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Role of Fibonacci, Blaise Pascal, Pierre de Fermat and Abraham de Moivre in the Devolopment of Number Patterns and Probability: A Historical Search

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Abstract

Theory of Probability is a branch of Applied Mathematics dealing with the effects of chances. The word 'chance' gives birth to the mathematical term 'Probability'. In this article particular emphasis is given to the works of Fibonacci (1170-1250), Pierre de Fermat (1607 -1665), Blaise Pascal (1623 -1662) and Abraham de Moivre (1667-1754) in the development of different number patterns and Probability Theory.

Keywords: The Problem of division of the stakes, Extended Binomial Coefficients.

1. INTRODUCTION

According to P.S. Laplace "It is remarkable that a science (Probability) which began with consideration of games of chance, should have become the most important object of human knowledge". He further mentioned "Probability has reference partly to our ignorance, partly to our knowledge… . The Theory of Chances consists in reducing all events of the same kind to a certain number of cases equally possible, i.e. we are equally undecided as to their existence and determining the number of these cases which are favourable to the event sought. The ratio of that number to the number of all the possible cases is the measure of the Probability… "[10].

James Clark Maxwell mentioned "The true logic of this world is to be found in theory of Probability" [10].

In ancient times, Plato (428-348 BC) and Aristotle (384-322 BC) used to discuss the word 'chance' philosophically. Antimenes (530-510 BC) first developed the process of 'insurance' which guaranteed a sum of money against wins or losses. In view of many uncertainties of everyday life such as health, weather, birth and death led to the concept of chance or random variables as output of an experiment. Almost all measurements in Mathematics have the fundamental property that the results vary in different trials. These results are random in nature.

Historically, the word 'Probability' was associated with the Latin word 'Probo' and the English word 'Probable'. In ancient times the concept of Probability arose in problems of gambling. In 1494, Fra Luca Pacioli, an Italian mathematician wrote the first printed book on Probability entitled 'Summa de arithmetica, geometria, proportioni e proportionalita'. In fact, in Europe, the first calculation on chance was recorded in this book. In 1550, an Italian physician Geronimo Cardano (1501-1575, he was also a mathematician and gambler) inspired by the book 'Summa' wrote a book about games of chance known as 'Liber de Ludo Aleae'. First mathematical treatment of Probability dealing with problems of mathematical expectation, addition of probability, frequency tables for throwing of a die for *n* successes in *n* independents trials was recorded in this book. His work attracted attention from other researchers and the idea of Probability between 0 and 1 to an event whose outcome is random was introduced [1, 3].

Galileo Galilei (1564-1642) published an article 'Sopra Le Scoperte dei Dadi' on the basis of his observations of random process for a long period [1]. In this article, in the context of rolling of 3 die he mentioned "…it is known that long observation has made dice-players consider 10 and 11 to be more advantageous than 9 and 12." Galileo explained the situation by taking the possible combinations of the 3 numbers composing sum. He was able to show that 10 will show up in 27 ways out of all possible 216 outcomes. Since 9 can be found in 25 ways out of all possible 216 outcomes, this explains why it is at a 'disadvantage' in comparison to 10. [1].

The 'Points Problem' proposed by Chevalier de Méré in 1654 is said be the starting point of famous correspondence between Pascal and Pierre de Fermat, the two main early stalwarts in the development of Probability theory. They continued to exchange their thoughts on mathematical principles and problems through a series of letters. So, Pascal and Fermat are the mathematicians credited with the founding of probability theory. Their main ideas were popularized by Christian Huygens, in his 'De ratiociniis in ludo ale', published in 1657.

2.1. Blaise Pascal's personal life:

Blaise Pascal was born on 19th June, 1623 in Clermont Ferrand, France. He lost his mother at the age of three years. He did his childhood education with his father. His father was É tienne Pascal, a local judge. Blaise Pascal had two sisters - Gilberte and Jacqueline. At the age of 16 years Pascal produced a short treatise 'Mystic Hexagram'. It is still famous as Pascal Theorem. It states that: "if a hexagon is inscribed in a circle or conic then the three intersection points of opposite sides lie on a line". The line is called 'Pascal Line'.

At the age of 19 years, Pascal constructed a mechanical calculator capable of addition and subtraction which is called as 'Pascal Calculator'. Pascal made significant contributions in the field of Fluid Dynamics also. He died on 16th August, 1962 at the age of 39 years before his work was published. In 1665, his work was published in 'Traite du triangle arithmetique'.

2.2. Pierre de Fermat's personal life:

Pierre de Fermat was born in 1607 at Beaumont-de-Lomagne, France. He was a French mathematician who is given credit for early developments that led to infinitesimal calculus. He made great contributions to Analytic Geometry, Probability Theory and Optics. He was famous for the Fermat's principle for light propagation and 'Fermat's Last Theorem', a celebrated problem in Number Theory. He died in 12th January, 1665 at Castres, France.

3. CORRESPONDENCE BETWEEN FERMAT AND PASCAL

Correspondence between Fermat and Pascal was the beginning of the development of modern concepts of Probability, and it started in the context of the 'Points Problem' or 'The Problem of Points' or 'The Problem of division of the stakes' proposed by Chevalier de Méré in 1654. This is a classical problem in Probability Theory, one of the famous problems that motivated the beginnings of the modern Probability Theory in the 17th century. It led Blaise Pascal to the first explicit reasoning about what today is known as an 'Expected Value'. The problem concerns a game of chance with two players who have equal chances of winning each round. The players contribute equally to a prize pot, and agree in advance that the first player to have won a certain number of rounds will collect the entire prize. Now suppose that the game is interrupted by external circumstances before either player has achieved victory. How does one then divide the pot fairly ?

It is tacitly understood that the division should depend somehow on the number of rounds won by each player, such that a player who is close to winning will get a larger part of the pot but the problem is not merely one of calculation; it also involves deciding what a 'fair' division actually is. Through discussions Pascal and Fermat not only provided a convincing, self-consistent solution to this problem, but also developed concepts that are still fundamental to Probability Theory. The starting insight for Pascal and Fermat was that the division should not depend so much on the history of the part of the interrupted game that actually took place, as on the possible ways the game might have continued, were it not interrupted. It is intuitively clear that a player with a 7-5 lead in a game to 10 has the same chance eventually winning as a player with a 17-15 lead in a game to 20 and Pascal and Fermat therefore thought that interruption in either of the two situations ought to lead to the same division of stakes. In other words, what is important is not the number of rounds each player has won yet, but the number of rounds each player still needs to win in order to achieve overall victory.

On 29th of July 1654, a letter from Pascal to Fermat contains, among many other mathematical problems, the following passage:

"M. de Méré told me that he had found a fallacy in the theory of numbers, for this reason: If one undertakes to get a six with one die, the advantage in getting it in 4 throws is as 671 is to 625. If one undertakes to throw 2 sixes with two dice, there is a disadvantage in undertaking it in 24 throws and nevertheless 24 is to 36 as 4 is to 6. This is what made him so indignant and made him say to one and all that the propositions were not consistent and arithmetic was self- contradictory: but you will very easily see that what I say is correct, understanding the principles as you do."

This famous problem mentioned above, one of the first recorded in the history of Probability and which challenged the intellectual giants of that time, can now be solved as shown in [5].

To throw a 6 with one die in 4 throws means to obtain the point '6' at least once in 4 trials. Define X_n, $1 \le n \le 4$ as follows:

$$P(X_n = k) = \frac{1}{6}, \qquad k = 1, 2, ..., 6$$

Further, assume that X_1, X_2, X_3, X_4 are independent. Put $A_n = \{X_n = 6\}$ then the event in question is $A_1 \cup A_2 \cup A_3 \cup A_4$. It is easier to calculate the probability of its complement which is identical to $A_1^c \times A_2^c \times A_3^c \times A_4^c$. The trials are assumed to be independent and the dice unbiased. We have

$$P(A_1^c A_2^c A_3^c A_4^c) = P(A_1^c)P(A_2^c)P(A_3^c)P(A_4^c) = \left(\frac{5}{6}\right)^4,$$

hence

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \left(\frac{5}{6}\right)^4 = 1 - \frac{625}{1296} = \frac{671}{1296}$$

This last number is approximately equal to 0.5177. Since 1296- 671=625, the odds are as 671 to 625 as stated by Pascal.

Next consider two dice, let ($X_n^{/}, X_n^{//}$) denote the outcome obtained in the *n*th throw of the pair and let $B_n = \{x_n^{/} = 6; \;\; X_n^{//} = 6\}$.

Then $P(B_n^c) = \dfrac{35}{36}$ and $P(B_1^c B_2^c B_3^c ....B_{24}^c) = \left(\dfrac{35}{36}\right)^{24}$

$$P(B_1 \cup B_2 \cup ... \cup B_{24}) = 1 - \left(\frac{35}{36}\right)^{24}$$

This last number is approximately equal to 0.4914, which confirms the disadvantage. One must give great credit to Chevalier de Méré for his sharp observation and long experience at gaming tables to discern the narrow inequality

$$P(A_1 \cup A_2 \cup A_3 \cup A_4) > \frac{1}{2} > P(B_1 \cup B_2 \cup ... \cup B_{24}).$$

His arithmetic went wrong because of a fallacious 'linear hypothesis'. Of course according to some historians the problem was not originated with de Méré.

In some situations the equally likely cases must be searched out. This point was illustrated by the aforesaid famous historical problem called the 'Problems of Points'[5].

For example: Suppose two players '*A*' and '*B*' play a series of games in which the probability of each winning a single game is equal to $\frac{1}{2}$. They may simply play head or tail by tossing a coin. Each player gains a point when he wins a game and nothing when he loses. Suppose, they stop playing when 'A' needs 2 more points and 'B' needs 3 more points to win the stake. How should they divide it fairly ? It is clear that the winner will be decided in 4 more games. For in those 4 games either A will have won more than equal to 2 points or B will have won more than equal to 3 points but not both. All the possible outcomes of these 4 games using the letter A or B to denote the winner of each game are

AAAA

AAAB *AABA* *ABAA* *BAAA*

AABB *ABAB* *ABBA* *BAAB* *BABA* *BBAA*

ABBB *BABB* *BBAB* *BBBA*

BBBB

These are equally likely cases on grounds of symmetry.

$\because \binom{4}{4}+\binom{4}{3}+\binom{4}{2}=11$ cases in which A wins the stake and

$\binom{4}{3}+\binom{4}{4}=5$ cases in which B wins the stake. Hence the stake should be divided in the ratio 11:5.

Objections were raised by learned contemporaries that the enumeration above was not reasonable because the series would have stopped as soon as the winner was decided and not have gone on through all four games in some cases. Thus, the real possibilities are as follows:

AA *ABBB*

ABA *BABB*

ABBA *BBAB*

BAA *BBB*

BABA

BBAA

But these are not equally likely cases. In modern terminology, if these ten cases are regarded as constituting the sample space then

$$P(AA)=\frac{1}{4}, \quad P(ABA)=P(BAA)=P(BBB)=\frac{1}{8}$$

$$P(ABBA)=P(BABA)=P(BBAA)=P(ABBB)=P(BABB)=P(BBAB)=\frac{1}{16}$$

As *A* and *B* are independent events with probability $\frac{1}{2}$.

By adding theses probabilities, we get

$$P(A \text{ wins the stake})=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}=\frac{11}{16}$$

$$P(B \text{ wins the stake})=\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{8}=\frac{5}{16}$$

Pascal did not explain his method this way. He said merely that "it is absolutely equal and indifferent to each whether they play in the natural way of the game, which is to finish as soon as one has his score or whether they play the entire four games".

4.1. Combinations and Binomials:

If we start with a set of *m* objects and ask how many ways can we select a subset of *n* objects, we are basically asking how many 'combinations' are possible. In this regards the order is not matter.

Let us consider an example: Suppose we are preparing a mixed vegetable in which there are 5 vegetables to choose from that include: brinjal, cabbage, carrot, cauliflower and potato.

Let consider : A = brinjal , B = cabbage, C = carrot, D = cauliflower , E = potato

1. When choosing zero vegetable for the preparation, there is only one combination.

No combination → 1 selection

2. When choosing only one vegetable, the list of combinations will look like:

A, B, C, D, E → 5 selections

3. When choosing two different kinds of vegetables the list of combinations will look like:

AB, AC, AD, AE, BC, BD, BE, CD, CE, DE → 10 selections.

4. In choosing three different vegetables the list of combinations will look like:

ABC, ABD, ABE, BCD, BCE, CDE, BDE, ACD, ACE, ADE, → 10 selections

5. In choosing four different kinds of vegetables the list of combinations will look like:

ABCD, ABCE, ABDE, ACDE, BCDE → 5 selections

6. And finally, when choosing all five vegetables there is only one combination.

ABCDE → 1 selection

WAY TO CHOOSE FROM 5 TOTAL VEGETABLES	$C\binom{m}{n}$	COMBINATIONS
No vegetable	$C\binom{5}{0}$	1
One vegetable	$C\binom{5}{1}$	5
Two vegetables	$C\binom{5}{2}$	10
Three vegetables	$C\binom{5}{3}$	10
Four vegetables	$C\binom{5}{4}$	5
Five vegetables	$C\binom{5}{5}$	1

Fig. 1

If we consider total number of vegetables m and are going to take n vegetables ($n \le m$) at a time then we can claim that $C\binom{m}{n}$ is the number of n-element subsets of a set of S that contains m elements for any n, $0 \le n \le m$.

4.2 Pascal's Triangle:

4.2.1. History of Pascal's Triangle:

Pascal's Triangle is one of the most famous and interesting patterns in Mathematics. The work on Pascal's triangle began at least 500 years before the birth of Blaise Pascal. Between the 10th and 11th centuries, Indian and Persian Mathematicians started to work on this pattern of numbers. During the 10th century, Arab Mathematicians developed a mathematical series for calculating the coefficients for $(1+x)^n$, where $n \in N$. In 1070, a Persian Mathematician Omar Khayyam worked on the Binomial Expansion and the Numerical Coefficients, which are the values of a row in Pascal's Triangle [3].

4.2.2. Symbolical Definition of Pascal's Triangle:

In the first part Blaise Pascal's publication entitled 'Traite du triangle arithmetique' (translated into English means, 'A Treatise on the Arithmetical Triangle'), Pascal defined the triangle as an unbounded rectangular array. Fig.4, given below is an example of Pascal's rectangular array [3].

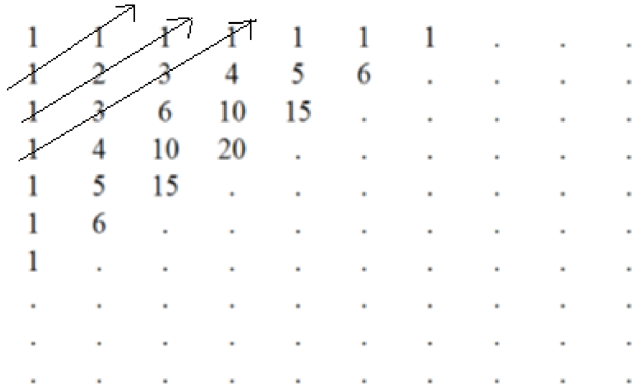

Fig. 2

Using the rectangular matrix Pascal defined the triangle symbolically using {$F(i\,,j)$, entry that occurs in the i^{th} row and the j^{th} column}

where $F(i, j) = F(i, j-1) + F(i-1, j)$, $\quad i, j = 2, 3, 4 \ldots$

$F(i, 1) = F(1, j) = 1$, $\quad i, j = 1, 2, 3 \ldots$

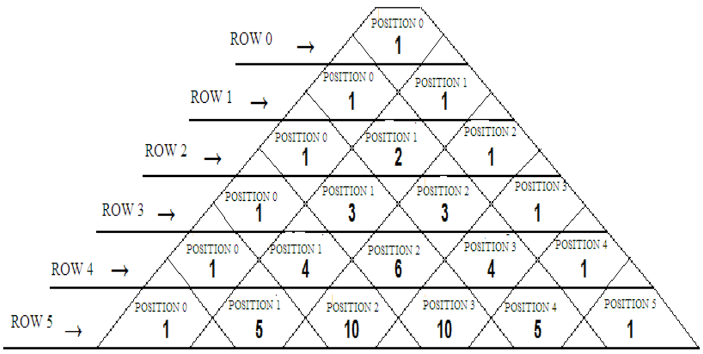

Fig. 3

$$C\binom{0}{0}$$
$$C\binom{1}{0}\ C\binom{1}{1}$$
$$C\binom{2}{0}\ C\binom{2}{1}\ C\binom{2}{2}$$
$$C\binom{3}{0}\ C\binom{3}{1}\ C\binom{3}{2}\ C\binom{3}{3}$$
$$C\binom{4}{0}\ C\binom{4}{1}\ C\binom{4}{2}\ C\binom{4}{3}\ C\binom{4}{4}$$

Fig. 4

'Pascal's Triangle' relates to combinatorics [2, 3]. As an example: if we notice at the last column in the table in Fig. 1 and compare that to Row 5 in Pascal's Triangle in Fig. 3, we notice at the last two columns in the table together, the third entry is $C\binom{5}{2}$ corresponds to the fifth row second position.

Thus, if one wants to find the number of two element subsets in a five element set, one can simply notice at Pascal's Triangle. In the combination format $C\binom{m}{n}$, the row number in Pascal's Triangle corresponds to the *m* or the total number of objects to choose from and the position within the row in Pascal's Triangle corresponds to the *n* or the number of objects to be chosen at a time.

POWER	BINOMIAL EXPANSION	PASCAL'S TRIANGLE
0	$(x+1)^0 = 1$	1 (Row 0)
1	$(x+1)^1 = x+1$	1,1 (Row 1)
2	$(x+1)^2 = x^2 + 2x + 1$	1,2,1 (Row 2)
3	$(x+1)^3 = x^3 + 3x^2 + 3x + 1$	1,3,3,1 (Row 3)
4	$(x+1)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$	1,4,6,4,1 (Row 4)
5	$(x+1)^5 = x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$	1,5,10,10,5,1 (Row 5)

Fig. 5

In Fig. 5, which contains a table showing the binomial expression $(x + 1)$ raised to the powers of zero to 5. We notice that for each expansion, the coefficients correspond to a row in Pascal's Triangle [2,3].

We notice at the coefficients and their relationship to the 'Binomial Theorem' which states that for any $m \geq 1$.

$$(x+y)^m = {}^mc_0x^m + {}^mc_1x^{m-1}y + {}^mc_2x^{m-2}y^2 + \ldots + {}^mc_nx^{m-n}y^n + {}^mc_my^m$$

4.2.3. Relation of Probability with Pascal's Triangle, Combination and Binomials:

Let us consider the following example which relates Pascal's Triangle, Combinations and Binomials with Probability**:**

Suppose the game is to roll a single die *m* times, and we consider it a win if a 6 occurs, but a loss if 1,2, 3, . . . ,5 occurs. Thus, we win with a probability of $\frac{1}{6}$. If we repeat the rolling *m* times, what is the probability of getting exactly *n* wins?

We consider the probability of success as *p* and the probability of failure as *q* such that

$$p + q = 1. \text{ So, } p = \frac{1}{6} \text{ and } q = \frac{5}{6}.$$

When we list the possible outcomes by using "*W*" for Win and "*L*" for Lose, the results of three repeats are shown below:

WWW, WWL, WLW, WLL, LWW, LWL, LLW, LLL.

For the three-repeat experiment, the chances of 0, 1, 2 and 3 wins are given by:

$P(0) = qqq = q^3$ which is equal to $C\binom{3}{0}p^0q^{3-0}$

$P(1) = pqq + qpq + qqp = 3pq^2$ which is equal to $C\binom{3}{1}p^1q^{3-1}$

$P(2) = ppq + pqp + qpp = 3p^2q$ which is equal to $C\binom{3}{2}p^2q^{3-2}$

$P(3) = ppp = p^3$ which is equal to $C\binom{3}{3}p^3q^{3-3}$

We notice that there is nothing special about repeating the experiment three times. So, looking at the above results we can conclude that if the experiment is repeated *m* times, the probability of obtaining exactly *n* wins is given by the formula:

$$P(n) = C\binom{m}{n}p^nq^{m-n}$$

4.3. Abraham de Moivre's Personal life:

Abraham de Moivre was born on 26th May 1667 at Vitry-le-Francois, a Kingdom in France. He was a great French mathematician known for 'de Moivre's formula'. This formula links complex numbers and trigonometry. He had a great contribution towards the development of Normal Distribution and Probability Theory. De Moivre wrote a book on Probability Theory entitled 'The Doctrine of Chances'. It is considered as a gift for the gamblers. He first discovered Binet's formula, the close- form expression for Fibonacci numbers linking the *n*th power of the golden ratio to the *n*th Fibonacci number. He was the first to postulate the Central Limit Theorem. He died on 27th November, 1754 in London, England.

4.3.1. Application of Binomial Coefficients in Probability Theory:

The extended Binomial Coefficients can be found in the work of Abraham de Moivre. A detailed theoretical discussion appeared in the third edition of the book 'The Doctrine of Chances or A Method of Calculating the Probabilities of Events in Play*'* (p.no.39-p.no.43) with illustrative examples. De Moivre's main result appeared in the form of a lemma stated as: "To find how many chances there are upon any number of dice, each of them of the same number of faces, to throw any given number of points."

In fact De Moivre dealt with the generalized problem in which a fair die has an arbitrary number of faces. He introduced the generating function for $C_m\binom{n}{r}$, calculated $C_m\binom{n}{r}$ numerically, and established the result

$$\left(\frac{1-t^m}{1-t}\right)^n = \sum_{r=0}^{(m-1)n} C_m\binom{n}{r} t^r$$

where $C_m\binom{n}{r}$ is the coefficient of t^r in $\left[\frac{(1-t^m)}{(1-t)}\right]^n$

To understand the utility of De Moivre's result in the context of probability let us consider a die with 'm' faces marked, $i = 0,1,2,\ldots m$-1. Assume that the turn up side probabilities are in geometric progression as follows [6]:

Face (i)	0	1	2	.	.	.	.	m-1
Probability (P_i)	q^{m-1}	pq^{m-2}	$p^2 q^{m-3}$	.	.	.	.	p^{m-1}

Here necessary and sufficient restriction on 'p' and 'q' are

$$q^{m-1} + pq^{m-2} + p^2q^{m-3} + \ldots + p^{m-1} = 1\ ,\quad 0 \le p \le 1,\qquad 0 \le q \le 1 \tag{1}$$

Note that the first restriction is equivalent to $q^m - p^m = q - p$.

Alternatively, parametrizations of (1) may yield other useful interpretations also. For instance, if $p < q$, then defining

$\theta = \frac{p}{q}$ one can easily see that (1) is equivalent to

$$P_i = \frac{(1-\theta)\theta^i}{(1-\theta^m)}\ ,\qquad i = 0,1,\ 2,\ldots m-1 \tag{2}$$

Here rolling the die is equivalent to generating a value of a geometric random variable constrained to the range $\{0,1,2,\ldots,m-1\}$ with $1-\theta$ and θ being the success and failure probabilities respectively.

Next, let us focus on the following event

$X_n^{(m)}$ = total score in 'n' rolls of 'm' sided- die with faces probabilities as described in (1)- (2).

It is clear that $X_n^{(m)}$ has the familiar Binomial Distribution with index 'n' and success probability 'p' when m = 2. For this reason, the distribution of $X_n^{(m)}$ is called the Extended Binomial Distribution of order m, index n and parameter p.

Note that, $X_n^{(m)}$ is simply the convolution of 'n' independent and identically distributed random variables corresponding to the scores of 'n' rolls of the die. Therefore, the probability generating function of $X_n^{(m)}$ can be written as

$$G(t) = E\left[t^{X_n^{(m)}}\right] = \left[\frac{q^m - (pt)^m}{q - pt}\right]^n$$

Expanding $G(t)$ in powers 't' yields an expression for probability mass function of $X_n^{(m)}$ as

$$P_r\left\{X_n^{(m)} = r; p\right\} = C_m\binom{n}{r} p^r q^{(m-1)n-r},$$

$$0 \le r \le (m-1)n \tag{3}$$

where $C_m\binom{n}{r}$ is the coefficient of t^r in $\left[\frac{(1-t^m)}{(1-t)}\right]^n$

Clearly the expression (3) is the probability mass function of extended binomial distribution.

Note that, $C_m\binom{n}{r}$ satisfies the following recurrence relation

$$C_m\binom{0}{r} = \begin{cases} 1\ , & for\quad r = 0 \\ 0\ , & for\quad r \ge 1 \end{cases}$$

$$C_m\binom{n}{r} = \sum_{j=0}^{m-1} C_m\binom{n-1}{r-j},\quad n \ge 1.$$

4.4. Different patterns of Pascal's triangle:

4.4.1. Hockey stick pattern in Pascal's triangle :

We can also look at sums of coefficients in Pascal's triangle along a hockey stick pattern [7]. As long as one start the "hockey stick" with a 1, the linear string of numbers along any size of diagonal totals up to equal the number that is offset from that diagonal below the last number, as shown below in Fig. 6 :

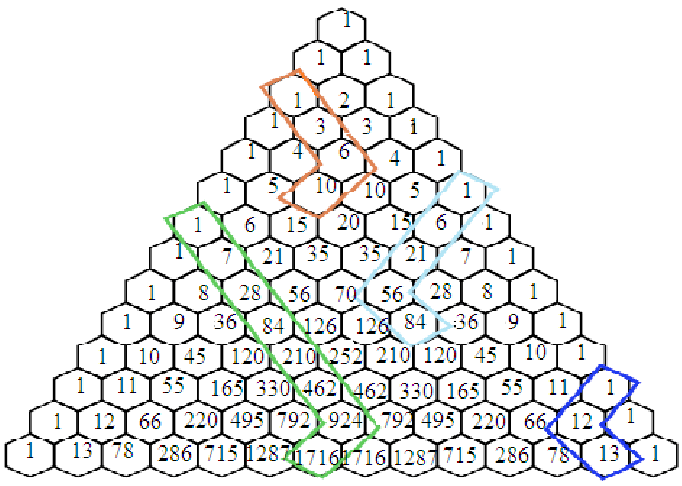

Fig. 6

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4.4.2 Triangular numbers in Pascal's triangle:

Triangular numbers are those numbers which are obtained by continued summation of natural numbers. In mathematical notation the *n*th triangular number is given by

$$T_n = 1 + 2 + 3 + ... + (n-1) + n$$

We can find these triangular numbers along certain diagonals in Pascal's triangle [7]. The diagonal marked below represent triangular numbers as shown in Fig. 7. Of course the same is not seen for other diagonals.

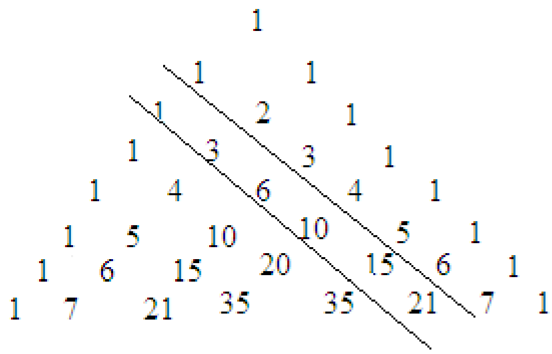

Fig. 7

Note that if we are adding an even number of natural numbers, we see that by pairing up the first and last terms of this series we get $\frac{n}{2}$ sets of (n+1) which means that the result comes out to be $\frac{n}{2}$ (n+1).

On the other hand if we are adding an odd number of natural numbers it comes out to be

$$\frac{(n-1)}{2}(n+1) + \frac{(n+1)}{2}$$

So, either way the addition comes out to be $\frac{n(n+1)}{2}$.

Note that $\frac{n(n+1)}{2} = \frac{(n+1)\,!}{2\,!\,((n+1)-2)\,!} = \binom{n+1}{2}$.

Thus, the numbers of the diagonal which is formed by triangular numbers must follow the above mathematical rule. This allows us to see why this diagonal is formed by triangular numbers.

4.4.3. Square Numbers in Pascal's triangle:

A square number or a perfect square is an integer that is the square of an integer. In Pascal's triangle square numbers can be found by adding up pairs of triangular numbers as shown in Fig. 8:

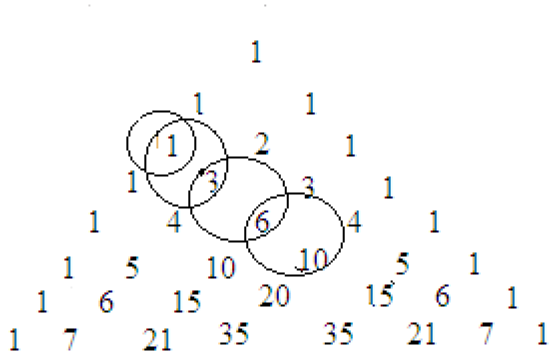

Fig. 8

We can see why the sum of 2 triangular numbers results in a square number as shown in below:

$$T_n + T_{n-1} = \frac{n(n+1)}{2} + \frac{(n-1)n}{2} = \frac{n^2}{2} + \frac{n}{2} + \frac{n^2}{2} - \frac{n}{2} = n^2$$

4.4.4 Fibonacci Numbers:

The Fibonacci Numbers are series of numbers where each of the number is the sum of the two preceding numbers starting from 0 and 1 and the *n*th Fibonacci Number is denoted by F_n. In particular, the first few Fibonacci Numbers are 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144 and so on to infinity. These numbers have been observed in many real life situations like the bee ancestry code, spiral structure in aloes peals of flower, phyllo taxi and so on. The Golden Ratio is a ratio between a Fibonacci number with its preceding one. Those structures which follows Golden ratio = 1.618034 is assumed to be perfect, long lasting and beautiful. Mathematically, the Fibonacci Numbers can be defined as follows:

$$F_0 = 0,\ F_1 = 1\ \ and\ \ F_n = F_{n-1} + F_{n-2},\ n \geq 1$$

In some books F_0, the 0 is removed and the Fibonacci Sequence starts with $F_1 = F_2 = 1$.

5.1. Fibonacci's personal life:

Fibonacci, an Italian Mathematician was born in 1170 in Pisa Republic. He was one of the most talented Western Mathematicians of the middle ages. He popularized the Hindu-Arabian numeral system in Europe. In 1202, he wrote the book 'Liber Abaci'. He also introduced Europe to the sequence of Fibonacci numbers. He died in 1250 in Pisa Republic.

5.2. Fibonacci sequence and it's prehistory:

Fibonacci numbers were first introduced in the western world by Italian Mathematician Fibonacci in his book "Liber Abaci" (1202). But these numbers appear to have first arisen as early as in 200 BC in the works of Pingala (Pingala was an ancient Indian Mathematician, who wrote the book "Chandaḫsastra" also called "Pingala-sutras") on enumerating possible patterns of sounds and rhythms in poetry and speech formed from syllables of two lengths. In Sanskrit as well as in Prakrit vowels are of two kinds- Long and Short. Virahanka's and other Prosodicists' analysis were to compute the number of matras (metre) of a given overall length that can be composed of these syllables. Denoting a short vowel *S* by 1 unit length and a long vowel *L* by 2 units length, the solution would be [9]:

Patterns of length n	Types	No. of patterns
1	*S*	1
2	*SS, L*	2
3	*SSS, SL; LS*	3
4	*SSSS, SSL, SLS ; LSS, LL*	5
5	*SSSSS, SSSL, SSLS, SLSS, SLL;LSSS , LSL, LLS*	8
6	*SSSSSS, SSSSL, SSSLS, SSLSS, SSLL, SLSSS, SLSL, SLLS; LSSSS, LSSL, LSLS, LLSS, LLL.*	13
	And so on.	

The patterns of length *n* arising out of those of length (*n*-1) and those of length (*n*-2) are differentiated by the semi-colon (;). It can be seen that a pattern of length *n* can be formed by prefixing *S* to a pattern of length (*n*-1) and prefixing *L* to a pattern of length (*n*-2).

5.3. Fibonacci Distribution:

Let us consider a pattern of length 2, say *HH*. Let *X* be a random variable representing the number of throws of getting this pattern for the first time and the experiment ends when this happens. The random variable *X* takes values *n* with probabilities:

$$P(X=n)=\left(\frac{F_{n-1}}{2^n}\right), \quad n=2,3,4,... \qquad (4)$$

The above expression (4) is known as probability mass function of 'Fibonacci Distribution' [9].

This can easily be verified from the fact that the number of trials *n* required for pattern *HH* can be obtained by adding the number of trials required $(n-1)$ with those of the number of trials required $(n-2)$ (as the case of the Sanskrit prosody example considered above).

5.4. Relationship between Pascal's triangle and the Fibonacci sequence:

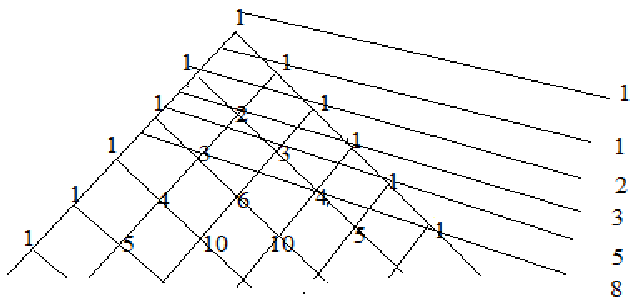

Figure no : 9

If we take Pascal's Triangle and draw the slanting lines as shown in Fig.9 and add the numbers that intersect each line, the sums turn out to be the values in the Fibonacci series [4]:

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, ...

5.5. Application of Fibonacci number in Probability Distribution:

Shane [8] provided a good application of Fibonacci Numbers in applied probability theory. Let X_n be the number of flips needed to advance the marker to position *'n'*. We would like to investigate the distribution of the random variable X_n .

For n = 1, the random variable is simply geometric

i.e. X_1 = number of trials until the first success occurs.

So we start with n = 2 and the probability of head $p=\frac{1}{2}$.

Let $P\left(X_2=k\right)=p_2(k), \ k=2,3,4...$

Now $p_2(2)=P(HH)=\frac{1}{2^2}$, $p_2(3)=P(THH)=\frac{1}{2^3}$,

$$p_2(k+3)=P(k \text{ trials with no run of two heads}).\ P(THH)$$

$$=\left(\frac{A_{2,k}}{2^k}\right).\left(\frac{1}{2^3}\right)=\frac{A_{2,k}}{2^{k+3}}, \qquad k=1,2,3,... \qquad (5)$$

where $A_{2,k}=$ number of arrangements of *k* heads and tails with no two consecutive heads.

To evaluate $A_{2,k}$, we note that we may classify the allowable arrangements according to whether the last tail is in the k^{th} or $(k\text{-}1)^{th}$ position.

Let $a_{2,k,i}=$ number of arrangements of *k* heads and tails having no two consecutive heads and having a tail in the i^{th} position, $i=k,k-1$, gives

$$A_{2,k} = a_{2,k,k} + a_{2,k,k-1}.$$

But $a_{2,k,k} = A_{2,k-1}$ and $a_{2,k,k-1} = A_{2,k-2}$ yielding

$$A_{2,k} = A_{2,k-1} + A_{2,k-2} \quad (6)$$

For k=1, the possible arrangements are *H* and *T*. Thus $A_{2,1}=2$.

For k=2, the possible arrangements are *TH* ,*HT* and *TT*. Thus $A_{2,2}=3$.

From (5) and (6) we get $p_2(k) = \frac{F_{k-2}}{2^k}, \quad k = 2, 3, 4,...$

where F_k = k^{th} *Fibonacci Numbers* with $F_0 = F_1 = 1$.

6. CONCLUSION

In this paper, we have compiled some of the very early works of Blaise Pascal and Pieree de Fermat which we found in our search for the chronological development of Probability theory from its inception. It is really interesting to note how a practical problem proposed by a gambler attracted the above named two giant mathematicians and how they arrived at a mathematical solution by mutual correspondence through letters. We further have compiled some of the works of Abraham de Moivre and Fibonacci and how some of the subsequent researchers have linked those to probability theory. Some of the number patterns observed by different researchers in case of Pascal's triangle and Fibonacci sequence have also been compiled. Though we do not claim any fundamental contribution through this paper, yet the compilations in the paper can give information to the readers about the early stage of development of Probability theory which has become such an indispensable tool in modern scientific research.

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