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Algebraic Topological Approach for Grid generation of certain manifolds

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Abstract

Grids (Mesh) find use in many application areas of scientific computing. This paper presents mesh generation algorithm that is based on diameter simplicies, which described in this model by vertices, edges, Triangles within advancing front approach. The present method is capable of triangulating a manifold shape like torus and cylinder is created. Several tests have been included into the algorithm to handle overlapping, edge crossing, and degenerate edges that may occur during the grid generation process. The algorithm is also developed to generate topological grid connectivity information needed during the grid generation process. Euler characteristic sketched for projective plane, and manifolds shapes like cube, cylinder and torus.

Keywords: Mesh Generation, Advancing Front, Numerical Simulation, Topology, Geometric Persistent Homology, Simplical Complex, Euler Characteristic, Manifolds.

INTRODUCTION:

Grid generation is a key factor in the numerical simulation of many applications in fluid dynamics, solid mechanics, and electrodynamics. Mesh provide an efficient way to compute geometric persistent homology therefore study the mesh of certain manifolds, such as cylinder, sphere and torus, is very important in topological study as simplical complex . Also, Data sets often have an intrinsic geometric and topological structure. The goal of many problems in geometric inference is to expose this intrinsic structure. In most application the term unstructured meshes refer to triangular and tetrahedral meshes in two and three dimension respectively. A primary application of unstructured meshes concerns the geometric modelling of complex real world objects, soft tissue modeling, and multi-resolution representation of complex shapes. Additionally unstructured meshes play a pivotal role in the numerical simulation of many physical problems in solid mechanics, geo-mechanics, and fluid dynamics [1]. Traditionally, meshes are used in scientific computing for finite element and finite volume analysis. Algorithmic ideas from mesh generation can also be applied to data analysis. Data sets often have an intrinsic geometric and topological structure. The goal of many problems in geometric inference is to expose this intrinsic structure. One important structure of a point cloud is its geometric persistent homology, a multiscale description of the topological features of the data with respect to distances in the ambient space.

The main advantages of unstructured grids are flexibility in fitting complicated domains, rapid grading from small to large elements, and relatively easy refinement and de-refinement. Unlike structured grid generation, unstructured grid generation has been part of main stream computational geometry for some years, and there is a large literature on the subject.

Mesh generation has a huge literature and there are excellent references on structured and unstructured mesh generation [1-5], most grid generation techniques currently in use can fit into one of the three basic methods; Advancing front method [5], Delaunay [6], and Quadtree [7-8]. Grid smoothing adjusts the locations of grid vertices in order to improve element shapes and overall grid quality. In grid smoothing, the topology of the grid remains invariant. Laplacian smoothing is the most commonly used smoothing technique. Laplacian[9-11] smoothing is computationally inexpensive and fairly effective, but it does not guarantee improvement in grid quality. Moustafa salama[12,13,14] presented a method for optimizing unstructured triangular meshes using a floating-point genetic algorithm, also using a hybrid algorithm which is a combination of rough set within genetic algorithm.

Simplicial complexes:

The term simplicial complex may refer to either of two seemingly unrelated concepts.The first concept is that of an abstract simplicial complex, which is a family of sets that is closed under deletion of elements. The second concept is that of a geometric simplicial complex, which is a geometric object in Euclidean space consisting of simplices1 of various dimensions (points, line segments, triangles tetrahedra, and so on), glued together according to certain rules. As we will see in a moment, the two concepts are in fact closely related: For every geometric simplicial complex, there is an underlying abstract simplicial complex describing its combinatorial structure. Conversely, one may realize any abstract complex as a geometric complex.

Geometric Realizations of Simplicial Complexes

One may realize a simplicial complex as a geometric object in

R^n, and the procedure is roughly the following. Identify each vertex with a point is 0-dimensional simplex. For each edge ab, draw a line segment between the points realizing the vertices a and b is 1-dimensional simplex. Next, for each 2-dimensional face abc, fill the triangle with sides given by the line segments realizing ab, ac, and bc. Continue in this manner in higher dimensions. For example, realize each 3-dimensional face abcd as the tetrahedron with sides given by the four filled triangles realizing the 2-dimensional faces contained in abcd is 3-dimensional simplex . Note that the full realization of an abstract simplicial complex is determined by how we realize the vertices of the complex.

Definition: Let V be a vector space over R^1 and let C be a subset of V. C is convex if

$$c_1, c_2, \in C \Rightarrow tc_1 + (1-t)\, c_2 \quad , \in C$$

for all $t \in I$.

A set $\{v_0, v_1, \ldots\ldots, v_k\}$ of vectors in a vector space V is convex-independent, or c-independent, if the set $\{v_1 - v_0, v_2 - v_0, \ldots\ldots, v_k - v_0\}$ is linearly independent. Note that this definition does not depend on which vector is called v0.

Theorem: Suppose $\{v_0, v_1, \ldots\ldots, v_k\}$ is a c-independent set. Let C be the

convex set generated by $\{v_0, v_1, \ldots\ldots, v_k\}$; that is, C is the smallest convex set containing $\{v_0, v_1, \ldots\ldots, v_k\}$. Then C consists of all vectors of the form $\sum_{i=1}^{k} a_i v_i$, where $a_i \geq 0$ for all i and $\sum_{i=1}^{k} a_i = 1$ Furthermore, each $v \in C$ is uniquely expressible in this form.

Definition:

Let V be a vector space over R^1 a convex set by c-independent vectors $\{v_0, v_1, \ldots\ldots, v_k\}$ is called a (closed) k-simplex and is denoted by $[v_0, v_1, \ldots\ldots, v_k]$. k is called the dimension of the simplex. If $v \in [v_0, v_1, \ldots\ldots, v_k]$, then the coefficients a_i, with $a_i \geq 0$ and $\sum_{i=1}^{k} a_i = 1$ such that $\sum_{i=1}^{k} a_i v_i$, are called *barycenteric coordinates* of v[15]

Definitions: Let $\{v_0, v_1, \ldots\ldots, v_k\}$ be a c-independent set. The set $[v \in [v_0, v_1, \ldots\ldots, v_k];\ a_i(v) > 0,\ i = 0, 1, \ldots, k]$ is called an open simplex and is denoted by $(v_0, v_1, \ldots\ldots, v_k)$. We shall also denote an open simplex by (S) and the corresponding closed simplex by [S]. Let $[S] = [v_0, v_1, \ldots\ldots, v_k]$ be a closed simplex. The vertices of [S] are the points $v_0, v_1, \ldots\ldots, v_k$. The closed faces of [S] are the closed simplices $[v_{j_0}, v_{j_1}, \ldots\ldots, v_{j_k}]$

where $\{j_0, j_1, \ldots\ldots, j_k\}$ is a non-empty subset of $\{0, 1, \ldots\ldots, k\}$. The open faces of

the simplex [S] are the open simplices $(v_{j_0}, v_{j_1}, \ldots\ldots, v_{j_k})$.

Remarks.

1. A vertex is a 0-dimensional closed face. It is also an open face.

2. An open simplex (S) is an open set in the closed simplex [S]. Its closure is [S].

3. The closed simplex [S] is the union of its open faces.

4. Distinct open faces of a simplex are disjoint.

5. The open simplex (S) is the interior of the closed simplex [S]; that is, it is the closed simplex minus its proper open faces (faces $\neq$ (S)).

A simplicial complex K (Euclidean) is a finite set of open simplices in some R^n such that

A) if $(S) \in K$, then all open faces of $[S] \in K$;

B) if $(S_1)(S_2) \in K$ and $(S_1) \cap (S_2) = \emptyset$ then $(S_1) = (S_2)$.

The dimension of K is the maximum dimension of the simplices of K.

Remarks. If K is a simplicial complex, let [K] denote the point set union of the

open simplices of K. Then $[K]$ is compact, and $[K] = \bigcup_{(S) \in K} (S) = \bigcup_{(S) \in K} [S]$.

If $[S]$ is a closed simplex, the collection of its open faces is a simplicial com-

complex which we denote by S.

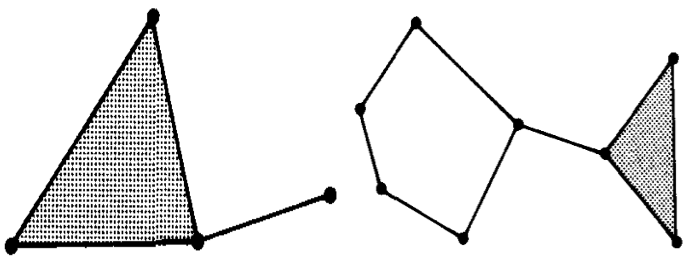

Figure 1 examples of simplicial complexes.

Examples. The following (Fig. 1) are examples of simplicial complexes.

Definitions. Let (S, P) be a metric space, and let Γ be a compact subset of S.

The diameter of T is $\operatorname{diam} T = \sup_{t_1, t_2 \in T} \rho(t_1, t_2)$.

Since T is compact and ρ is continuous, the maximum is assumed, and we may write: $\operatorname{diam} T = \max_{t_1, t_2 \in T} \rho(t_1, t_2)$.

Let K be a simplicial complex in R^n , where R^n is provided with the usual metric.

The mesh of К is the maximum diameter of simplices of К:

$$\text{mesh T} = \max_{s \in T} \text{diam } [S].$$

Mesh К is a discrete representation of geometric model in terms of its geometry, topology, and associated attributes.

The present mesh generation algorithm that is based diameter simplicies which describe in this model by vertices , edges, Triangles within advancing front approach.

Euler characterization χ

In topology, a branch of mathematics, we study shapes that are not rigid. We imagine that all shapes are made of rubber, so two shapes are the same type if one can be stretched or molded into the other, without breaking it. In 1752, the mathematician Leonard Euler discovered a simple formula that could tell shapes apart. Today we call this formula the Euler characteristic. This is what, in mathematics, we call an invariant (topological invariant). If two shapes have different Euler Characteristics, then they are different. If two shapes are the same, then they have the same Euler characteristic. All you need to compute the Euler characteristic is the number of vertices (V), the number of edges (E) and the number of faces F. Then the Euler characteristic is given by.

$$\chi = V - E + F \ .$$

- In case of projective plane $\chi = V - E + F = 1$,
- In case of Tourus and cylinder $\chi = V - E + F = 0$
- In case of sphere, tetrahedral, and cube $\chi = V - E + F = 2$

For example let us consider the Cube

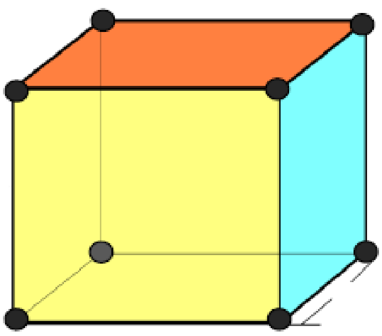

Figure 2. a Cubeas 3-dimensional examples of Euler characteristic

$V = 8$, $E = 12$, $F = 6$ then $\chi = V - E + F = 2$

We can show how can compute this formula easily in two-dimensional shapes for example projective plane is triangulate the shape into vertices, edges and triangles (faces).

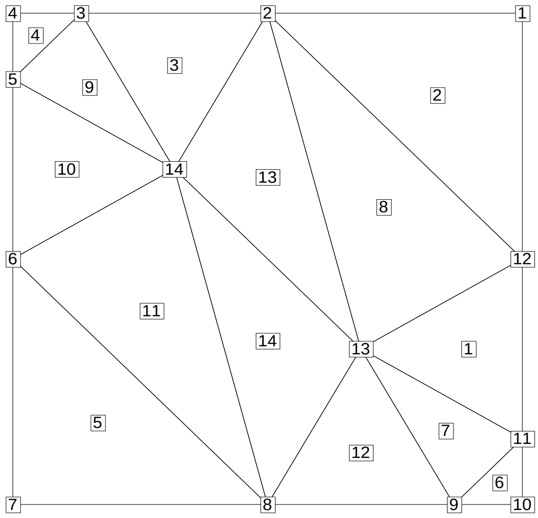

Figure 3. 2-dimensional examples Euler characteristic in case of projective plane.

$V = 14$, $E = 27$, $F = 14$ then $\chi = V - E + F = 1$

The objective of this research work is to develop and apply methods of algebraic topology to generate mesh on certain manifolds to study its' properties .

1 Sketching the Euler Characteristic for Projective Plane:

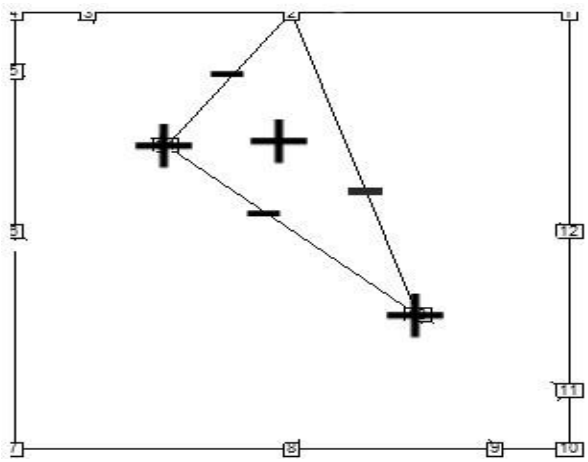

Figure 4 a 2-dimensional examples Euler characteristic in case create a first triangle in projective plane.

In general we can sketch the Euler characteristic in case of projective plane(we consider it as a square) is start with a vertex on the boundary and denote it by positive signal (+) now when create a triangle by beginning this vertex we need to add 2 vertices inside the domain and connect all three vertices together. Now we get three edges signed by negative (- - -) and two vertices signed by (++) and the inner of the triangle (face) inside signed by (+) see figure 4. Now we have inside the square three (+++) and three negative sign (- - -) so ‘ll be cancelled and it remain only the one + signal which we started the $\chi = V - E + F = 1$ by repeating this process many times until covering the whole square by triangles.

2 Simplical Complex and Euler Characterization Method

This method is characterized by a step-by-step procedure for the triangulation of keeping the geometric region has the same Euler characterization during triangulation in every step which created by advanced steps.

- Start with a region Ω described by a projective plane have Euler characterization $\chi = 1$.
- Discretize the boundary $\partial\Omega$ of a projective plane Ω which described by vertices and edges $F_0 = \partial\Omega \cup \Gamma$.
- Create triangles one by one, starting from the boundary and working inward or outward, by adding edge or edges and vertices.
- Select the vertices of each triangle according to a set of rules which keep the new region has the same Euler characterization $\chi = 1$.

In mathematical terms, if $\partial\Omega$ is the boundary of a planar domain Ω, and Γ is the set of boundary vertices, Advancing triangulation of the domain Ω and the interior vertices Γ is a collection of triangles $\{T_k\}$ such that:

Each T_k is formed by three vertices belonging to front ;

Each T_k lies completely inside the domain Ω; and front domain must be updated as $F_1^{\Omega} = F_0^{\Omega}/_{T_1}, F_1^{\Omega} = F_2^{\Omega}/_{T_2} \dots, F_k^{\Omega} = F_{k-1}^{\Omega}/_{T_1}$.

Ω is totally covered by $\{T_k\}$ and no two triangles of $\{T_k\}$ overlap (total region has the same Euler characterization $\chi = 1$.

The present algorithm can be graphically illustrated in bellow figure 5 which can be topologically as a projective plane.

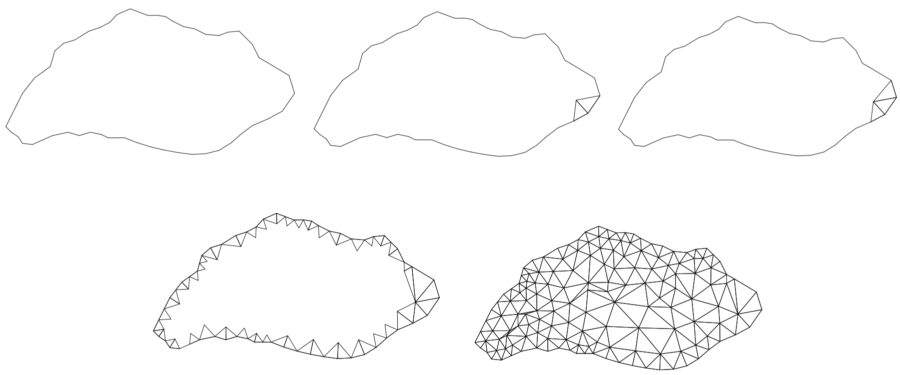

Figure 5. advancing front process.

The following pseudo code outlines the grid generation algorithm:

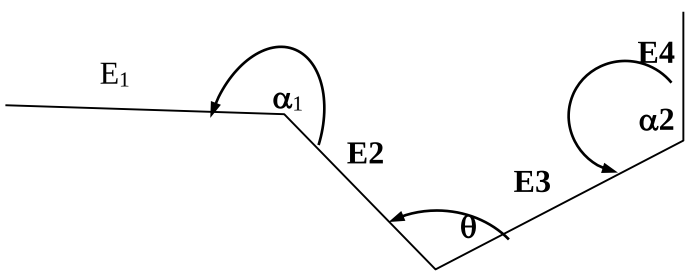

Figure 6. Section of the advancing front

(E_2 is the current segment, E_3 the next segment and E_1 the last segment)

The following pseudo code outlines the grid generation algorithm:

```
Do until the front F is empty when for some k front domain F^Ω_{k+1} = F^Ω_k
        Select E2 the smallest edge on the front to be the Current edge
         Check  the angles α1, θ, α2 of (E1, E2, E3, E4).
          IF (θ ≤ 90.0) then
               Create a triangle within  FIRSTCASE
               ELSE IF (θ ≤ 180.0) then
                    Create two triangle within  SECONDCASE
                    ELSE IF (θ  < 360) then
                         Create a triangle within  THIRDCASE
                    ENDIF
               ENDIF
         ENDIF
ENDDO
```

Figure 7. The advancing front algorithm

Details of the algorithm

As shown the algorithm in figure 7 containing three cases as follows:

To create a new triangle considering the left Front F_k still a simplicial complex (no overlapping between any two triangles in the domain Ω figure bellow

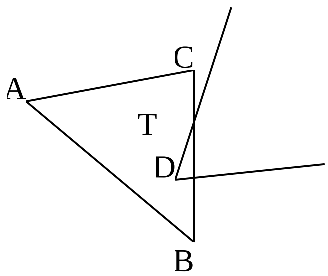

Figure 8. Overlapping not a simlicial complex

Case of (θ ≤ 90.0)

One triangle is created as in opposite figure and the front will be updated as follows:

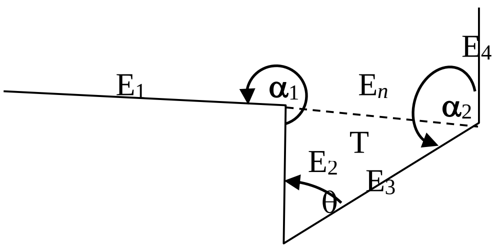

Figure 9. Case of (θ ≤ 90.0) one triangle is created

Updating the front

To update the front after constructing a triangle T_k, See figure 9,

The following steps are implemented:

-Add one Triangle to the set Triangles $\{T_k\}$

-Add one edge to the set of edges $\{E\}$.

- Remove two edges from front and add a new edge to the front and front domain updated as:

$F_k^{\Omega} = F_{k-1}^{\Omega} / T_k$.

Note: no changes on set of vertices $\{V\}$; and one more edge and one more face (triangles) i.e $\chi = 1$

Case of the angle π / 2 < θ ≤ π

In this case create two new triangles

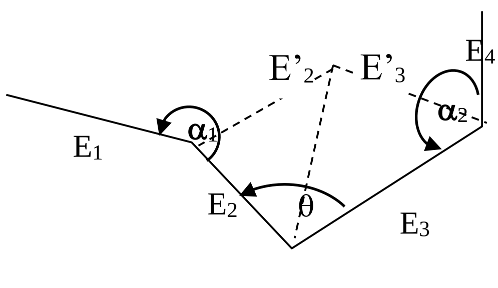

Figure 10. Case of (π/2 < θ ≤ π) two triangles are created

- ***Details of the second case:***

To create two new triangles the following step are undertaken:

Updating the front

To update the front after constructing two triangles See figure 10

The following steps are implemented:

- Add one vertex to the vertices' set $\{V\}$,

- Add two Triangles to the Triangles' set $\{T_k\}$,

- Add three edges to the edges' Set $\{E\}$,

- Remove two edges from front and add new two edges to the front; the front domain updated twice as $F_k^{\Omega} = F_{k-1}^{\Omega}/_{\mathrm{T}_k}$

Note: by adding one vertex; and three edges more and two faces more (triangles) i.e $\chi = 1$

Case of π < θ < 2π

- ***Details of the Third case:***

To create a new triangle as in bellow figure the following step are undertaken:

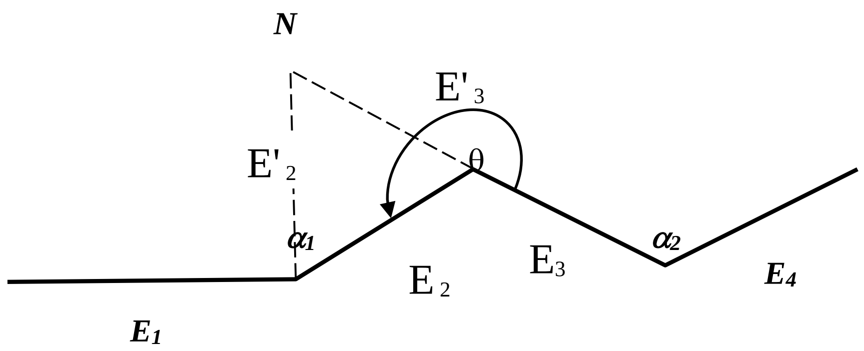

Figure 11. Case of (π < θ < 2π) One triangle is created.

Updating the front

To update the front after constructing one triangles See figure 11

The following steps are implemented:

- Add one vertex to the vertices' set $\{V\}$,

- Add one Triangles to the Triangles' set $\{T_k\}$,

- Add two edges to the edges' Set $\{E\}$,

- Remove one edges from front and add new two edges to the front; the front domain updated once as $F_k^{\Omega} = F_{k-1}^{\Omega}/_{\mathrm{T}_k}$

Note: by adding one vertex; and two edges more and one face more (triangles) i.e $\chi = 1$

Surfaces and Manifolds

- In brief, a (real) n-dimensional manifold is a topological space$\mathcal{M}$ for which every point $x \in \mathcal{M}$ has a neighborhood homeomorphic to Euclidean space $\mathbb{R}^n$.
- An n-manifold is a topological space that "locally looks like" the Euclidian space R^n . Both sphere and torus are 2-manifold , while A circle is a 1-manifold[15,16,17]

RESULTS

The developed grid generation algorithm is implemented into an efficient ForTran computer program that runs under Microsoft windows platform. The input data to the program is a file containing the coordinates of the object boundaries that constitute the initial advancing front in anti clockwise direction. The present grid generation algorithm that is based on diameter simplicies which described in this model by vertices, edges, Triangles within advancing front approach. Euler characterization χ

The program is tested for several applications and the results are displayed in figure here we input the data for projected plane after generating mesh we create a manifolds shapes like cylinder and tours with mathematica software package the figures11, 12 and 13 is implemented.

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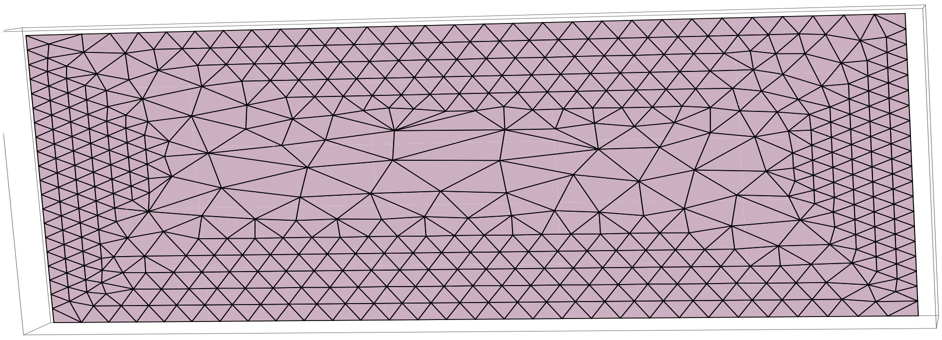

Figure 11. mesh of projective plane as rectangular shape

By connecting upper and lower edges the projective plane to obtain a cylinder which has Euler characteristic $\chi = 0$ the corner points A and C be the same point Also B and D, and we have to consider that all points on upper and lower edges of the projective plane be the same see in figure 12-13

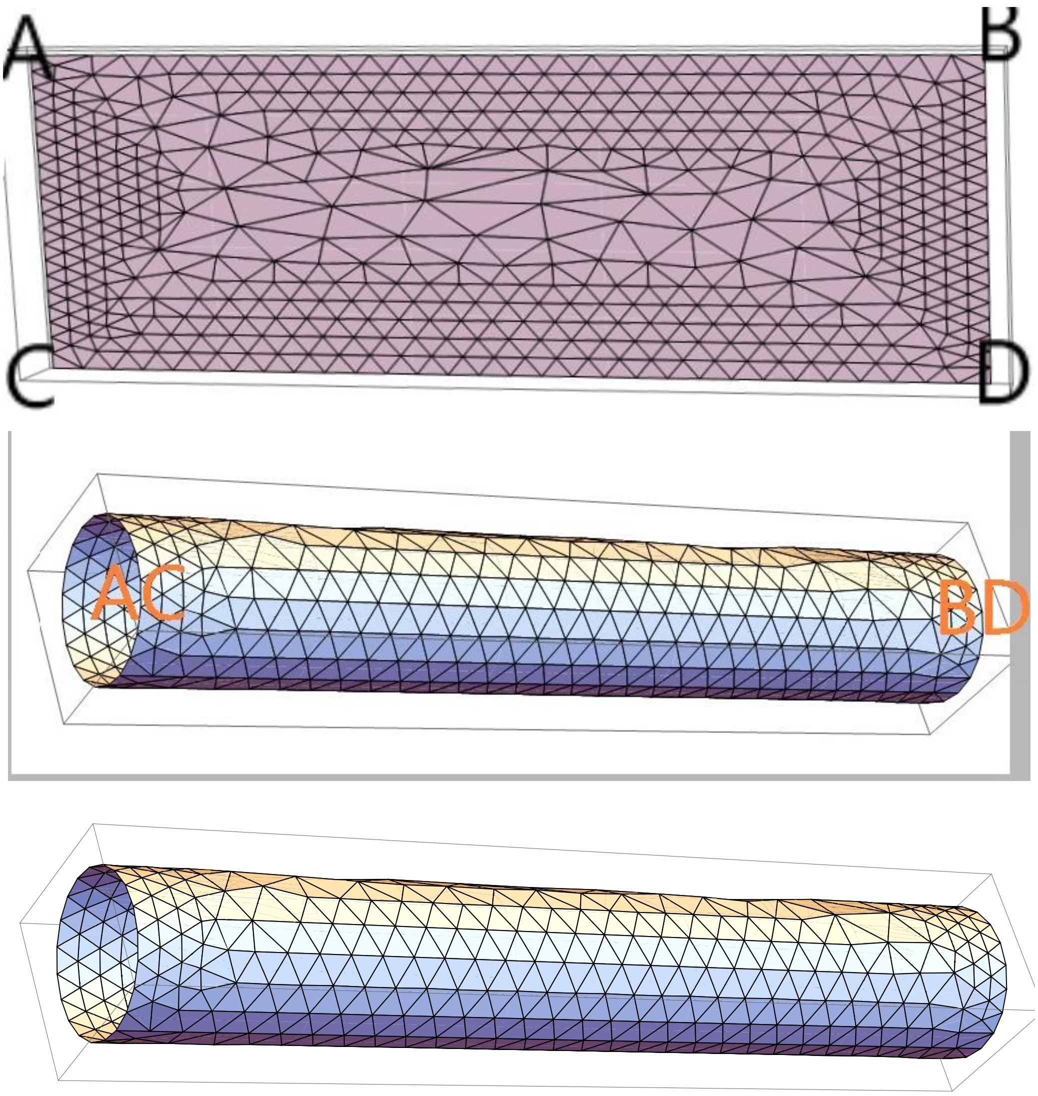

Figure 12. mesh cylinder shape from projective plane.

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Now we can rotate cylinder to obtain a torus obtain a cylinder which has Euler characteristic $\chi = 0$ as in figure 13. One can see that two bases of cylinder disappear when both of them touch each other.

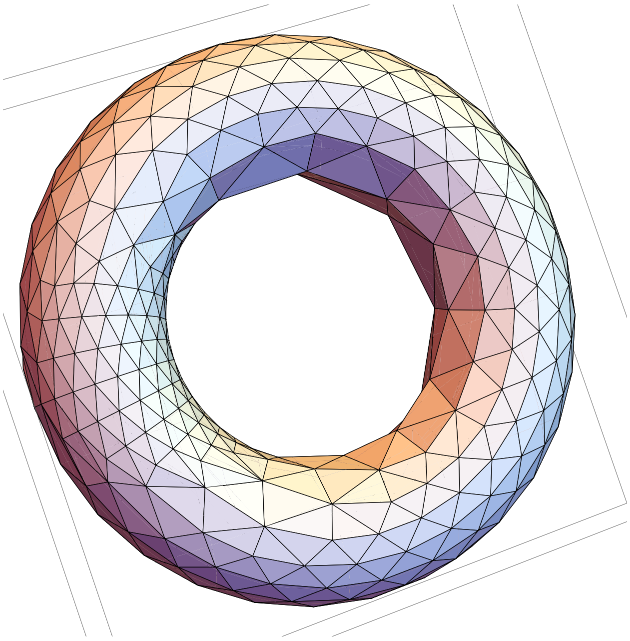

Figure 13. mesh torus shape

DISCUSSION

Euler characterization for two dimensional shapes are studied like figures 3 but figure 2 it is for cube as example for 3D shape. The figure 4 shows how sketching Euler characteristic for projective plane. The proposed results show that, an unstructured triangular grid generation algorithm based on topological concepts is developed for any arbitrary two-dimensional regions. Additionally, the algorithm uses an efficient branching strategy for triangle construction and selection which remaining the concepts of simplical complex. The grid generation algorithm also includes an efficient fast permutation algorithm to establish the connectivity of the triangular grid that is needed for the advancing front algorithm and for post-processing the developed grids. We can see that from 2D shape in figure 5, this figure reflects the efficiency and capability of proposed method.

Mesh is generated for projective plane as in figure 11. We can see the topology relation between projective plane and the others manifolds like cylinder and torus from figures 12 and 13. After generating these manifolds also Euler characteristic is studied for them. It can be noted from rotating cylinder to obtain a torus. It is clear that cylinder has Euler characteristic $\chi = 0$ as in figure 13. One can see that two bases of cylinder disappear when both of them touch each other i.e. Euler characteristic $\chi = 0$ for torus

CONCLUSION AND FUTURE WORK

In this paper, an unstructured triangular grid generation is developed for an arbitrary two-dimensional regions by keeping the geometric region has the same Euler characterization during triangulation in every step which created by advanced steps. The proposed algorithm based on diameter simplicies which described in this model by vertices, edges, Triangles within advancing front approach. The present method is capable of triangulating a manifold shape like torus and cylinder is created. The grid generation algorithm also includes an efficient fast permutation technique to establish the connectivity of the triangular grid that is needed for the advancing front algorithm and for post-processing the developed grids. Mesh generation algorithm has been tested and validated successfully for a number of test cases covering a wide range of applications. The results show that the grid generation algorithm is capable of generating high quality triangles for any complex two-dimensional region. As special case, the proposed method used for generating mesh for projected plane after generating mesh we create a manifolds shapes. A proposed technique used to transform two-dimensional mesh to three-dimensional mesh to create a manifold shapes like cylinder and torus. The topological properties of mesh for manifolds as evaluating euler characterization are studied and the results show that the efficiency of the proposed method.

For future work, the algorithm can be expanded for other manifolds like sphere.

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