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A Generalized fixed point theorem in 2-metric space

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Abstract

In this paper we have proved sufficient condition for the existence and uniqueness of fixed point theorem for three self independent maps in 2-mertic space. Our result generalizes and extends many previous results such as Singh and Lal[7], Khan, sastry and Rao[11] etc,

Keywords: 2-metric space, Contraction principle, Cauchy sequence, Convergent sequence, fixed point.

INTRODUCTION

There have been a number of generalization of a metric space. One such generalization is 2-metric space was initiated by Gahler[4],[5]. Geometrically in plane 2-metric function abstracts the properties of the area function for Euclidean triangle just as a metric function abstracts the length function for Euclidean segment. After the introduction of concept of 2-metric space, Many authors establishes an analogue of Banach's Contraction principle in 2-metric space. Iseki for the first time developed fixed point theorem in 2-metric space. Since then a quite number of authors establishes fixed point theorem in 2- metric space.

Lal and Singh [7] proved the following

Theorem (1.1) Let S and T are two self maps of a complete 2-metric space (X, d) such that:

$d(Sx, Ty, a) \leq a_1 d(x, y, a) + a_2 d(Sx, x, a) + a_3 d(Ty, y, a) + a_4 d(Sx, y, a) + a_5 d(Ty, x, a)$

for all x, y, a ϵ X, where a_i (i=1, 2, 3, 4,5) are positive integers such that

$(1-a_3-a_4) > 0$ and $(1-a_2-a_5) > 0$.

Then S and T have a unique common fixed point .

PRELIMINARIES:

Now we give some basic definitions and well known results that are needed in the sequel.

Definition (2.1)[4][5]: Let X be a non-empty set and d: X x X x X → R_+ . If for all x,y,z, and u in X.

We have

(d_1) d(x, y, z) = 0 if at least two of x, y, z are equal.

(d_2) for all x ≠ y, there exists a point z in x such that d(x, y, z) ≠ 0.

(d_3) d(x, y, z) = d(x, z, y) = d(y, z, x) =..and so on

(d_4) d(x, y, z) ≤ d(x, y, u) + d(x, u, z) + d (u, y, z).

Then d is called a 2-metric on X and the pair (X, d) is called 2-metric space.

Definition (2.2) : A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X,d) is said to be a cauchy sequence if $\lim_{m,n \to \infty} d(x_m, x_n, a) = 0$ for all a ϵ X

Definition (2.3) : A sequence $\{x_n\}_{n \in N}$ in a 2-metric space (X, d) is said to be a convergent at x ϵ X

if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all a ϵ X. The point x is called the limit of the sequence.

Definition (2.4) : A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Lemma (2.5):[10] Let $\{x_n\}_{n \in N}$ be a sequence in a complete 2-metric space (X,d) then there exists r ϵ(0,1) such that $d(x_n, x_{n+1}, a) \leq r(x_{n-1}, x_n, a)$ for all non negative integer n and every a in X then $\{x_n\}$ converges to a point in X.

MAIN RESULT:

Theorem (3.1):-If T,T_1 and T_2 are three operators mapping a complete 2-metric space (X,d) to itself be sequentially continuous and if for all x,y a in X.

(¡) min{d $(T_1^p(x), T_2^q(y), a)$, d$(Tx, T_1^p(Tx), a)$, d$(Ty, T_2^q(Ty), a)$, d$(T_1^p(Tx), T_2^q T_1^p(Tx), a)$, d $(Ty, T_2^q T_1^p(Tx), a)\}$ + Kmin{d$(Tx, T_2^q(Ty), a)$,d$(Ty, T_1^p(Tx), a)$,d $(Tx, T_1^p T_2^q(Ty), a)$, d$(T_2^q(Ty), T_2^q T_1^p(Tx), a)\}$ ≤r d(x,y,a), where r ∈ (0,1) and K is a real number.

(¡¡) d(Tx,Ty,a) ≤ d(x,y,a)

(¡¡¡) $TT_1^p = T_1^p T$

$T T_2^q = T_2^q T$

then there exists a unique common fixed point of T,T_1 and T_2 if k>r.

Proof:- Using condition (¡¡) & (¡¡¡),condition (¡) becomes min { d $(T_1{}^p(x), T_2{}^q(y), a)$, d $(x, T_1{}^p(y), a)$, d $(y, T_2{}^q(y), a)$, d$(T_1{}^p(x), T_2{}^q T_1{}^p(x), a)$, d$(y, T_2{}^q T_1{}^p(x), a)\}$ + K min{d$(x, T_2{}^q(y), a)$,d$(y, T_1{}^p(x), a)$,d$(x, T_1{}^p T_2{}^q(y), a)$, $(T_2{}^q(y), T_2{}^q T_1{}^p(x), a)\}$ ≤r d(x,y,a)

Now for given x_0 in X, we Consider a sequence $\{x_n\}_{n \in N}$ as x_0, x_1=$T_1{}^p(x_0)$, x_2=$T_1{}^q(x_1)$, … … … … .. x_n= $T_2{}^q(x_{2n-1})$, x_{2n+1}=$T_1{}^p(x_{2n})$

If for some m, x_m=x_{m+1}, then $T_1{}^p$ and $T_2{}^q$ have a common fixed point x_n in X .Thus we suppose that $x_m \neq x_{m+1}$ ∀ m=1,2,3,……

From the condition for x=x_{2n} & y=x_{2n+1}, we have

min{d $(T_1{}^p x_{2n}, T_2{}^q x_{2n+1}, a)$,d($x_{2n}$,$T_1{}^p(x_{2n})$,a), d($x_{2n+1}$,$T_2{}^q(x_{2n+1})$,a), d$(T_1{}^p(x_{2n}), T_2{}^q T_1{}^p(x_{2n}), a)$, d$(y, T_2{}^q T_1{}^p(x_{2n}), a)\}$ + Kmin{d$(x_{2n}, T_1{}^q(x_{2n+1}), a)$,d($x_{2n+}$,$T_1{}^p(x_{2n})$,a), d$(x_{2n}, T_1{}^p T_2{}^q(x_{2n+1}), a)$, d$(T_2{}^q x_{2n+1}, T_2{}^q T_1{}^p x_{2n}, a)\}$

≤r d(x_{2n}, x_{2n+1},a) for every non–negative integer n.

or, min{d(x_{2n+1}, x_{2n+2},a),d(x_{2n}, x_{2n+1},a)}

≤r d(x_{2n}, x_{2n+1},a) for every non–negative integer n.

Since (X ,d) is a 2-metric space, d(x_{2n}, x_{2n+1},a)≠0 for some a in X.

Hence if d(x_{2n}, x_{2n+1},a)< d(x_{2n}, x_{2n+2},a).

Then we have d(x_{2n}, x_{2n+1},a)≤r d(x_{2n}, x_{2n+1},a) ∀ r ∈ (0,1) which is impossible and so we have d(x_{2n+1}, x_{2n+2},a)≤ rd(x_{2n}, x_{2n+1},a).Similarly we have

d(x_{2n}, x_{2n+1},a)≤rd(x_{2n+1}, x_{2n},a),Therefore

d(x_m, x_{m+1},a)≤rd d(x_{m-1}, x_m,a) for every non–negative integer m and by lemma (2.5).The sequence $\{x_n\}$ converges to some point x_0 in X. i.e $\lim_{n\to\infty} x_n = x_0$

Now,

d(x_0,$T_1{}^p(x_0)$,a)≤ d(x_0,$T_1{}^p(x_0)$,x_{2n}) + d(x_0, x_{2n},a) + d(x_{2n+1},$T_1{}^p(x_0)$,a)

= d(x_0,$T_1{}^p(x_0)$,x_{2n}) + d(x_0, x_{2n},a) + d($T_1{}^p(x_{2n})$,$T_1{}^p(x_0)$,a)

→ 0 as n→ ∞

Therefore, d(x_0,$T_1{}^p(x_0)$,a)=0 ∀ a in X,thus $x_0$is a fixed point of $T_1{}^p$.Similarly x_0 is also a fixed point of $T_2{}^q$.i.e x_0 is the common fixed point of $T_1{}^p$ and $T_2{}^q$. Next let k>r and to prove the uniqueness of a common fixed point of $T_1{}^p$ and $T_2{}^q$ with $x_0 \neq y_0$.

Then d(x_0,y_0,a)≠0. For all a in X,

min{ d $(T_1{}^p(x_0), T_2{}^q(y_0), a)$, d $(x_0, T_2{}^q(y_0), a)$, d $(x_0, T_1{}^p(y_0), a)$, d$(T_1{}^p(x_0), T_2{}^q T_1{}^p(x_0), a)$, d$(x_o, T_2{}^q T_1{}^p(x_0), a)\}$ + K min d(x_0,$T_2{}^q(y_0)$,a), d(y_0,$T_1{}^p(x_0)$,a), d$(x_0, T_1{}^p T_2{}^q(y_0), a)$, d$(T_2{}^q(y_0), T_2{}^q T_1{}^p(x_0), a)\}$

≤r d(x_0, y_0,a)

or, K d(x_0, y_0,a)≤ r d(x_0, y_0,a)

¡,e d(x_0, y_0,a) ≤$\frac{r}{K}$ d(x_0, y_0,a),which is impossible. This proves that $T_1{}^p$ and $T_2{}^q$ have a unique common fixed point. $T_1^p(T_1(x_0))$= $T_1(T_1^p(x_0))$=$T_1(x_0)$, but x_0 is the unique fixed point of $T_1^p(x_0)$.

So $T_1(x_0)$= x_0.similiarly $T_2(x_0)$= x_0, and also x is also the unique fixed point of T_1 and T_2.

Now, d (x_0, Tx_0 , a) = d $(T_1{}^p(x_0), T_2{}^q(Tx_0), a)$

So,

min {d $(T_1{}^p(x_0), T_2{}^q(Tx_0), a)$, d (T$x_0$,$T_1{}^p(Tx_0), a)$, d$(T_2^q(x_0), T_2{}^q(T^q x_0), a)$,d($T_1{}^p(Tx_0)$,$T_2{}^q T_1{}^p(Tx_0)$,a), d($T^q x_0$,$T_2{}^q T_1{}^p T x_0$,a)} + K min{ d$(Tx_0, T_2{}^q(T^q x_0), a)$, d $(T^q x_0 , T_1{}^p(Tx_0), a)$, d(T$x_0$,$T_1{}^p T_2{}^q(T^q x_0)$,a), d($T_2{}^q T^q x_0$,$T_2{}^q T_1{}^p(Tx_0)$,a)}

≤ r d (x_0, T x_0, a)

or, K d(Tx_0,$T^q x_0, a)$ ≤ r d (x_0, T x_0, a)

or, d(Tx_0,$T^q x_0, a)$ ≤ $\frac{r}{K}$ d (x_0, T x_0, a) which gives

d (x_0, T x_0, a)=0 thus x_0=T x_0

Hence x_0 is the unique common fixed point of T,T_1&T_2

Remarks:

(¡) If we take T=I, theorem reduces to:

min { d $(T_1{}^p(x), T_2{}^q(x), a)$, d (x,$T_1{}^p(x), a)$, d (y,$T_2{}^q(y), a)$, d$(T_1{}^p(x), T_2{}^q T_1{}^p(x), a)$, d$(y, T_2{}^q T_1{}^p(x), a)\}$ + K min{ d (y,$T_2{}^q(y), a)$, d (y,$T_1{}^p(x), a)$, d$(x, T_1{}^p T_2{}^q(x), a)$, d$(T_2{}^q(x), T_2{}^q T_1{}^p(x), a)\}$ ≤ r d(x,y,a)

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