for symmetric matrix.

Step-3 For first assignment:

- If $\xi \leq 2$ choose min. of (min. cost from row/rows) for first assignment (in case of tie go down).
- If $\xi \geq 3$ and non-symmetric cost matrix; choose min. of (min. cost from row/rows) for first assignment (in case of tie go down).
- If $\xi \geq 3$ and symmetric cost matrix; choose min. of (max. cost from row/rows) for first assignment (in case of tie go down).

For other assignments: choose min. of (min. cost from row/rows) for assignment (in case of tie go down).

Step-4 Tie between two minimum cost can be settle as follows

- $\boldsymbol{\alpha}$. In case of tie between minimum cost preference should be given to that row having less row sum;
- $\boldsymbol{\beta}$. In case of tie between the minimum cost as well as row sum, preference should be given to that row having greater costs as compare to other;
- $\boldsymbol{\gamma}$. In case of tie between the minimum cost and row sum with identical costs; preference should be given to that row having less column sum as compare to other;
- $\boldsymbol{\delta}$. For $\xi_F \geq 3$ and symmetric cost matrix; In case of tie between the minimum cost, row sum with identical costs and column sum; then preference should be given to min.(next max. of row/rows) in these rows;

After tie break, the tie cost is suitable for second assignment in the other row if cost matrix is non-symmetric else choose the next min. cost of other row for next assignment for symmetric matrix. Again if there is a tie go to step-4. Repeat the steps 3 and 4 until all the row and column have an assignment. Now if the sequence contains all cities of the network, then it is optimal.

Inductive step:

Step-5 If the favorable sequence of cities is not optimal, then revised the absolute favorable cost ξ_F for all favorable cities using θ (cost of 1st assignment) as follows:

$$\xi_F = \begin{cases} |\xi_F - \theta|, & \xi_F > 0; \\ \xi_F + \theta, & \xi_F = 0. \end{cases} \qquad (2.1)$$

Step-6 Repeat the steps 3 and 4 until all the row and column have an assignment on the basis of modified absolute favorable cost. Now if the sequence contains all cities of the network; then it is optimal. Otherwise repeat step-5.

Step-7 Finally calculate the total assignment cost or time or distance.

Theorem 2.5. Let N be a network of n cities$(C_i,\ i = 1,2,\ldots n)$ obtained from a complete graph such that $|V(N)| = n$ and $\xi\colon |V(N)| \to Z$ is an absolute favorable cost for cities. Then using favorable set of cities and suitable assignment preference table; an optimal sequence can be obtained for the network.

Proof. The definitions (2.1); (2.2); (2.3); (2.4) and the proposed method will proof the result of the theorem.

3. NUMERICAL EXAMPLES

Three examples have been discussed to illustrate the process of calculation of the proposed method.

Example 3.1. Consider a network (complete graph) of five cities and a salesman has to visit these five cities $C_1; C_2; C_3; C_4$ and C_5. Solve the traveling salesman problem having $c_{12} = 8;$ $c_{13} = 4;$ $c_{14} = 9;$ $c_{15} = 9;$ $c_{23} = 6;$ $c_{24} = 7;$ $c_{25} = 10;$ $c_{34} = 5;$ $c_{35} = 6;$ $c_{45} = 4$ and $c_{ij} = c_{ji}$. There is no route between city i to j if a value for c_{ij} is not shown.

Solution: The network of the salesman is a complete graph with five cities, so the corresponding matrix will be symmetric matrix. Now the proposed method as follows:

Step-1 The given assignment cost of the traveling salesman problem is symmetric cost matrix. By step-1 and function in (2. 1); the following

$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \end{array} \begin{array}{c} \begin{array}{ccccc} C_1 & C_2 & C_3 & C_4 & C_5 \end{array} \\ \begin{bmatrix} \infty & 8 & 4 & 9 & 9 \\ 8 & \infty & 6 & 7 & 10 \\ 4 & 6 & \infty & 5 & 6 \\ 9 & 7 & 5 & \infty & 4 \\ 9 & 10 & 6 & 4 & \infty \end{bmatrix} \end{array} \begin{array}{c} \xi_F \\ 3 \\ 3 \\ 3 \\ 0 \\ 0 \end{array}$$

Step-2 Choose a suitable assignment preference table from proposed method by step-2 as follows

Table 4: Assignment preference table.

Absolute favorable cost (ξ_F)	0	3
Frequency (f)	2	3
Assignment Preference	1^{st}	2^{nd}

Step-3&4

- **1st Assignment:** By step-3 of the proposed method; choose min.(min. of R_4, min. of R_5)= min.(4; 4) = 4 makes a tie. Now by step- 4(α) choose the cell (4; 5) for first assignment. After assignment delete both the row R_4 and column C_5 .
- **2nd Assignment:** By step-4 of the proposed method; after tie break choose the next min. cost of row R_5 that is 6 in the cell (5; 3). After assignment delete both the row R_5 and column C_3.
- **3rd Assignment:** By step-3 of the proposed method; choose min. (min. of R_1 ; R_2 and R_3)= min.(4; 7; 8) = 4 in the cell (3; 1) for third assignment and delete the row R_3 and column C_1.
- **4th Assignment:** By step-3 of the proposed method; choose min. (min. of R_1 and R_2) = min. (7; 8) = 7 in the cell (2; 4) for fourth assignment and delete the row R_2 and column C_4.
- **5th Assignment**: By step-3 of the proposed method; choose min. of R_1 = 8 in the cell (1; 2) for fifth assignment and delete the row R_1 and column C_2.

Step-5 Now represent all the assignment in the given assignment cost matrix and on the basis of these assignments the optimal sequence for the salesman is $C_1 \to C_2 \to C_4 \to C_5 \to C_3 \to C_1$.

Step-6 The optimal assignment cost for the salesman is 8 + 7 + 4 + 4 + 6 = 29.

Example 3.2. Consider a network (complete graph) of five cities and a salesman has to visit these five cities $C_1; C_2; C_3; C_4$ and C_5. Solve the traveling salesman problem having $c_{12} = 20;$ $c_{13} = 4;$ $c_{14} = 10;$ $c_{23} = 5;$

$c_{25} = 10$; $c_{34} = 6$; $c_{35} = 6$; $c_{45} = 20$ and $c_{ij} = c_{ji}$. There is no route between city i to j if a value for c_{ij} is not shown.

Solution: The network of the salesman is a connected graph with five cities, so the corresponding matrix will be symmetric matrix. Now the proposed method as follows:

Step-1 The given assignment cost of the traveling salesman problem is symmetric cost matrix. By step-1 and function in definition (2.1) the following

$$\begin{array}{c|ccccc|c} & C_1 & C_2 & C_3 & C_4 & C_5 & \xi_F \\ C_1 & - & 20 & 4 & 10 & - & 6 \\ C_2 & 20 & - & 5 & - & 10 & 5 \\ C_3 & 4 & 5 & - & 6 & 6 & 3 \\ C_4 & 10 & - & 6 & - & 20 & 4 \\ C_5 & - & 10 & 6 & 20 & - & 4 \end{array}$$

Step-2 Choose a suitable assignment preference table from the proposed method by step-2 as follows

Table 5: Assignment preference table.

Absolute Favorable Cost(ξ_F)	3	4	5	6
Frequency (f)	1	2	1	1
Assignment Preference	4^{th}	1^{st}	3^{rd}	2^{nd}

Step-3&4

- **1st Assignment:** As per step-3; choose min. (max. of R_4, max. of R_5)= min.(20; 20) = 20 makes a tie. So choose min. of (next max. of R_4, next max. of R_5)= min(10; 10) = 10. Again it makes a tie. The cell (4; 1) is our choice for first assignment using step- 4(δ). After assignment delete both the row R_4 and column C_1.
- **2nd Assignment**: As per step-4; after tie break tie cost that is min. (max. of R_4, max. of R_5)= min.(20; 20) = 20 in the cell (5; 4) is suitable for second assignment. After assignment delete both the row R_5 and column C_4.
- **3rd Assignment:** As per step-3; choose min. (R_1) = 4 in the cell (1; 3) for making third assignment. Now delete the row R_1 and column C_3.
- **4th Assignment:** As per step-3; choose min. (R_2) = 10 in the cell (2; 5) for making fourth assignment. Now delete the row R_2 and column C_5.
- **5th Assignment:** As per step-3; choose min. (R_3) = 5 in the cell (3; 2) for making fifth assignment. Now delete the row R_3 and column C_2.

Step-5 Now represent all the assignment in the given assignment cost matrix and on the basis of these assignments optimal sequence for the salesman is $C_1 \rightarrow C_3 \rightarrow C_2 \rightarrow C_5 \rightarrow C_4 \rightarrow C_1$.

Step-6 The optimal assignment cost for the salesman is 4 + 10 + 20 + 10 + 5 = 49 units.

Example 3.3. Consider a network (complete digraph) of four cities and a salesman has to visit these four cities $C_1; C_2; C_3$ and C_4. Solve the traveling salesman problem having $c_{12} = 10$; $c_{13} = 15$; $c_{14} = 20$; $c_{21} = 5$; $c_{23} = 9$; $c_{24} = 10$; $c_{31} = 6$; $c_{32} = 13$; $c_{34} = 12$; $c_{41} = 8$; $c_{42} = 8$ and $c_{43} = 9$. There is no route between city i to j if a value for c_{ij} is not shown.

Solution: The network of the salesman is a complete digraph with four cities, so the corresponding matrix will be non-symmetric matrix. Now the proposed method as follows:

Step-1 By step-1 and function in definition (2.1) the following

$$\begin{array}{c|cccc|c} & C_1 & C_2 & C_3 & C_4 & \xi_F \\ C_1 & - & 10 & 15 & 20 & 5 \\ C_2 & 5 & - & 9 & 10 & 4 \\ C_3 & 6 & 13 & - & 12 & 5 \\ C_4 & 8 & 8 & 9 & - & 7 \\ \xi_T & 3 & 5 & 3 & 2 & \end{array}$$

Here for $f \geq 2$; $\sum f.\xi_F \geq \sum f.\xi_T$, so choose ξ_F for assignments.

Step-2 Choose a suitable assignment preference table from step-2 as follows

Table 6: Assignment preference table.

Absolute Favorable Cost(ξ_F)	4	5	7
Frequency(f)	1	2	1
Assignment Preference	3^{rd}	1^{st}	2^{nd}

Step-3&4

- **1st Assignment:** As per step-3; so choose min. (min. of R_1, min. of R_3)= 6 in the cell (3; 1) for first assignment. After assignment delete both the row R_3 and column C_1.
- **2nd Assignment:** As per step-3; min. (R_1) = 10 in cell (1; 2) for second assignments. After assignment delete both the row R_1 and column C_2.
- **3rd Assignment:** As per step-3; for third assignment choose min. (R_4) = 9 in the cell (4; 3): After assignment delete both the row R_4 and column C_3.
- **4th Assignment:** As per step-3; choose min. (R_2) = 10 in the cell (2; 4) for making fourth assignment. Now delete the row R_2 and column C_4.

Step-5 Now represent all the assignment in the given assignment cost matrix and on the basis of these assignments optimal sequence for the salesman is $C_1 \rightarrow C_2 \rightarrow C_4 \rightarrow C_3 \rightarrow C_1$.

Step-6 The optimal assignment cost for the salesman is 10 + 10 + 6 + 9 = 35 units.

4. ALGORITHM COMPLEXITY

The complexity of an algorithm is simply the number of computational steps that it takes to transform the input data to the result of a computation. The following result will estimate the complexity of the proposed method in worst cases.

Theorem 4.1. Suppose N is a network obtained from a complete graph by giving each edge an integer weight. Then the proposed method generates an optimal sequence containing all cities of N in time $O(|V(N)|^2)$ where $|V(N)|$ is cardinality of vertex set in N; under the assumption that all elementary arithmetic operations take constant time.

Proof. Suppose N is a network obtained from a complete graph by giving each edge an integer weight with cardinality of vertex set $|V(N)| = n.$ The algorithm proceeds by growing alternating favorable sequences for each cities of the network. Growing an alternating favorable sequence takes $5n$ time. Now growing n-alternating favorable sequences in a network having n cities takes $5n \times n = 5n^2$ time. Thus the total time spent on alternating favorable sequence is $O(5n^2) = O(n^2) = O(|V(N)|^2)$. This proves the Theorem.

5. RESULT ANALYSIS

In this section the results obtained by proposed method are compared with results obtained by other existing methods with their optimal solutions. The following table [Table 7] summarizes all the results. From the table it has been observed that, the solutions on table are gives optimal solution to the traveling salesman problems in a single attempt.

Table 7: Comparison table.

Methods	Optimal cost		
	Ex.1	Ex.2	Ex.3
Proposed Method	29	49	35
Optimal Solution	29	49	35

6. CONCLUSIONS

The traveling salesman problems have several applications even in its purest formulation, such as planning, logistic and the manufacture of microchips. It can be applied to anything from drilling holes in printed circuit board to designing fiber optic communications networks to coordinating military manure's to routing helicopters around oil rigs. Finally it can be claim that the proposed method is best method to generate optimal solution of the traveling salesman problems in a reasonable amount of time.

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