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# On the Convergence and Stability of New Hybrid Iteration Process in Banach spaces

**Naresh Kumar**

*Department of Mathematics, M. D. University, Rohtak 124001, Haryana, India.*

**Renu Chugh**

*Department of Mathematics, M. D. University, Rohtak 124001, Haryana, India.*

**Abstract :**
The purpose of this paper is to establish a new iterative process which is hybrid of Picard and SP iterative processes. In this paper, we show that our Picard-SP hybrid iterative process gives faster convergence results than existing Picard, Mann, Krasnoselski, Ishikawa and Picard-Krasnoselski hybrid iterative processes. In support of our claim, a numerical example with graph is presented using the MATLAB software. Moreover, we also show that Picard-SP hybrid iterative process is T-stable.

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## 1. Introduction and preliminaries

Approximation theory play an important role in finding the fixed points of a problem. There are some problems which are not solved exactly by well known methods.In this case, several authors used the approximation methods to find the solution of such problems.

Let $S$ be a nonempty convex subset of a normed space $M$ and $T : S \to S$ be a mapping. The mapping $T : S \to S$ is said to be a contraction if

$$\|Tx - Ty\| \leq \ell \|x - y\| \quad \text{for each } x, y \in S \ \text{ and } \ \ell \in (0,1) \tag{1.1}$$

The set of all fixed points of $T$ is denoted by $F(T)$.

In 1890, Picard [1] introduced the iteration process defined by the sequence $\{v_n\}$, known as Picard or successive or repeated function iteration process.

$$\begin{cases} v_1 = v \in S, \\ v_{n+1} = Tv_n, \quad n \in \mathbb{N} \end{cases} \tag{1.2}$$

Throught this paper, we use $\mathbb{N}$ for set of all positive integers.

In 1953, Mann [2] defined the following one step iteration process for sequence $\{u_n\}$.

$$\begin{cases} u_1 = u \in S \\ u_{n+1} = (1 - \alpha_n) u_n + \alpha_n Tu_n, \quad n \in \mathbb{N} \end{cases} \tag{1.3}$$

where the sequence $\{\alpha_n\}$ belongs to $(0,1)$. Mann showed that after taking $\alpha_n = 1$ in (1.3), it converts into Picard iterative process. Thus they claim that Mann iteration is generalization of Picard iteration process.

In 1955, an iterative process defined by the sequence $\{y_n\}$ is presented by Krasnoselski [3], called Krasnoselski iterative process as follows:

$$\begin{aligned} & y_1 \in S \\ & y_{n+1} = (1 - \lambda) y_n + \lambda Ty_n, \quad n \in \mathbb{N}, \end{aligned} \tag{1.4}$$

where $\lambda \in (0,1)$.

In 1974, Ishikawa [4] defined the following iteration process for sequence $\{s_n\}$:

$$\begin{aligned} & s_1 = s \in S \\ & s_{n+1} = (1 - \alpha_n) s_n + \alpha_n Ty_n \\ & y_n = (1 - \beta_n) s_n + \beta_n Ts_n \end{aligned} \tag{1.5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$
In 2013, Khan [5] introduced a new iterative process as follows:

$$\begin{aligned} & s_1 = s \in S \\ & s_{n+1} = Tz_n \\ & z_n = (1 - \alpha_n) s_n + \alpha_n Ts_n, \ n \in \mathbb{N}, \end{aligned} \tag{1.6}$$

where $\{\alpha_n\}$ is sequence of positive numbers in $(0,1)$.

In 2017, Godwin Amechi Okeke and Mujahid Abbas [6] introduced the Picard-Krasnoselski hybrid iterative process for sequence $\{w_n\}$ as follows:

$$\begin{aligned} & w_1 = w \in S \\ & w_{n+1} = Ty_n \\ & y_n = (1 - \lambda) w_n + \lambda Tw_n, \ n \in \mathbb{N}, \end{aligned} \tag{1.7}$$

where $\lambda \in (0,1)$.

Okeke and Abbas [6] show that for contraction mappings, their iteration process has faster rate of convergence than above discussed iterative processes.

In 2011, Withun Phuengrattana and Suthep Suantai [7] proposed the following iteration process:

$$\begin{aligned} & x_1 = x \in S \\ & z_n = (1 - \gamma_n) x_n + \gamma_n Tx_n \\ & y_n = (1 - \beta_n) z_n + \beta_n \ Tz_n \\ & x_{n+1} = (1 - \alpha_n) y_n + \alpha_n \ Ty_n \end{aligned} \tag{1.8}$$

for all $n \geq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$, known as SP-iteration process.

Motivated by the above works, we introduce a new hybrid iterative process, Picard-SP iteration process as follows:

$$\begin{aligned} x_{n+1} &= Tv_n \\ v_n &= (1-\gamma_n)z_n + \gamma_n Tz_n \\ z_n &= (1-\beta_n)y_n + \beta_n Ty_n \\ y_n &= (1-\alpha_n)x_n + \alpha_n Tx_n, \ \ n \in \mathbb{N}, \end{aligned} \tag{1.9}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequence in $[0,1]$. Our Picard-SP hybrid iterative process (1.9) gives faster rate of convergence than existing iterative processes (1.2), (1.3), (1.4), (1.5) and (1.7).We also show this by comparison tables using the MATLAB programming. Moreover, stability of our new Picard-SP hybrid iteration process is also shown.

Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes that converge to a fixed point $q$ of a given operator $T$.The sequence $\{x_n\}$ is better than $\{y_n\}$ in the sense of Rhoades [8] if $\|x_n - q\| \leq \|y_n - q\|$ for all $n \in \mathbb{N}$.
The definitions presented by Berinde [9] are as follows:

**Definition 1.1** ( [9]). *Let $\{u_n\}$ and $\{v_n\}$ be two sequences of real numbers converging to u and v, respectively. The sequence $\{u_n\}$ is said to converge faster than $\{v_n\}$ if*

$$\lim_{n\to\infty} \frac{|u_n - u|}{|v_n - v|} = 0 \tag{1.10}$$

**Definition 1.2** ( [9]). *Let $\{x_n\}$ and $\{y_n\}$ be two fixed point iteration processes that converge to a certain fixed point q of a given operator $T$. Suppose that the error estimates*

$$\|x_n - q\| \leq u_n \quad \text{for all } n \in \mathbb{N},$$

$$\|y_n - q\| \leq v_n \quad \text{for all } n \in \mathbb{N},$$

*are available, where $\{u_n\}$ and $\{v_n\}$ are two sequences of positive numbers converging to zero. If $\{u_n\}$ converges faster than $\{v_n\}$, then $\{x_n\}$ converges faster than $\{y_n\}$ to q.*

**Definition 1.3** ( [10]). *Let $\{s_n\}_{n=0}^{\infty}$ be an arbitrary sequence in S.Then an iteration procedure $x_{n+1} = f(T, x_n)$, converging to fixed point $q$, is said to be T-stable or stable if for $\in_n = \|s_{n+1} - f(T, t_n)\|$, with respect to T, if for $n = 0, 1, 2, 3, \ldots$, we have*
$\lim_{n\to\infty} \varepsilon_n = 0 \Leftrightarrow \lim_{n\to\infty} s_n = q.$

**Lemma 1.4** ( [11]). *Let $\{w_n\}$ be a sequence of positive real numbers which satisfies:*

$$w_{n+1} \leq (1-\mu_n) w_n.$$

*If $\{u_n\} \subset (0,1)$ and $\sum_{n=1}^{\infty} \mu_n = \infty$, then $\lim_{n\to\infty} w_n = 0$.*

**Lemma 1.5** ( [12]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be non-negative real sequences satisfying the following inequality.*

$$a_{n+1} \leq (1-c_n) a_n + b_n,$$

*where $c_n \in (0,1)$ for all $n \in N$, $\sum_{n=0}^{\infty} C_n = \infty$ and $\frac{b_n}{c_n} \to 0$ as $n \to \infty$.*
*Then $\lim_{n\to\infty} a_n = 0$.*

Several authors have presented the comparison of rate of convergence of various iterative processes (one can see [8, 13–22]).

## 2. Main Results

Now, we shall prove that the Picard-SP hybrid iterative process (1.9) gives the faster rate of convergence than all Picard,Krasnoselski,Mann,Ishikawa and Picard-Krasnoselski hybrid iterative processes.In favor of this, we also present an example using MATLAB Programming.We have also given a graphical representation for this. We showed that our iteration (1.9) converging to same fixed point as above discussed iterations do.And finally, we will also show that our Picard-SP hybrid iterative process (1.9) is stable.

**Theorem 2.1.** *Let S be a nonempty closed convex subset of a Banach space X and*
*$T : S \to S$ be a contraction mapping. Let $\{x_n\}_{n=0}^{\infty}$ be an iterative sequence generated by* (1.9) *with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \gamma_n = \infty$. Then $\{x_n\}_{n=0}^{\infty}$, converge strongly to a unique fixed point of $T$.*

*Proof.* The well-known Banach contraction principle guarantees the existence and uniqueness of a fixed point $q$.We shall show that $x_n \to q$ for $n \to \infty$. From (1.9) we have

$$\begin{aligned} \|y_n - q\| &= \|(1-\alpha_n)x_n + \alpha_n Tx_n - q\| \\ &= \|(1-\alpha_n)x_n + \alpha_n Tx_n - (1-\alpha_n+\alpha_n)q\| \\ &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n \|(Tx_n - Tq)\| \\ &\leq (1-\alpha_n)\|x_n - q\| + \alpha_n \ell \|x_n - q\| \\ &= 1 - \alpha_n(1-\ell)\|x_n - q\| \end{aligned} \tag{2.1}$$

Similarly,

$$\begin{aligned} \|z_n - q\| &\leq 1 - \beta_n(1-\ell)\|y_n - q\| \\ &= 1 - \beta_n(1-\ell)(1-\alpha_n(1-\ell)\|x_n - q\|) \ \ \text{(by 2.1)} \end{aligned} \tag{2.2}$$

$$\begin{aligned} \|v_n - q\| &\leq 1 - \gamma_n(1-\ell)\|z_n - q\| \\ &= 1 - \gamma_n(1-\ell)(1-\beta_n(1-\ell)(1-\alpha_n(1-\ell)\|x_n - q\|)) \ \ \text{(by 2.2)} \end{aligned} \tag{2.3}$$

Now,

$$\begin{aligned}\|x_{n+1}-q\| &= \|Tv_n-q\|\\ &\leq \ \ell\|v_n-q\|\\ &\leq \ell[1-\gamma_n(1-\ell)(1-\beta_n(1-\ell)(1-\alpha_n(1-\ell)\|x_n-q\|))] \text{ (by 2.3)}\\ &\leq \ell(1-\gamma_n(1-\ell)\|x_n-q\|) \end{aligned} \tag{2.4}$$

as $(1-\alpha_n(1-\ell))<1$ and $(1-\beta_n(1-\ell))<1$ for $\ell\in(0,1)$ and $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $[0,1]$. From (2.4), we have,

$$\begin{cases}\|x_{n+1}-q\| \leq (1-\gamma_n)(1-\ell)\|x_n-q\|\\ \|x_n-q\| \ \leq \ell(1-\gamma_{n-1})(1-\ell)\|x_{n-1}-q\|\\ \|x_{n-1}-q\| \ \leq \ell(1-\gamma_{n-2})(1-\ell)\|x_{n-2}-q\|\\ \qquad\vdots\\ \|x_1-p\| \ \leq \ell(1-\gamma_0)(1-\ell)\|x_0-p\|\end{cases} \tag{2.5}$$

Now, from (2.5) we can easily find

$$\|x_{n+1}-q\| \leq \|x_0-q\|\ell^{n+1}\prod_{r=0}^{n}(1-\gamma_n(1-\ell)) \tag{2.6}$$

where $(1-\gamma_r)(1-\ell)\in(0,1)$, because $\ell\in(0,1)$ and $\gamma_r\in[0,1]$, for all $n\in\mathbb{N}$.
As $1-x\leq e^{-x}$ for all $x\in[0,1]$, so from (2.6), we get

$$\|x_{n+1}-q\| \leq \frac{\|x_0-q\|\ell^{n+1}}{e^{(1-\ell)}\sum\limits_{r=0}^{n}\gamma_r} \tag{2.7}$$

Taking limits both sides inequality (2.7) yields $\lim\limits_{n\to\infty}\|x_n-q\|=0$, i.e $x_n\to q$ for $n\to\infty$ as required. ∎

**Theorem 2.2.** *Let $S$ be a non-empty closed convex subset of a normed space $N$ and $T:S\to S$ be a contraction mapping.Suppose that each of the iterative processes* (1.2)*,* (1.3)*,* (1.4)*,* (1.5)*,* (1.7) *and* (1.9) *converge to the same fixed point $q$ of $T$, where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequence in $(0,1)$ such that $0<\alpha\leq\lambda$, $\alpha_n$, $\beta_n$, $\gamma_n<1$, for all $n\in\mathbb{N}$ and for some $\alpha$.Then the Picard-SP hybrid iterative process* (1.9) *converges faster than all the other five iterative processes.*

*Proof.* Suppose that $q$ is the fixed point of the operator $T$.By using (1.1) and Picard iterative process (1.2), we have

$$\begin{aligned}\|v_{n+1}-q\| &= \|Tv_n-q\|\\ &\leq \ell\|v_n-q\|\\ &\ \vdots\\ &\leq \ell^n\|v_1-q\|,\end{aligned} \tag{2.8}$$

Let

$$a_n=\ell^n\|v_1-q\|. \tag{2.9}$$

Using (1.1) and Mann iterative process (1.3), we get

$$\begin{aligned}\|u_{n+1}-q\| &= \|(1-\alpha_n)(u_n-q)+\alpha_n(Tu_n-q)\|\\ &\leq (1-\alpha_n)\|u_n-q\|+\alpha_n\ell\|u_n-q\|\\ &= (1-(1-\ell)\alpha_n)\|u_n-q\|\\ &\leq (1-(1-\ell)\alpha)\|u_n-q\|\\ &\ \vdots\\ &\leq (1-(1-\ell)\alpha)^n\|u_1-q\|\end{aligned} \tag{2.10}$$

Let

$$b_n=(1-(1-\ell)\alpha)^n\|u_1-q\| \tag{2.11}$$

From (1.1) and Krasnoselski iterative process (1.4), it follows that

$$\begin{aligned}\|y_{n+1}-q\| &= \|(1-\lambda)(y_n-q)+\lambda(Ty_n-q)\| \\ &\leq (1-\lambda)\|y_n-q\|+\lambda\ell\|y_n-q\| \\ &= (1-(1-\ell)\lambda)\|y_n-q\| \\ &\leq (1-(1-\ell)\alpha)\|y_n-q\| \\ &\vdots \\ &\leq (1-(1-\ell)\alpha)^n\|y_1-q\| \end{aligned} \tag{2.12}$$

Set

$$C_n = (1-(1-\ell)\alpha)^n\|y_1-q\| \tag{2.13}$$

By (1.1) and Ishikawa iterative process (1.5), we get

$$\begin{aligned}\|y_n-q\| &= \|(1-\beta_n)(x_n-q)+\beta_n(Tx_n-q) \\ &\leq (1-\beta_n)\|x_n-q\|+\beta_n\ell\|x_n-q\| \end{aligned} \tag{2.14}$$

By equation (1.1), Ishikawa iterative process (1.5) and (2.14), we have

$$\begin{aligned}\|s_{n+1}-q\| &= \|(1-\alpha_n)(s_n-q)+\alpha_n(Ty_n-q)\| \\ &\leq (1-\alpha_n)\|s_n-q\|+\alpha_n\,\ell\|y_n-q\| \\ &\leq (1-\alpha_n)\|s_n-q\|+\alpha_n\,\ell[(1-\beta_n)\|y_n-q\|+\beta_n\,\ell\|y_n-q\|] \\ &= (1-\alpha_n)\|s_n-q\|+\alpha_n\,\ell(1-\beta_n)\|s_n-q\|+\alpha_n\beta_n\ell^2\|s_n-q\| \\ &\leq (1-\alpha_n)\|s_n-q\|+\alpha_n\,\ell\|s_n-q\| \\ &= (1-(1-\ell)\alpha_n)\|s_n-q\| \\ &\leq (1-(1-\ell)\alpha)\|s_n-q\| \\ &\vdots \\ &\leq (1-(1-\ell)\alpha)^n\|s_1-q\| \end{aligned} \tag{2.15}$$

Put

$$e_n = (1-(1-\ell)\alpha)^n\|s_1-q\| \tag{2.16}$$

By using (1.1) and Picard-Krasnoselski hybrid iterative process (1.7), we get

$$\begin{aligned}\|w_{n+1}-q\| &= \|Ty_n-q\| \\ &\leq \ell\|y_n-q\| \\ &\leq \ell\|(1-\lambda)(w_n-q)+\lambda(Tw_n-q)\| \\ &\leq \ell\|(1-\lambda)(w_n-q)+\lambda\ell(w_n-q)\| \\ &\leq \ell(1-(1-\ell)\lambda)\|w_n-q\| \\ &\leq \ell(1-(1-\ell)\alpha)\|w_n-q\| \\ &\vdots \\ &\leq [\ell(1-(1-\ell)\alpha)]^n\|w_1-q\| \end{aligned} \tag{2.17}$$

Put

$$h_n = [\ell(1-(1-\ell)\alpha)]^n\|w_1-q\| \tag{2.18}$$

Now, by using (1.1) and our new Picard-SP hybrid iterative process (1.9), we obtain that

$$\begin{aligned}
\|x_n - q\| &= \|Tv_n - q\| \\
&\leq \ell\|v_n - q\| \\
&= \ell\|(1-\gamma_n)(z_n - q) + \gamma_n(Tz_n - q)\| \\
&\leq \ell[(1-\gamma_n)\|z_n - q\| + \gamma_n\,\ell\|z_n - q\|] \\
&= \ell(1-(1-\ell)\gamma_n)\|z_n - q\| \\
&= \ell(1-(1-\ell)\gamma_n)\|(1-\beta_n)(y_n - q) + \beta_n(Ty_n - q)\| \\
&\leq [\ell(1-(1-\ell)\gamma_n)](1-\beta_n)\|y_n - q\| + \beta_n\ell\|y_n - q\| \\
&= [\ell(1-(1-\ell)\gamma_n)][1-(1-\ell)\beta_n]\|y_n - q\| \\
&= [\ell(1-(1-\ell)\gamma_n)][1-(1-\ell)\beta_n]\|(1-\alpha_n)(x_n - q) + \alpha_n(Tx_n - q)\| \\
&\leq [\ell(1-(1-\ell)\gamma_n)][(1-(1-\ell)\beta_n)][(1-\alpha_n)\|x_n - q\| + \alpha_n\ell\|x_n - q\| \\
&= [\ell(1-(1-\ell)\gamma_n][1-(1-\ell)\beta_n][1-(1-(1-\ell)\alpha_n]\|x_n - q\| \\
&\leq [\ell(1-(1-\ell)\alpha][1-(1-\ell)\alpha][1-(1-\ell)\alpha]\|x_n - q\| \\
&\;\;\vdots \\
&\leq [\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^n\|x_1 - q\| \\
&= [\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\|x_1 - q\| \qquad (2.19)
\end{aligned}$$

Set

$$I_n = [\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\|x_1 - q\| \qquad (2.20)$$

Now, we find the rate of convergence of our iterative process (1.9) as follows:

(i)
$$\begin{aligned}
\frac{I_n}{a_n} &= \frac{[\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\|x_1 - q\|}{\ell^n\|v_1 - q\|} \\
&= (1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\frac{\|x_1 - q\|}{\|v_1 - q\|} \to 0 \text{ as } n \to \infty \qquad (2.21)
\end{aligned}$$

Thus $\{x_n\}$ converges faster than $\{v_n\}$ to $q$. That is, the Picard-SP hybrid iterative process (1.9) converges faster than Picard iterative process (1.2). (ii) Similarly, $\frac{I_n}{b_n} = \frac{[\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\|x_1 - q\|}{[(1-(1-\ell)\alpha]^n\|u_1 - q\|}$

$$= [\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^n\frac{\|x_1 - q\|}{\|u_1 - q\|} \to 0 \text{ as } n \to \infty$$

Hence $\{x_n\}$ converges faster than $\{u_n\}$ to $q$.That is, the Picard-SP hybrid iterative process (1.9) converges faster than Mann iterative process (1.3).

(iii) Similarly, $\frac{I_n}{C_n} = [\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\frac{\|x_1 - q\|}{\|y_1 - q\|} \to 0$ as $n \to \infty$

Hence $\{x_n\}$ converges faster than $\{y_n\}$ to $q$.That is, the Picard-SP hybrid iterative process (1.9) converges faster than Picard-Krasnoselski iterative process (1.4).

(iv) $\frac{I_n}{e_n} = [\ell(1-(1-\ell)\alpha]^n[1-(1-\ell)\alpha]^{2n}\frac{\|x_1 - q\|}{\|s_1 - q\|} \to$ as $n \to \infty$ Hence $\{x_n\}$ converges faster than $\{s_n\}$ to $q$.That is, the Picard-SP hybrid iterative process (1.9) converges faster than Ishikawa iterative process (1.5).

(v) Finally,
$$\begin{aligned}
\frac{I_n}{h_n} &= \frac{[\ell(1-(1-\ell)\alpha_)]^n[1-(1-\ell)\alpha]^{2n}\|x_1 - q\|}{[\ell(1-(1-\ell)\alpha]^n\|w_1 - q\|} \\
&= [1-(1-\ell)\alpha]^{2n}\frac{\|x_1 - q\|}{\|w_1 - q\|} \to 0 \text{ as } n \to \infty
\end{aligned}$$

Hence $\{x_n\}$ converges faster than $\{w_n\}$ to $q$.That is, the Picard-SP hybrid iterative process (1.9) converges faster than Picard-Krasnoselski hybrid iterative process (1.7). This complete the proof of Theorem 2.2. ∎

**Example 2.3.** *Let $s = [0,1] \subseteq X = \mathbb{R}$ and $T : S \to S$ be defined by $Tx = \frac{x+2}{3}$ for all $x \in S$. Choose $\alpha_n = \beta_n = \lambda = \frac{1}{2}$ for each $n \in \mathbb{N}$ with the initial value $x_1 = 3$. Clearly $T$ is a contraction mapping with contractive constant $\ell = \frac{1}{3}$ and $F(T) = \{1\}$. Table 1,*

*Table 2 and graphical representation given below show that our iterative process* (1.9) *converges faster than all of Picard, Mann, Krasnoselski, Ishikawa and Picard-Krasnoselski hybrid iterative processes.*

**Table 1**

| S.No. | Picard | Krasnoselski | Mann |
|---|---|---|---|
| 1. | 3.000000000000000 | 3.000000000000000 | 3.000000000000000 |
| 2. | 1.666666666666667 | 2.333333333333333 | 2.333333333333333 |
| 3. | 1.222222222222222 | 1.777777777777778 | 1.777777777777778 |
| 4. | 1.074074074074074 | 1.425925925925926 | 1.425925925925926 |
| 5. | 1.024691358024691 | 1.225308641975309 | 1.225308641975309 |
| 6. | 1.008230452674897 | 1.116769547325103 | 1.116769547325103 |
| 7. | 1.002743484224966 | 1.059756515775034 | 1.059756515775034 |
| 8. | 1.000914494741655 | 1.030335505258345 | 1.030335505258345 |
| 9. | 1.000304831580552 | 1.015320168419448 | 1.015320168419448 |
| 10. | 1.000101610526851 | 1.007710889473149 | 1.007710889473149 |
| 11. | 1.000033870175617 | 1.003872379824383 | 1.003872379824383 |
| 12. | 1.000011290058539 | 1.001941834941461 | 1.001941834941461 |
| 13. | 1.000003763352846 | 1.000972799147154 | 1.000972799147154 |
| 14. | 1.000001254450949 | 1.000487026799051 | 1.000487026799051 |
| 15. | 1.000000418150316 | 1.000243722474684 | 1.000243722474684 |
| 16. | 1.000000139383439 | 1.000121930929061 | 1.000121930929061 |
| 17. | 1.000000046461146 | 1.000060988695104 | 1.000060988695104 |
| 18 | 1.000000015487049 | 1.000030502091076 | 1.000030502091076 |
| 19. | 1.000000005162350 | 1.000015253626713 | 1.000015253626713 |
| 20. | 1.000000001720783 | 1.000007627673748 | 1.000007627673748 |
| 21. | 1.000000000573595 | 1.000003814123671 | 1.000003814123671 |
| 22. | 1.000000000191198 | 1.000001907157435 | 1.000001907157435 |
| 23. | 1.000000000063733 | 1.000000953610584 | 1.000000953610584 |
| 24. | 1.000000000021244 | 1.000000476815914 | 1.000000476815914 |
| 25. | 1.000000000007081 | 1.000000238411498 | 1.000000238411498 |
| 26. | 1.000000000002361 | 1.000000119206929 | 1.000000119206929 |
| 27. | 1.000000000000787 | 1.000000059603858 | 1.000000059603858 |
| 28. | 1.000000000000262 | 1.000000029802060 | 1.000000029802060 |
| 29. | 1.000000000000087 | 1.000000014901074 | 1.000000014901074 |
| 30. | 1.000000000000029 | 1.000000007450551 | 1.000000007450551 |
| 31. | 1.000000000000010 | 1.000000003725281 | 1.000000003725281 |
| 32. | 1.000000000000003 | 1.000000001862642 | 1.000000001862642 |
| 33. | 1.000000000000001 | 1.000000000931322 | 1.000000000931322 |
| 34. | **1.000000000000000** | 1.000000000465661 | 1.000000000465661 |
| 35. | 1.000000000000000 | 1.000000000232830 | 1.000000000232830 |
| 36. | 1.000000000000000 | 1.000000000116415 | 1.000000000116415 |
| 37. | 1.000000000000000 | 1.000000000058208 | 1.000000000058208 |
| 38. | 1.000000000000000 | 1.000000000029104 | 1.000000000029104 |
| 39. | 1.000000000000000 | 1.000000000014552 | 1.000000000014552 |
| 40. | 1.000000000000000 | 1.000000000003638 | 1.000000000007276 |
| 41. | 1.000000000000000 | 1.000000000001819 | 1.000000000003638 |
| 42. | 1.000000000000000 | 1.000000000000909 | 1.000000000001819 |
| 43. | 1.000000000000000 | 1.000000000000455 | 1.000000000000909 |
| 44. | 1.000000000000000 | 1.000000000000227 | 1.000000000000455 |
| 45. | 1.000000000000000 | 1.000000000000114 | 1.000000000000227 |

Continued on next page

**Table 1 – continued from previous page**

| S.No. | Picard | Krasnoselski | Mann |
|---|---|---|---|
| 46. | 1.000000000000000 | 1.000000000000057 | 1.000000000000114 |
| 47. | 1.000000000000000 | 1.000000000000028 | 1.000000000000057 |
| 48. | 1.000000000000000 | 1.000000000000014 | 1.000000000000028 |
| 49. | 1.000000000000000 | 1.000000000000007 | 1.000000000000014 |
| 50. | 1.000000000000000 | 1.000000000000004 | 1.000000000000007 |
| 51. | 1.000000000000000 | 1.000000000000002 | 1.000000000000004 |
| 52. | 1.000000000000000 | 1.000000000000001 | 1.000000000000002 |
| 53. | 1.000000000000000 | **1.000000000000000** | 1.000000000000001 |
| 54. | 1.000000000000000 | 1.000000000000000 | **1.000000000000000** |
| 55. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 56. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 57. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 58. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 59. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 60. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 61. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 62. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 63. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 64. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 65. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 66. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 67. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 68. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 69. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 70. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 71. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 72. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 73. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 74. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |
| 75. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |

**Table 2**

| S.No. | Ishikawa | Picard-Krasnoselski | Picard-SP |
|---|---|---|---|
| 1. | 3.000000000000000 | 3.000000000000000 | 3.000000000000000 |
| 2. | 2.222222222222222 | 1.444444444444445 | 1.197530864197531 |
| 3. | 1.746913580246914 | 1.098765432098765 | 1.019509221155312 |
| 4. | 1.456447187928670 | 1.021947873799726 | 1.001926836657315 |
| 5. | 1.278939948178631 | 1.004877305288828 | 1.000190304855044 |
| 6. | 1.170463301664719 | 1.001083845619740 | 1.000018795541239 |
| 7. | 1.104172017683995 | 1.000240854582164 | 1.000001856349752 |
| 8. | 1.063660677473552 | 1.000053523240481 | 1.000000183343185 |
| 9. | 1.038903747344949 | 1.000011894053440 | 1.000000018107969 |
| 10. | 1.023774512266358 | 1.000002643122987 | 1.000000001788441 |
| 11. | 1.014528868607218 | 1.000000587360664 | 1.000000000176636 |
| 12. | 1.008878753037745 | 1.000000130524592 | 1.000000000017446 |

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**Table 2 – continued from previous page**

| S.No. | Ishikawa | Picard-Krasnoselski | Picard-SP |
|---|---|---|---|
| 13. | 1.005425904634177 | 1.000000029005465 | 1.000000000001723 |
| 14. | 1.003315830609775 | 1.000000006445659 | 1.000000000001723 |
| 15. | 1.002026340928196 | 1.000000001432369 | 1.000000000000170 |
| 16. | 1.001238319456120 | 1.000000000318304 | 1.000000000000017 |
| 17. | 1.000756750778740 | 1.000000000318304 | 1.000000000000002 |
| 18 | 1.000462458809230 | 1.000000000070734 | **1.000000000000000** |
| 19. | 1.000282613716752 | 1.000000000015719 | 1.000000000000000 |
| 20. | 1.000172708382459 | 1.000000000003493 | 1.000000000000000 |
| 21. | 1.000105544011503 | 1.000000000000776 | 1.000000000000000 |
| 22. | 1.000064499118141 | 1.000000000000173 | 1.000000000000000 |
| 23. | 1.000039416127753 | 1.000000000000038 | 1.000000000000000 |
| 24. | 1.000024087633627 | 1.000000000000009 | 1.000000000000000 |
| 25. | 1.000014720220550 | 1.000000000000002 | 1.000000000000000 |
| 26. | 1.000008995690336 | **1.000000000000000** | 1.000000000000000 |
| 27. | 1.000005497366316 | 1.000000000000000 | 1.000000000000000 |
| 28. | 1.000003359501638 | 1.000000000000000 | 1.000000000000000 |
| 29. | 1.000002053028779 | 1.000000000000000 | 1.000000000000000 |
| 30. | 1.000001254628698 | 1.000000000000000 | 1.000000000000000 |
| 31. | 1.000000766717537 | 1.000000000000000 | 1.000000000000000 |
| 32. | 1.000000468549606 | 1.000000000000000 | 1.000000000000000 |
| 33. | 1.000000286335871 | 1.000000000000000 | 1.000000000000000 |
| 34. | 1.000000174983032 | 1.000000000000000 | 1.000000000000000 |
| 35. | 1.000000106934075 | 1.000000000000000 | 1.000000000000000 |
| 36. | 1.000000065348601 | 1.000000000000000 | 1.000000000000000 |
| 37. | 1.000000039935256 | 1.000000000000000 | 1.000000000000000 |
| 38. | 1.000000024404879 | 1.000000000000000 | 1.000000000000000 |
| 39. | 1.000000014914093 | 1.000000000000000 | 1.000000000000000 |
| 40. | 1.000000009114168 | 1.000000000000000 | 1.000000000000000 |
| 41. | 1.000000005569769 | 1.000000000000000 | 1.000000000000000 |
| 42. | 1.000000003403748 | 1.000000000000000 | 1.000000000000000 |
| 43. | 1.000000002080068 | 1.000000000000000 | 1.000000000000000 |
| 44. | 1.000000001271153 | 1.000000000000000 | 1.000000000000000 |
| 45. | 1.000000000776816 | 1.000000000000000 | 1.000000000000000 |
| 46. | 1.000000000474721 | 1.000000000000000 | 1.000000000000000 |
| 47. | 1.000000000290107 | 1.000000000000000 | 1.000000000000000 |
| 48. | 1.000000000177288 | 1.000000000000000 | 1.000000000000000 |
| 49. | 1.000000000108342 | 1.000000000000000 | 1.000000000000000 |
| 50. | 1.000000000066209 | 1.000000000000000 | 1.000000000000000 |
| 51. | 1.000000000040461 | 1.000000000000000 | 1.000000000000000 |
| 52. | 1.000000000024726 | 1.000000000000000 | 1.000000000000000 |
| 53. | 1.000000000015111 | 1.000000000000000 | 1.000000000000000 |
| 54. | 1.000000000009234 | 1.000000000000000 | 1.000000000000000 |
| 55. | 1.000000000005643 | 1.000000000000000 | 1.000000000000000 |
| 56. | 1.000000000003449 | 1.000000000000000 | 1.000000000000000 |
| 57. | 1.000000000002108 | 1.000000000000000 | 1.000000000000000 |
| 58. | 1.000000000001288 | 1.000000000000000 | 1.000000000000000 |
| 59. | 1.000000000000787 | 1.000000000000000 | 1.000000000000000 |
| 60. | 1.000000000000481 | 1.000000000000000 | 1.000000000000000 |

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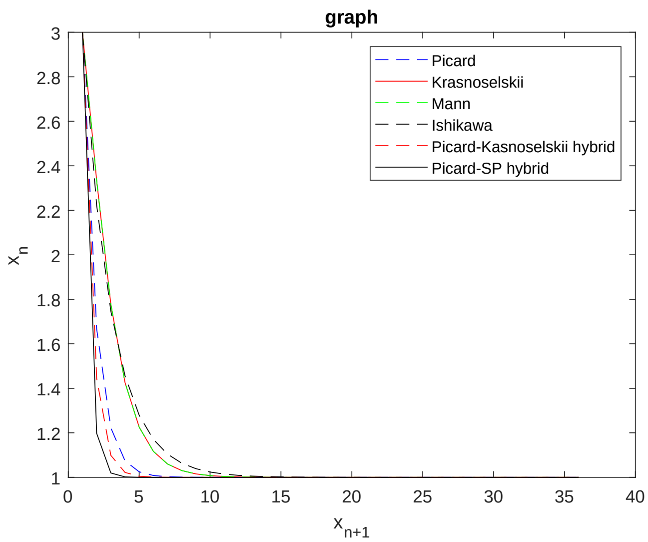
**Table 2 – continued from previous page**

| S.No. | Ishikawa | Picard-Krasnoselski | Picard-SP |
|---|---|---|---|
| 61. | 1.000000000000294 | 1.000000000000000 | 1.000000000000000 |
| 62. | 1.000000000000180 | 1.000000000000000 | 1.000000000000000 |
| 63. | 1.000000000000110 | 1.000000000000000 | 1.000000000000000 |
| 64. | 1.000000000000067 | 1.000000000000000 | 1.000000000000000 |
| 65. | 1.000000000000041 | 1.000000000000000 | 1.000000000000000 |
| 66. | 1.000000000000025 | 1.000000000000000 | 1.000000000000000 |
| 67. | 1.000000000000015 | 1.000000000000000 | 1.000000000000000 |
| 68. | 1.000000000000009 | 1.000000000000000 | 1.000000000000000 |
| 69. | 1.000000000000006 | 1.000000000000000 | 1.000000000000000 |
| 70. | 1.000000000000004 | 1.000000000000000 | 1.000000000000000 |
| 71. | 1.000000000000002 | 1.000000000000000 | 1.000000000000000 |
| 72. | 1.000000000000001 | 1.000000000000000 | 1.000000000000000 |
| 73. | 1.000000000000001 | 1.000000000000000 | 1.000000000000000 |
| 74. | **1.000000000000000** | 1.000000000000000 | 1.000000000000000 |
| 75. | 1.000000000000000 | 1.000000000000000 | 1.000000000000000 |




Now, we prove the stability of our iteration process (1.9).

**Theorem 2.4.** *Let $S$ be a nonempty closed convex subset of a Banach space $X$ and $T : S \to S$ be a contraction mapping.Let $x_n$ be an iterative sequence generated by (1.9) with real sequences $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta\}_{n=0}^{\infty}$ in $[0,1]$ satisfying $\sum_{n=0}^{\infty} \gamma_n = \infty$. Then the iterative process (1.9) is $T$-stable.*

*Proof.* Let $\{s_n\}_{n=0}^{\infty} \subset X$ be any arbitrary sequence in $S$. Let the sequence generated by (1.9) is $x_{n+1} = f(T,x_n)$ converging to unique fixed point $q$ (by Theorem 2.1) and $\varepsilon_n = \|s_{n+1} - f(T,s_n)\|$. We will prove that $\lim_{n\to\infty} \varepsilon_n = 0 \iff \lim_{n\to\infty} s_n = q$.

Let $\lim_{n\to\infty} \varepsilon_n = 0$. By using (2.4) we get

$$\begin{aligned}\|s_{n+1} - q\| &\leq \|s_{n+1} - f(T,s_n\| + \|f(T,s_n) - q\| \\ &= \varepsilon_n + \|T[(1-\gamma_n)[(1-\beta_n)[(1-\alpha_n)s_n + \alpha_n T s_n] \\ &\quad + \beta_n T[(1-\alpha_n)s_n + \alpha_n T s_n]] \\ &\quad + \gamma_n T[(1-\beta_n)[(1-\alpha_n)s_n + \alpha_n T s_n] + \beta_n T[(1-\alpha_n)s_n \\ &\quad + \alpha_n T s_n]]]\| \\ &\leq \ell(1-\gamma_n(1-\ell)\|s_n - q\|) + \varepsilon_n \quad (2.22)\end{aligned}$$

Define $a_n = \|s_n - q\|$, $C_n = \gamma_n(1-\ell) \in (0,1)$ and $b_n = \varepsilon_n = 0$ which implies that $\frac{b_n}{c_n} \to 0$ as $n \to \infty$. Thus all the conditions of Lemma 1.5 we get $\lim_{n\to\infty} s_n = q$.

Conversely, let $\lim_{n\to\infty} s_n = q$, we have

$$\begin{aligned}\varepsilon_n &= \|s_{n+1} - f(T,s_n)\| \\ &\leq \|s_{n+1} - q\| + \|f(T,s_n) - q\| \\ &\leq \|s_{n+1} - q\| + \ell(1-\gamma_n(1-\ell)\|s_n - q\|\end{aligned}$$

This implies that $\lim_{n\to\infty} \varepsilon_n = 0$. Hence our Picard-SP hybrid iterative process (1.9) is stable with suspect to $T$. ∎

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