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Dominating Functions and Cayley Graphs

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ABSTRACT

Let $G = (V, E)$ be a graph. A function $f : V \to [0, 1]$ is called a dominating function if $\sum_{u \in N[v]} f(u) \geq 1$ for every vertex $v \in V$. Let f and g be any two functions from V to $[0, 1]$, $f \neq g$. We say f is less than g and we write $f < g$ if $f(u) \leq g(u)$ for all $u \in V$. A dominating function f of G is said to be minimal dominating function if whenever $g < f$, g is not a dominating function of G. In this Paper we study these functions for certain classes of graphs which includes Quadratic Residue Cayley graphs in particular.

Keywords: Domination, Dominating function, Domination number

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1. INTRODUCTION AND PRELIMINARIES

Dominating sets in a graph plays an important role in the study of locations. It deals with neighbourhood of a vertex and there by finds application in the location of service sectors such as hospitals, post office, banks, markets etc. In order to maximizer the beneficiaries of a source, it needs to study the location of such a source in the setup of socio-economic systems. It has a lot of application in networks such as electrical network, communication networks, intelligence networks etc. The problem of finding minimal dominating sets is related to having minimum number of resources to serve the whole community effectively there by maximizing the benefits and minimizing the cost. This study will help also in setting up of cost effective models of residential layouts, sales outlets, alarm systems etc. The study of domination sets and domination number of graphs was been initiated by Theresa Haynes et.al. [8].

Let A be any group. Let S be a subset of A containing no identity element of A. The Cayley graph(Directed Cayley graph) of A, denoted by $Cay(A, S)$, is the graph whose vertex set is A and there is an edge from vertex x to a vertex y whenever $y = xs$, for some $s \in S$. The edge xy is then said to be generated by the color s. The identity element is excluded from the set S in the definition above just to avoid loops at every vertex. The definition of Cayley graph is given by Cayley in his seminal work [1] and called each directed cycle as a p-gone. The length of each directed colored cycle in the Cayley graph denotes the order of the element of the group A which corresponds to that particular color. Further, if we take $S = A - \{e\}$, where e is the identity, then the Cayley graph is just denoted by $Cay(S)$. In the graph $Cay(S)$, the set of colors in any Hamiltonian cycle corresponds to a generating set of A.

Graph theoretic terminologies are considered from [2]. Let G and H be any two graphs. The corona product of G and H, denoted by $G \circ H$, is the graph obtained by $|V(G)|$ copies of H and adding edges between each vertex of i^{th} copy of H to the i^{th} vertex of G.

2. DOMINATING SETS AND DOMINATING FUNCTIONS

Let $G = (V, E)$ be a simple connected undirected graph. A subset D of V is called *dominating set* if every vertex in $V - D$ is adjacent to a vertex in D. A dominating set D is called minimal dominating set if no proper subset of D is a dominating set. The minimum cardinality of a minimal dominating set is called *domination number* of G and denoted by $\gamma(G)$ and the *upper domination number* of G is the maximum cardinality of a minimal dominating set of G, denoted by $\Gamma(G)$. A function $f : V \to [0, 1]$ is called *dominating function* if $\sum_{u \in N[v]} f(u) \geq 1$ for every vertex $v \in V$. Let f and g be any two functions from V to $[0, 1]$, $f \neq g$. We say f is less than g and we write $f < g$ if $f(u) \leq g(u)$ for all $u \in V$. A dominating function f of G is said to be minimal dominating function if whenever $g < f$, g is not a dominating function of G.

Let $G = (V, E)$ be a graph without isolated vertices. A subset T of V is called a *total dominating set* of G if every vertex in V is adjacent to at least one vertex in T. If no proper subset of T is total dominating set of G, then T is called a *minimal total dominating set* of G. The minimum cardinality of a minimal total dominating set of G is called *total domination number* of G and denoted by $\gamma_t(G)$. The maximum cardinality of a minimal total dominating set of G is called the *upper total domination number of* G and is denoted by $\Gamma_t(G)$. A function $f : V \to [0,1]$ is called a *total dominating function of* G if $\sum_{u\in N(v)} f(u) \geq 1$ for every vertex $v \in V$. A total dominating function f of G is said to be minimal total dominating function if whenever $g < f$, g is not a total dominating function of G. For the similar work on domination we refer [1, 3, 4, 5, 6, 7].

We recall the following for immediate reference;

Definition 2.1. *Let p be an odd prime and $gcd(a,p) = 1$. If the quadratic congruence $x^2 \equiv a\ (modp)$ has a solution, then a is said to be a quadratic residue of p, otherwise a is called quadratic non-residue of p.*

Theorem 2.2. *If p is a prime number of the form $4n + 3$, then for $0 < a < p$, $x^2 \equiv a\ (mod\ p)$ if and only if $x^2 \not\equiv p - a\ (modp)$. If p is a prime number of the form $4n + 1$, then for $0 < a < p$, $x^2 \equiv a\ (mod\ p)$ if and only if $x^2 \equiv p - a\ (modp)$.*

Theorem 2.3. *Let p be an odd prime and $gcd(a,p) = 1$. Then a is a quadratic residue of p if and only if $a^{\frac{p-1}{2}} \equiv 1\ (mod\ p)$; In otherwords, $x^2 \equiv a\ (mod\ p)$ has a solution if and only if $a^{\frac{p-1}{2}} \equiv 1\ (mod\ p)$.*

Remark 2.4. *It is known that if $S = \{a : 0 < a < p, x^2 \equiv a\ (mod\ p)$ has a solution$\}$, then $|S| = (p-1)/2$ and $|S\cup S^{-1}| = (p-1)/2$ if $p = 4n+1$, and $p-1$ if $p = 4n+3$.*

Theorem 2.5. *If p is an odd prime and $gcd(a,p) = 1$, then the congruence $x^2 \equiv a\ (mod\ p^n)$, $n \geq 1$ has a solution if and only if a is a quadratic residue of p.*

Consider the group Z_p with respect to addition modulo p and a graph $G = (V,E)$, where $V = Z_p$ and two vertices a and b are adjacent in G if and only if $ab^{-1} \in S \cup S^{-1}$, $S = \{a \in Z_p \mid a^{\frac{p-1}{2}} \equiv 1\ (mod\ p)\}$. This graph G is called quadratic residue Cayley graph and usually denoted by $Cay(Z_p, S)$.

3. DOMINATING FUNCTIONS OF CAYLEY GRAPHS AND THEIR PRODUCT

In this section we find minimal dominating function and minimal total dominating functions of quadratic residue Cayley graphs and Cartesian product of Cayley graphs and some other general graphs.

Theorem 3.1. *For any group A and the Cayley graph $G = (V,E) = Cay(A,S)$, a function $f : V \to [0,1]$ defined by $f(v) = \frac{1}{q}$, for all $v \in V$, is a dominating function of G, where $1 \leq q \leq |S\cup S^{-1}|+1$ and f is a minimal dominating function if $q = |S \cup S^{-1}| + 1$.*

Proof. The neighborhood of $v \in V$ consists of $m = |S \cup S^{-1}| + 1$ vertices and $f(v) = \frac{1}{q}$, therefore $\sum_{u\in N[v]} f(u) = (1/q) \times \deg(v) = \frac{m}{q}$.

Case 1 $q = m$

In this case, $\sum_{u\in N[v]} f(u) = \frac{m}{m} = 1$, for all $v \in V$. Therefore f is a dominating function. Now we check minimality of f. Suppose $g : V \to [0,1]$ is a function defined by

$$g(v) = \begin{cases} 1/r, & \text{for } v = v_k \text{ and } r > m \\ 1/m, & \text{for } \forall\, v \in V - \{v_k\} \end{cases}$$

Then,

$$\sum_{u\in N[v_k]} g(u) = \frac{1}{r} + \frac{1}{m}[m-1] < \frac{1}{m} + \frac{m-1}{1} = 1$$

Thus g is not a dominating function. Therefore f is a minimal dominating function.

Case 2 $1 \leq m < q$

In this case, $\sum_{u\in N[v]} f(u) = \frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q} = \frac{m}{q} > 1$, $\forall v \in V$. Thus f is a dominating function. To prove f is not minimal dominating function, we consider $g : V \to [0,1]$ defined by

$$g(v) = \begin{cases} 1/r, & \text{for } v = v_k \\ 1/m, & \forall\, v \in V - \{v_k\} \end{cases}$$

Then for $r > q$, (There exits such r.) we have $f > g$ and

$$\begin{aligned} \sum_{u\in N[v]} g(u) &= \begin{cases} \frac{1}{r} + \frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}, & \text{if } v_k \in N[v] \\ \frac{1}{q} + \frac{1}{q} + \frac{1}{q} + \cdots + \frac{1}{q}, & \text{if } v_k \notin N[v] \end{cases} \\ &= \begin{cases} \frac{1}{r} + \frac{m-1}{q}, & \text{if } v_k \in N[v] \\ \frac{m}{q}, & \text{if } v_k \notin N[v] \end{cases} \end{aligned}$$

Here, $\frac{1}{r} + \frac{m-1}{q} > \frac{1}{q} + \frac{m-1}{q} = \frac{m}{q}$.

In any case $\sum_{u\in N[v]} g(u) \geq \frac{m}{q} > 1$. Hence g is also dominating function. Therefore f is not minimal dominating function. $\square$

Illustration: For example consider $p = 7$. Then $S = \{1,2,4\}$ and $S \cup S^{-1} = \{1,2,3,4,5,6\}$. Therefore $|S \cup S^{-1}| = 6$ and the graph $G = Cay(Z_7, S)$ is a complete graph. We define $f(v) = \frac{1}{7}$ for all $v \in V$, so that f is a minimal dominating function, if we define $f(v) = \frac{1}{6}$ for all $v \in V$, then f becomes a minimal total dominating function of G. If $p = 13$ then $S = \{1,3,4,9,10,12\}$ and $S \cup S^{-1} = \{1,3,4,9,10,12\}$. Therefore $|S \cup S^{-1}| = 6$ and the graph $G = Cay(Z_13, S)$ is a 6-regular graph. We define $f(v) = \frac{1}{7}$ for all $v \in V$, so that f is a minimal dominating function, if we define $f(v) = \frac{1}{6}$ for all $v \in V$, then f becomes a minimal total dominating function of G.

Corollary 3.2. *For a prime number p, let $G = (V,E) = Cay(Z_p, S)$, where S the class of quadratic residue modulo p, then a function $f : V \to [0,1]$ defined by $f(v) = \frac{1}{q}$, where*

$$q = \begin{cases} (p+1)/2 & \text{if } p \equiv 1 (mod\ 4) \\ p & \text{if } p \equiv 3 (mod\ 4) \end{cases}$$

is a minimal dominating function of G.

Corollary 3.3. *For a prime number p, let $G = (V,E) = Cay(Z_{p^n}, S)$, where S the class of quadratic residue modulo p, then a function $f : V \to [0,1]$ defined by $f(v) = \frac{1}{q}$, where*

$$q = \begin{cases} [n(p-1)/2] + 1 & \text{if } p \equiv 1 (mod\ 4) \\ np - n + 1 & \text{if } p \equiv 3 (mod\ 4) \end{cases}$$

is a minimal dominating function of G.

Theorem 3.4. *If f is a total dominating function of $G = (V,E)$ then $F : V \times V \to [0,1]$ defined by $F(v_i, v_j) = \alpha f(v_i) + \beta f(v_j)$ for any $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ is a total dominating function for $G \Box G$.*

Proof. Let $(u_i, u_j) \in V \times V$. Then $N((u_i,u_j)) = \{(v_i, v_j) : v_i = u_i \text{ and } v_j \in N(u_j) \text{ or } v_j = u_j \text{ and } v_i \in N(u_i)\}$.

$$\begin{aligned} \sum_{(v_i,v_j)\in N((u_i,u_j))} F(v_i,v_j) &= \alpha[f(v_i)\deg(u_j) \\ &\quad + \sum_{u\in N(u_i)} f(u)] + \beta[f(v_j)\deg(u_i) \\ &\quad + \sum_{u\in N(u_j)} f(u)] \\ &= \alpha f(v_i)\deg(u_j) + \beta f(v_j)\deg(u_i) \\ &\quad + \alpha \sum_{u\in N(u_i)} f(u) + \beta \sum_{u\in N(u_j)} f(u) \end{aligned}$$

Since, $\sum_{u\in N(u_i)} f(u) \geq 1$ and $\sum_{u\in N(u_j)} f(u) \geq 1$, the above equation simplifies to,

$$\begin{aligned} \sum_{(v_i,v_j)\in N((u_i,u_j))} F(v_i,v_j) &\geq \alpha f(v_i)\deg(u_j) + \beta f(v_j)\deg(u_i) + \alpha + \beta \\ &> \alpha + \beta = 1 \end{aligned}$$

Thus F is a total dominating function for $G \Box G$. □

Corollary 3.5. *If f is a dominating function of $G = (V,E)$ then $F : V \times V \to [0,1]$ defined by $F(v_i, v_j) = \alpha f(v_i) + \beta f(v_j)$ for any $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ is a dominating function for $G \Box G$.*

Proof. For any $v \in V$, we have

$$\begin{aligned} \sum_{u\in N[v]} f(u) \geq 1 \quad &\Rightarrow \sum_{u\in N(v)} f(u) + f(v) \geq 1 \\ &\Rightarrow \sum_{u\in N(v)} f(u) \geq 1 - f(v) \end{aligned}$$

Thus, for any vertex $(a,b) \in V \times V$, we get

$$\begin{aligned} \sum_{(v_i,v_j)\in N[a,b]} F(v_i,v_j) &= \alpha\left[f(v_i)(\deg(v_j)+1) + 1 - f(v_i)\right] \\ &\quad + \beta\left[f(v_j)(\deg(v_i)+1) + 1 - f(v_j)\right] \\ &= \alpha\left[f(v_i)\deg(v_j) + 1\right] \\ &\quad + \beta\left[f(v_j)\deg(v_i) + 1\right] \\ &> \alpha + \beta = 1 \end{aligned}$$

Thus F is a dominating function for $G \Box G$. □

We further show that two such total dominating functions in a graph G can induce a total dominating function in the Cartesian product $G \Box G$ and establish the convexity of such functions in the product graphs.

Theorem 3.6. *Let $G = (V,E)$ be a regular graph and $f : V \to [0,1]$ and $g : V \to [0,1]$ total dominating functions of G. Let $F_1 : V \times V \to [0,1]$ be defined by $F_1((u,v)) = \alpha_1 f(u) + \beta_1 f(v)$, α_1, $\beta_1 \geq 0$, $\alpha_1 + \beta_1 = 1$ and $F_2 : V \times V \to [0,1]$ be defined by $F_2((u,v)) = \alpha_2 f(u) + \beta_2 f(v)$, α_2, $\beta_2 \geq 0$, $\alpha_2 + \beta_2 = 1$. Then the convex combination $H = \alpha F_1 + \beta F_2$ is also total dominating function of $G \Box G$, where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.*

Proof. Let $N(u) = \{u_1, u_2, u_3, \ldots, u_k\}$ and $N(v) = \{v_1, v_2, v_3, \ldots, v_k\}$. Then for $(u,v) \in V \times V$, $N((u,v)) = \{(u,v_1), (u,v_2), \ldots, (u,v_k), (u_1,v), (u_2,v), \ldots, (u_n,v)\}$ and $\sum_{(x,y)\in N(u,v)} H(x,y) = H(u,v_1) + H(u,v_2) + \cdots + H(u,v_k) + H(u_1,v) + \cdots + H(u_k,v) = \alpha F_1(u,v_1) + \beta F_2(u,v_1) + \alpha F_1(u,v_2) + \beta F_2(u,v_2) + \cdots + \alpha F_1(u,v_k) + \beta F_2(u,v_k) + \alpha F_1(u_1,v) + \beta F_2(u_1,v) + \cdots + \alpha F_1(u_k,v) + \beta F_2(u_k,v) = \alpha(\alpha_1 f(u) + \beta_1 f(v_1)) + \beta(\alpha_2 g(u) + \beta_2 g(v_1)) + \alpha(\alpha_1 f(u) + \beta_1 f(v_2)) + \beta(\alpha_2 g(u) + \beta_2 g(v_2)) + \cdots + \alpha(\alpha_1 f(u) + \beta_1 f(v_k)) + \beta(\alpha_2 g(u) + \beta_2 g(v_k)) + \alpha(\alpha_1 f(u_1) + \beta_1 f(v)) + \beta(\alpha_2 g(u_1) + \beta_2 g(v)) + \alpha(\alpha_1 f(u_2) + \beta_1 f(v)) + \beta(\alpha_2 g(u_2) + \beta_2 g(v)) + \cdots + \alpha(\alpha_1 f(u_k) + \beta_1 f(v)) + \beta(\alpha_2 g(u_k) + \beta_2 g(v)) = \alpha\alpha_1[k\ f(u) + \sum_{i=1}^{k} f(u_i)] + \alpha\beta_1[\sum_{i=1}^{k} f(v_i) + k\ f(v)] + \beta\alpha_2[k\ g(u) + \sum_{i=1}^{k} g(u_i)] + \beta\beta_2[\sum_{i=1}^{k} g(v_i) + k\ g(v)] \geq \alpha\alpha_1[k\ f(u) + 1] + \alpha\beta_1[1 + k\ f(v)] + \beta\alpha_2[k\ g(u) + 1] + \beta\beta_2[1 + k\ g(v)] \geq k[\alpha\alpha_1 f(u) + \alpha\beta_1 f(v) + \beta\alpha_2 g(u) + \beta\beta_2 g(v)] + \alpha\alpha_1 + \alpha\beta_1 + \beta\alpha_2 + \beta\beta_2$.

Thus, $\sum_{(u_i,v_j)\in N(u,v)} H(u_i,v_j) = k\alpha[\alpha_1 f(u) + \beta_1 g(v)] + k\beta[\alpha_2 g(u) + \beta_2 g(v)] + \alpha(\alpha_1 + \beta_1) + \beta(\alpha_2 + \beta_2) \geq k\left[\alpha F_1((u,v)) + \beta F_2((u,v))\right] + \alpha + \beta = kH((u,v)) + 1 \geq 1$. Therefore H is a total dominating function of $G \Box G$ □

4. DOMINATING FUNCTIONS OF CORONA PRODUCT OF GRAPHS

In this section we find minimal total dominating function for the graph $K_1 \circ P_n$ and generalize the result of corona product of any two paths.

Theorem 4.1. *Let $G = K_1 \circ P_n$ and $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$ and v be the vertex of K_1. A function $f : V \to [0,1]$ defined by $f(v) = 1 - \frac{1}{n}$, $f(v_i) = \frac{1}{n}$ for all i, $1 \le i \le n$. Then f is a minimal total dominating function.*

Proof. $f(N(v)) = \sum_{i=1}^{n} f(v_i) = \sum_{i=1}^{n} \frac{1}{n} = 1$, $f(N(v_1)) = f(v_2) + f(v) = \frac{1}{n} + 1 - \frac{1}{n} = 1$, $f(N(v_n)) = f(v_{n-1}) + f(v) = \frac{1}{n} + 1 - \frac{1}{n} = 1$, and for i, $2 \le i \le n-1$, $f(N(v_i)) = f(v_{i-1}) + f(v_{i+1}) + f(v) = \frac{1}{n} + \frac{1}{n} + 1 - \frac{1}{n} = 1 + \frac{1}{n} > 1$. Therefore f is a total dominating function. We now prove f is a minimal total dominating function of G. Let $g : V \to [0,1]$ be such that $g < f$. Then $g(u) \le f(u)$ for all $u \in V(G)$ and $g(w) < f(w)$ for some $w \in V$.

Case 1 Let $g(v_i) < f(v_i)$ for some i. Then $g(N(v)) = g(v_1) + g(v_2) + \cdots + g(v_i) + \cdots + g(v_n) < f(v_1) + f(v_2) + \cdots + f(v_i) + \cdots + f(v_n) = 1$. Which implies g is not a total dominating function.

Case 2 Let $g(v_i) = f(v_i)$ for all i and $g(v) < f(v)$. Then $g(N(v_1)) = g(v) + g(v_2) < f(v) + f(v_2) = 1 - \frac{1}{n} + \frac{1}{n} = 1$. Which implies g is not a total dominating function.

Thus f is a total dominating function of G. □

In the view of the arguments made in the above theorem, the proofs of the following theorems become trivial.

Theorem 4.2. *Let $G = P_m \circ P_n$ and $\{v_1, v_2, v_3, \ldots, v_m\}$ be the vertices of P_m, such that v_i is adjacent to v_{i+1}, $1 \le i \le m$. For $i \le k \le m$, $\{w_{k_1}, w_{k_2}, w_{k_3} \ldots, w_{k_n}\}$ be the vertices of k^{th} copy of P_n, such that w_{k_i} is adjacent to $w_{k_{i+1}}$ for each i, $1 \le i \le n$. Then function $f : V \to [0,1]$ defined by $f(v_i) = 1$, $1 \le i \le m$ and $f(w_{k_j}) = 0$ for all k, j, $1 \le k \le m$, $1 \le j \le n$ is a minimal total dominating function of G. Also f is a characteristic function of G.*

Theorem 4.3. *Let $G = P_m \circ P_n$ and $\{v_1, v_2, v_3, \ldots, v_m\}$ be the vertices of P_m, such that v_i is adjacent to v_{i+1}, $1 \le i \le m$. For $i \le k \le m$, $\{w_{k_1}, w_{k_2}, w_{k_3} \ldots, w_{k_n}\}$ be the vertices of k^{th} copy of P_n, such that w_{k_i} is adjacent to $w_{k_{i+1}}$ for each i, $1 \le i \le n$. Then function $f : V \to [0,1]$ defined by $f(v_i) = 1 - \frac{1}{n+1}$, $1 \le i \le m$ and $f(w_{k_j}) = \frac{1}{n+1}$ for all k, j, $1 \le k \le m$, $1 \le j \le n$ is a minimal total dominating function of G.*

5. DOMINATING FUNCTIONS OF POWER GRAPHS

Theorem 5.1. *Let $G = P_n^2$, $n \ge 8$ and $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$. Then a function $f : V \to [0,1]$ defined by*

$$f(v_i) = \begin{cases} 1/2 & i = 2, 3, n-2, n-1 \\ 1/4 & otherwise \end{cases}$$

is a minimal total dominating function.

Proof. By the definition of f, we observe that (i) $f(N(v_1)) = f(v_2) + f(v_3) = \frac{1}{2} + \frac{1}{2} = 1$, (ii) $f(N(v_n)) = f(v_{n-2}) + f(v_{n-1}) = \frac{1}{2} + \frac{1}{2} = 1$, (iii) $f(N(v_2)) = f(v_1) + f(v_3) + f(v_4) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, (iv) $f(N(v_3)) = f(v_1) + f(v_2) + f(v_4) + f(v_4) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{5}{4} > 1$, (v) $f(N(v_{n-1})) = f(v_n) + f(v_{n-2}) + f(v_{n-3}) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$, (vi) $f(N(v_{n-2})) = f(v_n) + f(v_{n-1}) + f(v_{n-3}) + f(v_{n-4}) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} = \frac{5}{4} > 1$ and (vii) $f(N(v_i)) = f(v_{i-2}) + f(v_{i-1}) + f(v_{i+1}) + f(v_{i+2}) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$, for all i, $4 \le i \le n-3$. Hence f is a total dominating function of G.

We now prove that f is a minimal total dominating function of G. Let $g : V \to [0,1]$ be such that $g < f$. Then $g(v_i) \le f(v_i)$ for all i, $1 \le i \le n$ and $g(v_k) < f(v_k)$ for some k, $1 \le k \le n$.

If $g(v_1) < f(v_1)$, then $g(N(v_2)) = g(v_1) + g(v_3) + g(v_4) < f(v_1) + f(v_3) + f(v_4) = \frac{1}{4} + \frac{1}{2} + \frac{1}{4} = 1$. Similarly, if $g(v_n) < f(v_n)$, then $g(N(v_{n-1})) < 1$.

If $g(v_2) < f(v_2)$, then $g(N(v_1)) = g(v_2) + g(v_3) < f(v_2) + f(v_3) = \frac{1}{2} + \frac{1}{2} = 1$. Similarly, if $g(v_{n-1}) < f(v_{n-1})$, then $g(N(v_n)) < 1$.

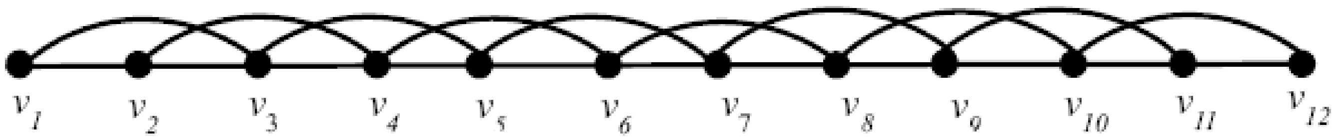
If $g(v_3) < f(v_3)$, then $g(N(v_1)) = g(v_2) + g(v_3) < f(v_2) + f(v_3) = \frac{1}{2} + \frac{1}{2} = 1$. Similarly, if $g(v_{n-1}) < f(v_{n-1})$, then $g(N(v_n)) < 1$.

If $g(v_i) < f(v_i)$ for all i, $4 \le i \le \lceil \frac{n}{2} \rceil$, then $g(N(v_{i+2})) = g(v_i) + g(v_{i+1}) + g(v_{i+3}) + g(v_{i+4}) < f(v_i) + f(v_{i+1}) + f(v_{i+3})) + f(v_{i+4})) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} = 1$. Similarly, if $g(v_i) < f(v_i)$ for all i, $\lceil \frac{n}{2} \rceil \le i \le n-3$, then $g(N(v_{i-2})) < 1$.

Thus in any case g is not a total dominating function. □

Remark 5.2. *In the above theorem, we have seen a minimal total dominating function for P_n^2 with $n \ge 8$. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$. The graphs P_n^2, with $3 \le n < 8$ can be dealt individually as follows;*

The graph P_3^2 is a complete graph and the function $f : V \to [0,1]$ defined by $f(v_i) = \frac{1}{2}$, for all i is a minimal total dominating function.



i	1	2	3	4	5	6	7	8	9	10	11	12
$f(v_i)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{4}$
$f(N(v_i))$	1	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{5}{4}$	1	1	$\frac{5}{4}$	$\frac{3}{2}$	$\frac{5}{4}$	1	1

Figure 1: A minimal total dominating function f for the graph P^2_{12}.

For the graph P^2_4, the function $f : V \to [0,1]$ defined by $f(v_1) = f(v_4) = \frac{1}{4}$ and $f(v_2) = f(v_3) = \frac{1}{2}$ is a minimal total dominating function.

For the graph P^2_5, the function $f : V \to [0,1]$ defined by $f(v_1) = f(v_5) = 0$ and $f(v_2) = f(v_3) = f(v_4) = \frac{1}{2}$ is a minimal total dominating function.

For the graph P^2_6, the function $f : V \to [0,1]$ defined by $f(v_1) = f(v_6) = 0$ and $f(v_2) = f(v_3) = f(v_4) = f(v_5) = \frac{1}{2}$ is a minimal total dominating function.

For the graph P^2_7, the function $f : V \to [0,1]$ defined by $f(v_1) = f(v_7) = f(v_4) = \frac{1}{4}$ and $f(v_2) = f(v_3) = f(v_5) = f(v_6) = \frac{1}{2}$ is a minimal total dominating function.

The graph P^2_{12} and a minimal total dominating function for it is shown in Figure 1.

The above idea can be extended to define a minimal total dominating function for P^k_n and to prove it is minimal total dominating function.

Theorem 5.3. *Let $G = P^k_n$, $n \geq 4k$ and $V = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of P_n such that v_i is adjacent to v_{i+1} for each $i = 1, 2, \ldots, n-1$. Then a function $f : V \to [0,1]$ defined by*

$$f(v_i) = \begin{cases} \frac{1}{k} & 2 \leq i \leq k+1 \text{ and } n-k \leq i \leq n-1 \\ \frac{1}{2k} & otherwise \end{cases}$$

is a minimal total dominating function.

Proof. In the graph P^k_n, we observe

$$\deg(v_i) = \begin{cases} k & \text{for } i = 1, n \\ 2i-1 & \text{for } 2 \leq i \leq k \\ 2(n-i)+1 & \text{for } n-k+1 \leq i \leq n \\ 2k & otherwise \end{cases}$$

Now by the definition of function f, we observe,

$$\begin{aligned} f(N(v_1)) &= f(v_2) + \cdots + f(v_k) = \frac{1}{k} + \cdots + \frac{1}{k} = 1 \\ f(N(v_2)) &= f(v_1) + f(v_3) + \cdots + f(v_{k+1}) + f(v_{k+2}) \\ &= \frac{1}{2k} + \frac{1}{k} + \cdots + \frac{1}{k} + \frac{1}{2k} \\ &= \frac{1}{2k} + \frac{k-1}{k} + \frac{1}{2k} = 1 \end{aligned}$$

$$f(N(v_i)) = \begin{cases} 1 + \frac{i-2}{2k} > 1 & \text{if } 3 \leq i \leq k+2 \\ 2 - \frac{i-2}{2k} > 1 & \text{if } k+3 \leq i \leq 2k+1 \\ 1 & \text{if } 2k+2 \leq i \leq \lceil \frac{n}{2} \rceil \end{cases}$$

and for all i, $\lceil \frac{n}{2} \rceil \leq i \leq n$ $f(N(v_i)) \geq 1$ follows by the symmetry of the path. Hence f is a total dominating function.

To prove f is a minimal total dominating function; Let $g : V \to [0,1]$ be a function so that $g < f$. Then $g(v_i) \leq f(v_i)$ for all i $1 \leq i \leq n$ and $g(v_j) < f(v_j)$ for some j $1 \leq j \leq n$.

If $g(v_1) < f(v_1)$, then $g(N(v_2)) < f(N(v_2)) = 1$, which implies that, g is not a total dominating function. If $g(v_j) < f(v_j)$ for some j, $2 \leq j \leq k+1$, then $g(N(v_1)) < f(N(v_1)) = 1$, which again implies that g is not a total dominating function. If $g(v_j) < f(v_j)$ for some j, $k+2 \leq j \leq \lceil \frac{n}{2} \rceil$, then $g(N(v_{2j+2})) < f(N(v_{2j+2})) = 1$, which again implies that g is not a total dominating function. If $g(v_j) < f(v_j)$ for some j, $\lceil \frac{n}{2} \rceil \leq j \leq n$, then by replacing j by $n-j+1$ and applying the above statements we can prove that g is not a total dominating function of G. Hence f is a minimal total dominating function of G. □

Remark 5.4. *In the above theorem we have shown that P^k_n has a minimal total dominating function for $n \geq 4k$. For smaller values of n we can find minimal total dominating function individually as in the case of P^2_n.*

Remark 5.5. *The graph $G = C^k_n$ is a $2k$-regular graph and the function $f : V \to [0,1]$ defined by $f(u) = \frac{1}{2k}$ for all $u \in V$, is a minimal total dominating function of G.*

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