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Solving Differential Equation by Extended form of Laplace Transform

Jitendra Kumar Pati

Associate Professor, Department of Math,
C V Raman College of Engineering,
Mahura, Janla, Bhubaneswar, 752054, India.

Abstract

In this paper a generalized transform is designed. This transform is found to be an ideal generalized of both Elzaki and Laplace transform. We applied this to solve different types of differential equations namely Bessel's, Legender, heat wave equation etc.

Keywords*:* Laplace Transform; Wave equation; Elzaki Transform; Legender equation; Bessel's equation.

1. INTRODUCTION

Keeping in view of Elzaki transform we introduce an extended form of Laplace transform as

$$E(f) = U^n \int_0^\infty f(t) e^{-Ut} dt \text{, U>0 , } n \in \text{N} \qquad (1.1)$$

The new transform preserves the law of exponential order and linearity property. Using this differentiation and Integral properties of Laplace transform we also have an extension for the concerned properties as follows.

2. DERIVATION OF DIFFERENTIAL FORMULA USING EXTENDED FORM OF LAPLACE TRANSFORM

Using (1.1) we can find the following properties

(1) $E\,(t^n) = \dfrac{n!}{U^{n-1}}$

(2) $E(f') = U^n \int_0^\infty f'(t)\, e^{-Ut}\, dt = -U^n f(o) + U\,E(f)$ So for a special case n=o we have $E(f') = L(f')$

(3) $E(f'') = -U^n f'(o) + U\,E(f') = -U^n f'(o) - U^{n+1} f(o) + U^2 E(f)$

(4) $E(e^t) = \dfrac{U^n}{U-1}$

(5) $E\,(e^{at}) = \dfrac{U^n}{U-a}$ and $E\,(e^{at}) = \dfrac{U^n}{U+a}$

(6) $E(\cosh at) = \dfrac{U^{n+1}}{U^2 - a^2}$ and $E(\sin at) = \dfrac{aU^n}{U^2 - a^2}$

(7) $E(\cos at) = \dfrac{U^{n+1}}{U^2 + a^2}$ and $E(\sin at) = \dfrac{aU^n}{U^2 + a^2}$

(8) $E(tf(t)) = U^n \int_0^\infty tf(t) e^{-Ut} dt - U^n \int_0^\infty tf(t) U^{-Ut} dt = \frac{n}{U} E(f) - F'(U)$ Now $E(t\, f(t)) = \dfrac{n}{U}\, E(f) - F'(U)$

(9) $E(tf'(t)) = (n-1)E(f) - U\frac{d}{dU}E(f)$

(10) $E(tf''(t)) = (n-1)E(f') - U\frac{d}{dU}E(f') = -(n-1)U^n f(0) + (n-1)UE(f) + nU^n f(0)$

$-UE(f) - U^2\frac{d}{dU}E(f) = U^n f(0) + (n-2)UE(f) - U^2\frac{d}{dU}E(f)$

(11) $E(t^2 f(t)) = \frac{n}{U}E(tf(t)) - \frac{d}{dU}E(tf(t)) = \frac{n}{u}(\frac{n}{u}E(f) - \frac{d}{du}E(f)) - \frac{d}{du}(\frac{d}{dU}E(f) - \frac{d}{dU}E(f))$

$= \frac{n^2}{U^2}E(f) - \frac{2n}{U}\frac{d}{dU}E(f) \quad + \frac{n}{U^2}E(f) - \frac{d^2}{dU^2}E(f)$

(12) $E(f(t-a)\,\text{u}^*(t-a)) = U^n\int_0^\infty f(t-a)e^{-Ut}\,dt = U^n\int_0^\infty f(\alpha)e^{-U(a+\alpha)}d\alpha = e^{-U\alpha}E(f)$

(13) $E(J_0(at)) = \sum\frac{(-1)^k}{(k!)^2}\left(\frac{a}{2}\right)^{2k}E(t^{2k}) = \sum\frac{(-1)^k}{(k!)^2}\left(\frac{a}{2}\right)^{2k}\frac{(2k)!}{U^{1-2k}}$

3. VARIOUS APPLICATIONS USING EXTENDED FORM OF LAPLACE TRANSFORM.

Earlier so many works have been done by using Tarig and Elzaki transform. Here we introduce the extended Laplace transform to solve ordinary as well as partial differential equation. Herewith we cited few applications of extended Laplace transform.

Solving Ordinary Differential equation:

Example 3.1

Solve $y'' - y = t$, $y(0) = 1$, $y'(0) = 1$

Applying the Laplace transform both sides we get

$$s^2 y(s) - sy(0) - y(s) = \frac{1}{s^2}$$

So $y(s) = \frac{1}{s-1} + \frac{1}{s^2-1} - \frac{1}{s^2}$

Hence $y(t) = e^t + \sinh t - t$

The same example we sole by using extended form of Laplace transform as follows.

Taking transform both sides we have

$$E(y'') - E(y) = E(t)$$

So $-U^n y'(0) - U^{n+1}y(0) + U^2 E(y) - E(y) = \frac{1}{U}$

$$\Rightarrow E(y)(U^2 - 1) = \frac{1}{U} + U^n + U^{n+1}$$

$$\Rightarrow E(y) = \frac{A}{U} + \frac{BU + C}{U^2 - 1} + \frac{U^n}{U-1^2} + \frac{U^{n+1}}{U^2 - 1}$$

So $E(y) = -\frac{1}{U} + \frac{U}{U^2-1} + \frac{U^n}{U-1}$

$$y = E^{-1}(-\frac{1}{U} + \frac{U}{U^2-1} + \frac{U^n}{U-1}) = -t + \sinh t + e^t$$

Hence the same answer we obtain.

Solving Bessel Equation:

Example 3.2

$t^2 y'' + ty' + (t^2 - v^2)y = 0$, $y(0) = a$, $y'(0) = b$

Taking transform both sides, we have

$$E(t^2 y'') + E(ty') + E(t^2 y) - E(v^2 y) = 0$$

$$\Rightarrow \frac{n}{U}\left(U^n y(0) + (n-2)UE(y) - U^2 \frac{d}{dU}E(y)\right)$$

$$-\frac{d}{dU}\left(U^n y(0) + (n-2)UE(y) - U^2 \frac{d}{dU}E(y)\right)$$

$$+(n-1)E(y) - U\frac{d}{dU}E(y) + \frac{n^2}{U^2}E(y) - \frac{2n}{U}\frac{d}{dU}E(y)$$

$$+\frac{n}{U^2}E(y) - \frac{d^2}{dU^2}E(y) - v^2 E(y) = 0$$

So, $(U^2-1)\frac{d^2}{dU^2}E(y) + (-2nU + 3U - \frac{2n}{U})\frac{d}{dU}E(y) + E(y)(n^2 - 2n + 1 + \frac{n^2}{U^2}E(y) + \frac{n}{U^2} - v^2) = 0$ **(3.1)**

Now we know

$\frac{d}{dU}E(y) = E(y') = -U^n y(0) + UE(y)$, and $\frac{d^2}{U^2}E(y) = -U^n y'(0) - U^{n+1} y(0) + U^2 E(y) = 0$ so substituting the initial conditions, the above equation (3.1) simplifies to

$$E(y)(U^4 - U^2 - 2nU^2 + 3U^2 - 2n + (n-1)^2 - v^2 + \frac{n^2}{U^2} + \frac{n}{U^2}) = U^{n+2}b + U^{n+3}a - U^n b - U^{n+1}a$$

$$+3U^{n+1}a - 2nU^{n-1}a$$

So,

$$E(y) = \frac{(U^{n+5} + 2nU^{n+3} - 2nU^{n+3} - 2nU^{n+1})a + (U^{n+4} - U^{n+2})b}{U^2(U^2 + (n-1)^2 - 2n - v^2) + (n^2 + n)}$$

Then

$$E(y) = \frac{1}{n^2+n}((U^{n+5} + 2U^{n+3} - 2nU^{n+1})a + (U^{n+4} - U^{n+2})b)\left[1 - \frac{U^4 + U^2(n-1)^2 - 2nU^2 - v^2U^2}{n^2+n}\right]^{-1}$$

Using binomial expansion and neglecting the higher degree terms we have

$$E(y) = \frac{1}{n^2+n}\left[(U^{n+5} + 2U^{n+3} - 2nU^{n+1})a\right] + \frac{1}{n^2+n}\left[(U^{n+4} - U^{n+2})b\right]$$

So $E(y)=\frac{1}{n^2+n}\left[(U^{n+6-1}+(2-2n)U^{n+4-1})a\right]-\frac{1}{n^2+n}\left[(2nU^{n-1+2})a\right]+\frac{1}{n^2+n}\left[(U^{n+5-1}-U^{n+3-1})b\right]$

$$\Rightarrow y=\frac{1}{n^2+n}\left[(\frac{t^{n+6}}{(n+6)!}+(2-2n)\frac{t^{n+4}}{(n+4)!})a\right]-\frac{1}{n^2+n}\left[(2n\frac{t^{n+2}}{(n+2)!})a\right]+\frac{1}{n^2+n}\left[(\frac{t^{n+5}}{(n+5)!}-\frac{t^{n+3}}{(n+3)!})b\right]$$
$+----$

We may claim that the above result is the solution of Bessel differential equation and supported to be expressed in terms of power series also.

Example 3.3

Solve the equation $ty''+y'+a^2ty=0$, $y(0)=1$

Taking transform both sides, we obtain

$$(n-2)UE(y)-U^2\frac{d}{dU}E(y)+UE(y)+n\frac{a^2}{U}E(y)-a^2\frac{d}{dU}E(y)=0$$

$$\Rightarrow\frac{\frac{d}{dU}E(y)}{E(y)}=\frac{U^2(n-1)+na^2}{U(U^2+a^2)}$$

Integrating

$$\ln E(y)=\ln(U^2+a^2)^{\frac{n-1}{2}}+\ln U^n-\ln(U^2+a^2)^{\frac{n}{2}}$$

So, $E(y)+U^n(U^2+a^2)^{\frac{-1}{2}}=U^{n-1}\left[1-\frac{1}{2}\frac{a^2}{U^2}+---\right]$

Using (13) and inverse transform both sides, we get

$y=AJ_0(at)$ by taking $2k=n$

Solving Legender Differential equation:

Example 3.4

$(1-t^2)\,y''-2t\,y'+\lambda(\lambda+1)y=0$, $y(0)=a$, $y'(0)=b$

Taking transform both sides

$E((1-t^2)\,y'')-E(2t\,y')+E(\lambda(\lambda+1)y)=0$

$E(y'')-E(t^2\,y'')-2E(t\,y')+\lambda(\lambda+1)E(y)=0$

So, $U^n\,y'(0)-U^{n+1}y(0)-U^2\frac{d^2}{dU^2}E(y)+nU\frac{d}{dU}E(y)+\left[(-n^2+n+U^2+\lambda(\lambda+1)\right]E(y)=0$

$$\Rightarrow -U^2\left[-U^n y'(0)-U^{n+1}y(0)+U^2E(y)\right]+nU\left[(-U^n y(0)+UE(y)\right]+\left[(-n^2+n+U^2+\lambda(\lambda+1)\right]E(y)$$

$$-U^n y'(0)-U^{n+1}y(0)+0$$

Substituting the initial conditions we have,

$$U^{n+2}b+U^{n+3}a-U^4E(y)-U^{n+1}na+nU^2E(y)+\left[(-n^2+n+U^2+\lambda(\lambda+1)\right]E(y)-U^n b-U^{n+1}a=0$$

$$E(y)=\frac{\left[a(nU^{n+1}+U^{n+1}-U^{n+3})\right]}{\left[\lambda(\lambda+1)-U^4+(n+1)U^2-n^2+n\right]}+\frac{\left[b(U^n-U^{n+2})\right]}{\left[\lambda(\lambda+1)-U^4+(n+1)U^2-n^2+n\right]}$$

$$E(y)=\frac{1}{\lambda(\lambda+1)}\left\{1+\frac{-U^4+(n+1)U^2-n^2+n}{\lambda(\lambda+1)}\right\}^{-1}\left[a(nU^{n+1}+U^{n+1}-U^{n+3})+b(U^n-U^{n+2})\right]$$

Using binomial expansion we have

$$E(y)=\frac{1}{\lambda(\lambda+1)}\left[a\left\{(n+1)U^{n+1}-U^{n+3}\right\}+\left\{U^n-U^{n+2}\right\}\right]$$, Neglecting the higher terms

$$E(y)=\frac{1}{\lambda(\lambda+1)}\left[a\left\{(n+1)U^{n-1+2}-U^{n-1+4}\right\}\right]+\frac{1}{\lambda(\lambda+1)}\left[b\left\{(U^{n-1+1}-U^{n-1+3}\right\}\right]$$

Hence

$$y(t)=\frac{1}{\lambda(\lambda+1)}\left[a\left\{(n+1)\frac{t^{n+2}}{(n+2)!}-\frac{t^{n+4}}{(n+4)!}\right\}\right]+\frac{1}{\lambda(\lambda+1)}\left[b\left\{\frac{t^{n+1}}{(n+1)!}-\frac{t^{n+3}}{(n+3)!}\right\}\right]$$

We may take the result as the solution of Legender differential equation and in power series also.

Solving Partial Differentiation equation:

Introducing the concept of partial differentiation in Extended form of Laplace transform we have,

$E(y)=U^n\int_0^\infty f(t)e^{-Ut}dt, U>0, n\in N$, and

$$E\left(\frac{\partial f(x,t)}{\partial t}\right)=U^n\int_0^\infty \frac{\partial f}{\partial t}e^{-Ut}\,dt=\lim_{p\to\infty}\int_0^\infty U^n\frac{\partial f}{\partial t}e^{-Ut}\,dt$$

$$=\lim_{p\to\infty}\left|(U^n e^{-Ut}f(x,t)_0^p+U^{n+1}\int_0^\infty e^{-Ut}f(x,t)dt\right|=UE(f)-U^n f(x,0)$$

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Example:

Solve $\frac{\partial w}{\partial x}+x\frac{\partial w}{\partial t}=x$, $w(x,0)=1$, $w(0,t)=1$

Taking transform both sides,

$$E(\frac{\partial w}{\partial x})+E(x\frac{\partial w}{\partial t})=E(x)$$

$$\Rightarrow E(\frac{\partial w}{\partial x})+x(UE(w)-U^n w(x,0))=\frac{x}{U}$$

So, $\frac{\partial E(w)}{\partial x}+xUE(w)-xU^n=xU^{n-1}$

$$\Rightarrow \frac{\partial w(x,U)}{\partial x}+xUw(x,U)=x(U^{n-1}-U^n)$$

Using the concept of linear differential equation, we obtain

$$w(x,U)=U^{n-1}+U^{n-2}+c(U)e^{\frac{-Ux^2}{2}} \quad \textbf{(3.2)}$$

Now, $w(0,t)=1$, $E(w(0,t))=E(1)$, $w(0,U)=U^{n-1}$

Applying the boundary condition $w(o,U)=U^{n-1}$ in equation (3.2), we have,

$$c(U)=-U^{n-2}$$

So, $w(x,U)=U^{n-2}+U^{n-1}-U^{n-2}e^{\frac{-Ux^2}{2}}$

$$\Rightarrow w(x,t)=E^{-1}(U^{n-2})+E^{-1}(U^{n-1})-$$

$$E^{-1}(U^{n-2}e^{\frac{-Ux^2}{2}})$$

$$=\frac{t^{n-1}}{(n-1)!}+\frac{t^n}{n!}-\frac{(t-\frac{x^2}{2})}{(n-1)!}u^*(t-\frac{x^2}{2})$$

Same result we obtain for n=1 using Laplace transform to th this problem.

Solving One dimensional wave equation:

$w_{tt}=c^2w_{xx}$, $w(0,t)=0$, $w(l,t)=0$, $w(x,0)=x$, $w_t(x,0)=0$

Taking transform both the sides,

$$E(w_{tt})=c^2E(w_{xx})$$

$$\Rightarrow -U^n w_t(x,0)-U^{n+1}w(x,0)+U^2E(w)=c^2\frac{\partial^2}{\partial x^2}E(w)$$

So, $\frac{\partial^2}{\partial x^2}-U^2Bw=B(-U^{n+1}x)$, where $\frac{1}{c^2}=B$,

$E(w(x,t))=w(x,U)$

This is a second order non-homogeneous differential equation, general solution is,

$$w=w_c+w_p$$

Hence $w(x,U)=c_1e^{UBx}+c_2e^{-UBx}+U^{n-1}x$

Using the boundary conditions, we get

$c_1+c_2=0$, $c_1e^{UBl}+c_2e^{-UBl}=U^{n-1}l$

So, $w(x,u)=\frac{-U^{n-1}l}{e^{UBl}-e^{-UBl}}e^{UBx}+\frac{U^{n-1}l}{e^{UBl}-e^{-UBl}}e^{-UBx}+U^{n-1}x$

$$\Rightarrow w(x,t)=E^{-1}[\ \frac{-U^{n-1}l}{e^{UBl}-e^{-UBl}}e^{UBx}x\ ]+E^{-1}\left[\frac{U^{n-1}l}{e^{UBl}-e^{-UBl}}e^{-UBx}\right]+E^{-1}\left[U^{n-1}x\right]$$

$$w(x,t)=E^{-1}\left[\frac{-U^{n-1}l\{1+UBx+-----\}}{UBl+\frac{U^3B^3l^3}{3!}+--------}\right]+E^{-1}\left[\frac{U^{n-1}l\{1-UBx+-----\}}{UBl+\frac{U^3B^3l^3}{3!}+--------}\right]+E^{-1}\left(U^{n-1}x\right)$$

$$w(x,t)=E^{-1}\left[\frac{-U^{n-1}l\{1+UBx+-----\}}{UBl(1+\frac{U^2B^2l^2}{6}+--------}\right]+E^{-1}\left[\frac{U^{n-1}l\{1-UBx+-----\}}{UBl(1+\frac{U^2B^2l^2}{6}+--------}\right]+E^{-1}\left(U^{n-1}x\right)$$

$$=E^{-1}\left\{\frac{-U^{n-2}}{B}[1+UBx+----]\left[1-\frac{U^2B^2l^2}{6}+------\right]\right\}+E^{-1}\left\{\frac{U^{n-2}}{B}[1+UBx+----]\left[1-\frac{U^2B^2l^2}{6}+------\right]\right\}$$

$$+E^{-1}\left(U^{n-1}x\right)$$

$$=E^{-1}\left\{\frac{-U^{n-2}}{B}+\frac{-U^{n-2}}{B}-2U^{n-1}x+--------\right\}+E^{-1}\left(U^{n-1}x\right)$$

$$=-2x\frac{t^n}{n!}+x\frac{t^n}{n!}+------=-x\frac{t^n}{n!}+--------$$

This may be the solution of one dimensional wave equation and expressed in power series also.

4. CONCLUSION

Here we emphasize the new transform using the generalization of Laplace as well as Elzaki which helps to solve a differential equation with constant coefficient and variable coefficient. This concept may be used to solve two dimensional wave equation and telegraphic equation etc.

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