International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 12 (2019) pp. 2960-2967

Generalized Invexity and Semi – Continuity In Mathematical Programming

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Abstract:
In this paper, a class of non-smooth generalized invexity for a nonlinear programming problem. Here we will derive optimality conditions and duality results with generalized invexity conditions.

Keywords: Generalized invexity, optimization problem, optimality, Duality.

1. INTRODUCTION

Convexity and generalized convexity play an impotent role in the study of optimality and duality concepts of mathematical programming, game theory, network analysis and other related fields of O.R.

Since last four decades many attempts have been made to weaken convexity hypothesis [1,2]. First, Hanson and Mond [11] introduced a new class of functions called type-I functions for a generalized scalar optimization problem, it was further generalized to pseudo type-I and quasi type-I by Rueda and Hanson [21] consequently, different generalizations on type-I functions have been introduced by different authors.

Further, Anurag (2014) introduced another set of new class of functions known as generalized α-univex type-I for a vector – valued functions. Also, derived K-T type sufficient optimality conditions under generalized invexity and obtained various type of duality results for a chosen multi objective programming problem with inequality constraints. Further, Suneja et al. [24, 25] introduced generalized type-I functions using cones obtained sufficient optimality in this non set up and also established different duality results by considering vector optimization problem.

More recently, Yu et al [31, 32] established optimality and duality results for a differentiable vector optimization problem with a set of inequality constraints with respect to generalized type-I functions by using Banaach Spaces. Consequently Christian Niculesen obtained a set of optimality and duality results for a non linear fractional multiobjective programming problem using η- semi differentiability under type-I functions. Further, Lupsa et al., [15] introduced semi E-prinivex maps using Banach spaces and studied different properties. Very recently, Huhua a Jiao [12] introduced a non differentiable vector optimization problem may semi E- type-I maps on Banach spacon and obtained different optimality and duality results.

Motivated by the above ideas, in this paper we introduce a generalized η-V-E-semi differentiable invex functions. We obtain optimality and duality results under different types of generalized η-V-E-semi differentiable invex functions.

2. PRELIMINARY NOTATIONS AND DEFINITIONS

Here, Let X, Y and Z_j, j∈M = {1, 2, ...,m} be real Banach spaces.

(MP) min (f(x)) = (f_1(x), ...,f_p(x))

subject to constraints :

- g (x) = g_j (x), j – 1, 2, ..., m

and x ∈ K ⊂ X ∈ (D_1 X ..., D_m)

Here, the functions f_i: X→Y, g_j(x) : X→Y, i=1,2,….,p; j=1,2,….,m

Maps from x to Dj, there are subsets of X and Z.

Let us define the feasible set as

$$F(x)=\{x\in KCX:g_j(x)\in D_j, j\in M\}.$$

Usually the, the notations have their usual meanings

Definition 2.1 ([12]) The problem (MP) satisfies the slater regularity condition if $\exists x_1 \in F \exists$

$$g_j(x_1)<0, x_1\in D_j,\ j\in m,\ j=1,2,...,m$$

Definitions 2.2 ([12]) Let $x_1 \in F$ be a weekly efficient solution for the problem (MP) if ∃ no $x\in F$ ∋

$$f_i(x) <_C f_i(x_1),\ i=1,2,\ldots,p$$

Definition 2.3. ([12])

A set K⊂X is said in the E – invex with respect to η if

$$E(y)+\lambda\eta^T(E(x),E(y))\in K, \forall x,y\in K, \lambda\in[0,1].$$

Definition 2.4 ([12, 20])

If K⊂X be an invex set w.r.t. η then the map $F:X\rightarrow Y$ is said to be semi E – prinvex on K w.r. to η if

$$f_i(E(y)+\geq\lambda\eta^T(E(x),E(y))\leq\lambda f_i(x)+(1-\lambda)f_i(y)),$$
$$\forall x,y\in K,\ \lambda\in[0.1].$$

Lemma 2.1. D⊂Y∈R^m be a convex cone with int $D\neq Q$. Then,

i) $\forall\mu_1^x\in D^*\searrow\{0_y^*\},\ x_1\in\text{int } D\Rightarrow\left|(\mu_1^*-x_1)\right|>0.$

ii) $\forall\ \mu_1^*\in\text{int } D^*,\ x_1\in D\searrow\{0_y\}\Rightarrow(\mu_1^*-x_1)>0.$

Let us define the concepts required in the sequel.

International Journal of Applied Engineering Research ISSN 0973-4562 Volume 14, Number 12 (2019) pp. 2960-2967
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Definition 2.5. Suppose $f_i : D \to V$ is a map, where $D \subset X$ is an E-invex set w.r.t. η then the map f is said to be $\eta - V - E -$ Semi – differentiable at $(\bar{x}_1) \in D$ if

$$f_i\left(E(\bar{x}), \eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) = \lim_{\substack{\lambda_1 \to 0^+ \\ \rho_1 \to 0^+}} \frac{1}{\lambda}\left[f_i\left(E(\bar{x}_1) + \lambda\eta^T\left(E(x_1), E(\bar{x}_1)\right) - f_i\left(E(\bar{x}_1)\right)\right)\right] + \rho \| \theta(x_1, \bar{x}_1) \|^2,$$

for each $x_1 \in D$.T.

Definition 2.6 A function $f : D \to Y$ is called semi –E-V-invex at $\bar{x}$, on $D \subset X$ w.r. to y. If f_i is E-η-V-semi differentiable at $\bar{x}_1 \in D$ and $E(\bar{x}_1) = \bar{x}_1$,

$$f_i(x_1) - f_i(\bar{x}_1) \geq \lambda\ f_i\left(\bar{x}_1; \eta^T E(x_1), E(\bar{x}_1)\right) + \rho \| \theta(x_1, \bar{x}_1) \|^2$$

We generalize the concepts of [12, 20] as follows.

Definitions 2.7

The function (f_i, g) is said to be semi E-v-ρ-type-I at $\bar{x} \in D$ w.r. to η and x, if for each $x \in D$, ∃ two maps E, η and θ such that $E(\bar{x}_1) = \bar{x}_1$, and for all $\mu^* \in D^*, v_j \in D_j, j \in M,$ we have

$$\mu_1^*, f_i(x_1) - f_i(\bar{x}_1) \geq (\mu_1^* of)\ f_i\left(\bar{x}_1, \eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho \left\| \theta(x_1 - \bar{x}_1) \right\|^2$$

$$-\sum_{j=1}^{m}\left(v_j^* o\, g_j(\bar{x}_1)\right) \geq (v_j\, o\, gj) g_j\left(\bar{x}, \eta^T\left(E\left(x_1, E(\bar{x})\right)\right)\right) + \rho \| \theta(x_1, \bar{x}_1) \|^2.$$

Definition 2.8

The functions (f_i, g) is said to be semi – quasi – E-v-ρ Type-I at $\bar{x}_1$ w.r. to η, θ if for each $x_1 \in D$, ∃ maps E, η and θ such that $E(\bar{x}_1) = \bar{x}_1$ and for all $\mu_1^* \in D^*, v_j^* \in D_j^*, J \in M.$ we have,

$$\left(\mu_1^*, f_i(x_1)\right) \leq \left(\mu_1^*, f_i(\bar{x}_1)\right) \Rightarrow (\mu_1^* of) f_i\left(\bar{x}_1 : \eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho \| \theta(x_1, \bar{x}_1) \| \leq 0. \quad (2.3)$$

and

$$-\sum_{j=1}^{m}\left(v_j^{\alpha}, g_j(\bar{x}_1)\right) \leq 0 \Rightarrow \sum_{j=1}^{m}\left(v_j^*, g_j\right) g_j\left(\bar{x}, \eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho \left\| \theta(x_1, \bar{x}_1) \right\|^2 \leq 0. \quad (2.4)$$

Definition 2.9

The functions (f_i, g) is said to be semi pseudo-E-V-type-I-invex at $\bar{x}_1 \in D$ w.r. to η, θ if each $x_1 \in D$ ∃ maps E, η and θ ∍ $E(\bar{x}_1) = \bar{x}_1$ and for all $\mu \in D$, $u_j \in D_j$, $j \in M$, we have $(\mu_1^* of) f_i\left(\left(\bar{x}_1, \eta^T\left(E(x_1)\right), E(\bar{x}_1)\right)\right) \geq 0$

$$\Rightarrow \left(\mu_1^*, f_i(x_1)\right) \geq \left(\mu_1^*, f_i(\bar{x}_1)\right);\ \sum_{j-1}^{m}\left(v_j^* o\, g_j\right) g_j\left(\bar{x}; \eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho \| \theta(x_1, \bar{x}_1) \|^2 \geq 0$$

$$\Rightarrow \sum_{j=1}^{m}\left(v_j\, o\, g_j(\bar{x},)\right) \geq 0$$

Definition 2.10

The functions (f_i, g) is said to be semi quasi-E-V-type-I-invex at $\bar{x}_1 \in D$ w.r. to η, θ if each $x_1 \in D$ ∃ maps E, η and θ ∍ $E(\bar{x}_1) = \bar{x}_1$ and for all $\mu \in D$, $u_j \in D_j$, $j \in M$, we have

$$\Rightarrow \left(\mu_1^*, f_i(x_1)\right) \geq \left(\mu_1^*, f_i(\bar{x}_1)\right);\ \sum_{j-1}^{m}\left(v_j^* o\, g_j\right) g_j\left(\bar{x}; \eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho \| \theta(x_1, \bar{x}_1) \|^2 \geq 0$$

$$\Rightarrow \sum_{j=1}^{m}\left(v_j\, o\, g_j(\bar{x},)\right) \geq 0 \text{, we have } (\mu_1^* of) f_i\left(\left(\bar{x}_1, \eta^T\left(E(x_1)\right), E(\bar{x}_1)\right)\right) \geq 0$$

Definition 2.11

The functions (f_i, g) is said to be quasi pseudo-E-V-type-I-invex at $\bar{x}_1 \in D$ w.r. to η, θ if each $x_1 \in D$ $\exists$ maps E, η and θ $\ni$ $E(\bar{x}_1) = \bar{x}_1$ and for all $\mu \in D$, $u_j \in D_j$, $j \in M$, we have $(\mu_1^* of) f_i ((\bar{x}_1, \eta^T (E(x_1)), E(\bar{x}_1))) \le 0$

$$\Rightarrow (\mu_1^*, f_i(x_1)) \ge (\mu_1^*, f_i(\bar{x}_1)); \ \sum_{j-1}^{m} (v_j^* o g_j) g_j (\bar{x}; \eta^T (E(x_1), E(\bar{x}_1))) + \rho \| \theta (x_1, \bar{x}_1) \|^2 \le 0$$

$$\Rightarrow \sum_{j=1}^{m} (v_j o g_j (\bar{x},)) \le 0$$

3. OPTIMALITY CONDITIONS

Here, we will state and prove suffering optimality conditions for the optimization (MP).

In the sequel, we need generalized Gordan type alternative theorem to get the required necessary optimality conditions.

We will state the following Lemma, which is useful in the sequel of our work.

Lemma 3.1 Suppose $f: X \to Y$ is a semi E-V-preinvex on E-V-invex set $D \subset X$ w.r.t. η, θ, if $D \subset Y$ is a convex cone with non empty interior. Then, either

i) $\exists \ x_1 \in D, \text{s.t} - f_i(x_1) \in \text{int } D$

(or)

ii) $\exists \ q \in D^* \searrow \{0\} \text{s.t} (\text{gotf}_i)(D) < R+$

where $R_+ = \{\alpha \in R, \ \alpha \ge 0\}$

3.1 Necessary Optimality Conditions

Theorem 3.1: Let f_i and g_j, $j \in M$ be semi-preinvex maps on a E-V-invex set $D \subset X$ with respect to η, θ and all $\eta - \theta - E -$ semi-differentiable at $\bar{x}_1 \in D$, where $E(\bar{x}_1) = \bar{x}_1$. Suppose $\bar{x}_1$ is a weakly efficient solution of the problem (MP), then

$\exists \ \mu_1 \in D, \ v_j \in D_j$, not all zero, $\exists \ (\mu_1^* of_i) f_i (\bar{x}_1; \eta^T (E(x_1), E(\bar{x}_1)))$

$$+ \sum_{j=1}^{m} (u_j o g_j) g_j (\bar{x}; \eta^T (E(x_1), E(\bar{x}_1))) > 0 \ \forall x_1 \in F \tag{3.10}$$

$$\text{and } \sum_{i=1}^{m} (v_j^* o g_j (\bar{x}_1)) = 0 \tag{3.11}$$

Proof : By using proposition 3 in [12, 20], it follows that the corresponding feasible set $F_1 = \{x_1 \in D : g_j(x_1) \in D_j, j \in M\}$ is E-V-invex set w.r. to η and θ.

Suppose $\bar{x}_1$ is a weakly efficient solution of the problem (MP).

In this context, the system being

- $[(f_i(x_1)) - f_i(\bar{x}_1) \times g_j(x_1)] \in \text{int} \in D \times D_j, j \in M$, has no solution.

But from Lemma 3.1,

$\exists$ q= (τ*, $(\tau^*, v_j^*) \in (D^*, D_j^*) - \{0, 0\} \ni \tau_0^* [f_i(r_1) - f_i(\bar{r}_1)] + v_j^* \delta_j (r_j) \ge 0, j \in M, x \in F_1$.

Implies that $v_j^* o g_j (\bar{x}_1) \ge 0, j \in M$.

Further, if $\bar{x}_1 \in F$, which implies that $v_j^* o g_j (\bar{x}_1) \le 0, j \in M$.

From the above, that is $v_j^* o g_j (\bar{x}_1) = 0, \ j \in M$. (3.13)

Since F_1 is E-v-invex set and f_i, g_j, $j \in M$ and η-E-v-semi-differentiable at $(\bar{x}_1)$, where $E(\bar{x}_1) = \bar{x}_1$, from the above equations (3.12) and (3.13), it follows that

$$\tau_1^* o f_i\left(\bar{x}_1 + \lambda\eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho\,\|\,\theta\left(E\left((\bar{x}_1), \bar{x}_1\right)\right)\|^2$$

$$\frac{\tau^* o f_i(\bar{x}_1) + v_j^* o g_j\left(\bar{x}_1 + \lambda\eta^T\left(E(x_1), E(\bar{x}_1)\right)\right) + \rho\,\|\,\theta\left(E(\bar{x}_1, \bar{x}_1)\right)\|^2 - v_j^* o g_j(\bar{x}_1)}{\lambda}$$

$$\left(\tau^* o f_i\right) f_i\left(\bar{x}_1; \eta^T\left(E(\bar{x}_1), \bar{x}_1\right) + \rho\eta.\left(E(x_1), \bar{x}_1\right)\eta^2 + \left(v_j^* o\delta j\right)^1\left(\bar{x}_1; \eta^T E(\bar{x}_1), \bar{x}_1\right)\right) +$$

$$\rho\left\|\theta\left(E(\bar{x}_1), \bar{x}_1\right)\right\|^2 \geq 0\ j \in m, x_1 \in f \qquad (3.14)$$

From the above equation (3.13) and (3.14), we obtain $\sum_{j=1}^{m}\left(v_j^*, g_j(\bar{x}_1)\right) = 0$

and consequently $m\left(\tau_i^* o f_i\right) f_i\left(\bar{x}_1; \eta^T\left(E(\bar{x}_1), \bar{x}_1\right)\right) + \rho\left\|\theta\left(E(x_i), \bar{x}_1\right)\right\|^2$

$$+\sum_{j=1}^{m}\left(v_j^* o g_j\right) g_j\left(\bar{x}; \eta^T\left(E(\bar{x}_1, \bar{x}_1)\right)\right) + \rho\left\|\theta\left(E(x_1, \bar{x}_1)\right)\right\|^2 \geq 0. \qquad (3.15)$$

By setting $m\tau = m\tau^* = \mu_1^*$ in the equation (3.15), we obtain the required result. Hence proved. □

We will give sufficient optimality conditions by using the concept of semi-E-V- type-I invex functions.

Theorem 3.2: Suppose there exists $\bar{x}_1 \in F_1$ and $\mu_1^* \in D^* \searrow \{0y^*\}$ or $\mu_1^* \in \operatorname{int} D^*$ $v_j^* \in D_j^*$, $j \in M$, the equations (3.10) and (3.11) holds. Further, if any of the following conditions hold :

i) (f_i, g_j) is semi-E-V-ρ-invex- type-I w.r.t. the same η and θ;
ii) (f_i, g_j) is pseudo quasi semi-E-V-ρ-type-I invex at $\bar{x}_1 \in F$, w.r. to the same η and θ; and
iii) (f_i, g_j) is quasi strictly pseudo semi-E-V-ρ-type-I-invex at $\bar{x}_1 \in F$ w.r. to the same η and θ. Thus $\bar{x}_1$ is a weakly efficient solution for the problem (MP).

Proof : We prove this by contradiction.

Let us assume that $\bar{x}_1$ is not a weakly efficient solution for the problem (MP). Then, ∃ a feasible solution $\hat{x}_1$ of the problem (MP)

i.e., $f_i(\hat{x}_1) <c\ f_i(\bar{x}_1)$, i=1,2,….,p.

But $\mu_1^* \in D^* \searrow \{0_{y^*}\}$ and by Lamma 2.1,

we obtain $\left(\mu_1^*\ f_i(\hat{x}_1) - f_i(\bar{x}_1)\right) < 0 \qquad (3.16)$

$$\left(\mu_1^* o f_i\right) f_i\left(\bar{x}_1; \eta^T E\left((\hat{x}_1, \bar{x}_1)\right) + \rho\left\|\theta(\hat{x}_1, \bar{x}_1)\right\|^2\right) < 0 \qquad (3.17)$$

Based on the relation (3.11) and condition (i) in Th. 3.2, we set

$$\sum_{j=1}^{m}\left(v_j^* o v_j\right) v_j\left(\bar{x}_1; \eta^T\left(E(\hat{x}_1), \bar{x}_1\right)\right) + \rho\left\|\theta(\hat{x}_1), \bar{x}_1\right\|^2 < 0 \qquad (3.18)$$

Now, adding (3.17) and (3.18), we obtain

$$\left(\mu_1^* o f_i\right) f_i\left(\bar{x}_1; \eta^T\left(E(\hat{x}_1), \bar{x}_1\right) + \rho\left\|\theta\left(E(\hat{x}_1), \bar{x}_1\right)\right\|^2\right) + \sum_{i=1}^{m}\left(v_j^* o g_j\right) g_j\left(\bar{x}_1; \eta^T\left(E(\hat{x}_1), \bar{x}_1\right)\right) +$$

$$\rho\left\|\theta(\hat{x}_1), \bar{x}_1\right\|^2 < 0.$$

Which is a contradiction to equation (3.10) into by applying condition (ii) in Th. 3.2, one can obtain the relations as

$$\sum_{j=1}^{m}\left(v_j^* o g_j\right) g_j\left(\bar{x}_1; \eta^T\left(E(\hat{x}_1), \bar{x}_1\right)\right) + \rho\left\|\theta\left(E(\hat{x}_1), \bar{x}_1\right)\right\|^2 \leq 0.$$

Now, consider equation (3.10), we obtain

$$\left(\mu_1^* o f_i\right) f_i\left(\bar{x}_1; \eta^T\left(E\left(\hat{x}_1\right), \bar{x}_1\right) + \rho\left\|\theta\left(E\left(\hat{x}_1\right), \bar{x}_1\right)\right\|^2\right) \geq 0.$$

But, by condition (ii) of Th. 3.2, we obtain again, as $\left(\mu_1^* o f_i\left(\hat{x}_1\right)\right) - f_i\left(\bar{x}_1\right) \geq 0$, which is contradiction to eqn. (3.16).

Finally, let us apply condition (iii) of The 3.2 and relation (3.16), we set

$$\left(\mu_1^* o f_i\right) f_i\left(\bar{x}_1; \eta^T\left(E\left(\hat{x}_1\right), \bar{x}_1\right) + \rho\left\|\theta\left(E\left(\hat{x}_1\right), \bar{x}_1\right)\right\|^2\right) \leq 0.$$

Let us combine the above inequality with equation (3.10), we obtain

$$\sum_{j=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(\bar{x}_1; \eta^T\left(E\left(\hat{x}_1\right), \bar{x}_1\right)\right) + \rho\left\|\theta\left(\hat{x}_1, \bar{x}_1\right)\right\|^2 \geq 0.$$

By applying condition (iii) of Th. 3.2, it leads to the following as

$$-\sum_{j=1}^{m}\left(\nu_j^* o g_j\left(\bar{x}_1\right)\right) > 0,$$

which is a contradiction to eqn. (3.11). Hence, the result proved. □

4. DUALITY

Here, we consider the Mond – Weir type dual problem as follows :

Let us consider the dual problems (MDP) as follows :

(MDP) max f(u)

s.t.c :

$$\left(\mu_1^* o f_i\right) f_i\left(u; \eta^T\left(E(x), u\right)\right) + \sum_{j=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(E(x), u\right)\right) \geq 0, \ \forall x \in F_1,$$

$$\sum_{j=1}^{m}\left(\nu_j^* o g_j\right) g_j(u) \geq 0,$$

$$u \in D, \ \mu^* \in D^*, \ u_j^x \in D_j^x, j \in M..$$

Let us define the feasible solution of the above problem as follows :

$$G_1 = \left\{\left(u, \mu_1^*, v_j^*\right) : \left(\mu_1^* o f_i\right) f_i \left| u; \eta^T\left(E(x), u\right) \right. + \sum_{j=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(E(x), u\right)\right) \geq 0\right\}$$

$$\sum_{j=1}^{m}\left(\nu_j^* o g_j(x)\right) \geq 0, \ \forall x_1 \in F_1,$$

$$y \in D, \ \mu^*, \ u_j^* \in D_j^*; j \in M\Big\} \qquad (4.1)$$

For the above dual problem, we will state and prove weak, strong and converse quality, theorems as follows :

Theorem 4.1 (weak quality)

Let $x_1 \in F_1\left(u, \mu_j^*, u_j^*\right) \in G_1$ and $\mu_1^* \in D_1^* \searrow \left\{0_{y*}\right\}$ or $\mu_1^* \in$ and D^*. Further, suppose if any one of the following conditions is holds :

i) (f_i, g_j) is semi-E-u-ρ-type-I–invex at u ∈ F_1 w.r.t. the same η and θ;
ii) (f_i, g_j) is pseudo quasi semi E-u-ρ-type-I-invex w.r.t. the same η and θ; and
iii) (f_i, g_j) is quasi strictly pseudo semi-E-u-ρ-type-2-invex at u∈F_1 w.r.t. to the same η and u.

Then

$$f_i\left(x_j\right) \nless_c f_i(u) \text{ or}$$

$$f_i\left(x_1\right) \nless_c f_i(u)$$

Proof : We prove this by contradiction. Let $\hat{x}_1 \in F_1$, $\left(u, \mu_1^*, v_j^*\right) \in G_1$ ∍

Let $\quad f_i\left(\hat{x}_1\right) < c\, f_i\left(u\right)$

Since $\mu_1^* \in D^* \searrow \left\{0_y^*\right\}$ and by Lemma 2.1, where

$$\left(\mu_1^*, f_i\left(\hat{x}_1\right) - f_i\left(u\right)\right) < 0 \tag{4.2}$$

Since $\left(u, \mu_1^*, \nu_j^*\right) \in G_1$, we have $-\sum_{j=1}^{m}\left(\nu_j^* o g_j\left(u\right)\right) \le 0$ (4.3)

Based on the inequality in (4.1) and since $\hat{x}_1 \in F_1$, we set

$$\left(\mu_1^* o f_i\right) f_i\left(u; \eta^T\left(E\left(\hat{x}_1\right)u\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2\right)$$
$$+\sum_{i=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(E\left(\hat{x}_1\right), u\right)\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2 \ge 0 \tag{4.4}$$

By using the above relations (4.2) and (4.3), and applying condition (i) in Th. 4.1, we get

$$\left(\mu_1^* o f_i\right) f_i\left(u; \eta^T\left(E\left(\hat{x}_1\right), u\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2\right) < 0$$

and $\sum_{i=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(\hat{x}_1, u\right)\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2 \le 0$

on summation, the above two inequalities gives

$$\left(\mu_1^* o f_i\right) f_i\left(u; \eta^T\left(E\left(\hat{x}_1\right), u\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2\right) + \sum_{i=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(E\left(\hat{x}_1\right), u\right)\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2 < 0$$

which is a contradiction to the inequality (4.4)

Suppose, if condition (ii) holds, that

$$-\sum_{i=1}^{m}\left(\nu_j^* o g_j\left(x\right)\right) < 0$$

which implies that

$$\sum_{i=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(E\left(\overline{x}_1\right), u\right)\right) + \rho\left\|\theta\left(\hat{x}_1, u\right)\right\|^2 \le 0$$

Now by using eqn. (4.4), we obtain

$$\left(\mu_1^* o f_i\right) f_i\left(u; \eta^T\left(E\left(\hat{x}_1\right), u\right) + \rho\left\|\theta\left(\hat{x}_1\right), u\right\|^2\right) \ge 0$$

Now apply condition (ii) we get

$\left(\mu_1^* o f_i\left(\overline{x}_1\right)\right) - \left(f_i\left(x\right)\right) \ge 0$, which is a contradiction to (4.4).

If condition (iii) of theorem 4.2 is applied then equation (4.2) becomes to $\left(\mu_1^* o f_i\right) f_1\left(u; \eta^T\left(E\left(\hat{x}_1, u\right)\right) + \rho \,||\, \theta\left(\hat{x}_1, u\right)\right)||^2 \le 0$

Again applying equation (4.4), we obtain

$$\sum_{i=1}^{m}\left(\nu_j^* o g_j\right) g_j\left(u; \eta^T\left(E\left(\overline{x}_1\right), u\right)\right) + \rho\left\|\theta\left(\hat{x}_1, u\right)\right\|^2 \ge 0$$

Again, apply the condition (iii) of theorem 4.2 we obtain

$-\sum_{i=1}^{m}\left(\nu_j^* o g_j\left(x\right)\right) > 0$, which contradicts with condition (4.3).

Hence the result proved. □

Theorem (4.2) (Strong Duality)

Let $\tilde{x} \in F_1$, $\left(u, \mu_1^*, v_j^*\right) \in G_1\, \mu_1^* \in D^* \searrow \left\{o_y^*\right\}$ be an efficient solution and assume that $\tilde{x}$ satisfies a constrain qualification for (MP). Then $\exists \mu_1^*, v_j^*$ such that $\left(u, \mu^*, v_j^*\right) \in G_1\, \mu_1^* \in D^* \searrow \left\{o_y^*\right\}$ is feasible for (MD). Moreover, if weak duality theorem 4.1 holds between (MP) and (MDP) then $\left(u, \mu^*, v_j^*\right)$ is an efficient solution for (MDP).

Proof: Proof follows from theorem 4.1.

CONCLUSIONS

In this paper we derived optimality conditions duality results with respective generalized E-V-ρ-univexity. These results are generalizations of Hehua Jiao [12] and Youness [27].

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