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Common Coupled Fixed Point in C*-algebras Valued Metric Spaces

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Abstract

In this paper we introduced a Common coupled fixed-point theorem in the C*-algebras valued metric spaces with certain contraction condition.

INTRODUCTION

We begin with the some basic definitions and facts about structures of C*-algebra and Fixed Point Theory. We also give some facts which play a central role in the C*-algebra valued metric spaces.

A C*-algebra $\mathcal{A}$ is a complex Banach algebra with a conjugate-linear involution$*: \mathcal{A} \to \mathcal{A}$, such that

$(x^*)^* = x$, $(xy)^* = y^*x^*$, ,$(x + y)^* = x^* + y^*$, $\|x^*x\| = \|x\|^2$

for all x, y *in* $\mathcal{A}$. *The* C^**-condition* $\|x * x\| = \|x\|^2$ *implies that the involution is an isometry in the sense that* $\|x^*\| = \|x\|$ *for all* x *in* $\mathcal{A}$.

A C^**-algebra is called unital if it possesses a unit. It follows easily that* $\|1\| = \|1\|$.

In general C^**-algebra is non-commutative, for the commutative* C^**-algebra its completely determined by Gelfand, as in the following*

Theorem 1.1 *[Gelfand,3]. If* $\mathcal{A}$ *is a non-zero commutative* C^**-algebra, then the Gelfand representation*

$\varphi: \mathcal{A} \to C_0(\Omega(\mathcal{A}))$, $a \mapsto \hat{a}$ *is an isometric* $*$*-isomorphism.*

Theorem 1.2 *[2] Every* C^**-algebra has a faithful representation on some Hilbert space.*

This theorem was proved in [2] and means that every C^*-algebra is isometrically isomorphic to a normclosed $*$-algebra in $\mathcal{B}(\mathcal{H})$, for some Hilbert space $\mathcal{H}$. This is one of the most important results in the theory of C^*-algebras. We now introduce some basics of Positive C^*-algebras, we refer to [2] and [3] for more details and proofs. An element a in a C^*-algebras $\mathcal{A}$ is called self adjoint if $a = a*$, denote $\mathcal{A}_{sa}$ the set of all self adjoint elements in $\mathcal{A}$, a $\in \mathcal{A}$ is called positive element if $a \in \mathcal{A}_{sa}$ and $Sp(a) \subset \mathbb{R}^+$. We write $a \geq 0$ if a is positive. And denote by $\mathcal{A}_+$ the set of all positive elements in $\mathcal{A}$. The set $\mathcal{A}_+$ is a closed cone in the sense that

$$(a + b \in \mathcal{A}_+, if\ a, b \in \mathcal{A}_+ \ and\ \mathcal{A} + \cap (-\mathcal{A}_+) = \{0\})$$

Lemma 1.3 [3] *Let* $\mathcal{A}$ *be a unital* C^**-algebra and let* $a \in \mathcal{A}$. *Then the following are equivalent:*

(i) $a \geq 0$,
(ii) $a = b^2\ for\ some\ b \in \mathcal{A}_{sa}$
(iii) $a = bb^* for\ some\ b \in \mathcal{A}$

For a given $a, b \in \mathcal{A}_+$, we denote $a \leq b\ if\ b - a \geq 0$, $\mathcal{A}_+$ becomes a partially ordered vector space.

***Lemma*1.4 [3] .** *Suppose that* $\mathcal{A}$ *is unital* C^**_algebra with a unit* I.

(1) *If* $a \in \mathcal{A}$ *with*$\|a\| < \frac{1}{2}$, *then* $I - a$ *is invertible and* $\|a(I - a)^{-1}\| < 1$;
(2) *Suppose that* $a, b \in \mathcal{A}$ *with* $a, b \geq \theta$ *and* $ab = ba$, *then* $ab \geq \theta$
(3) *Suppose that* $a, b \in \mathcal{A}$ *with* $a, a \leq b$ $\|a\| \leq \|b\|$.
(4) *Suppose that* $c \geq 0$ *and* $a \in \mathcal{A}$ *then,* $a^*ca \geq 0$

C*-ALGEBRAS VALUED METRIC SPACE

In the next we introduced the definition of the C*-algebras valued metric spaces and give some examples. Moreover we introduced the meaning of Cauchy sequence and convergent.

The main reference in this section is [6] and there is a generalization for these results was introduced in [4]

Definition 2.1 . Let X be a nonempty set. Suppose the mapping d: X × X → A satisfies:

(1) $0 \leq d(x, y)$ *for all* $x, y \in X$ *and* $0 = d(x, y) \Leftrightarrow x = y$
(2) $d(x, y) = d(y, x)$*for all* $x, y \in X$
(3) $d(x, y) \leq d(x, z) + d(z, y)$*for all* $x, y, z \in X$

Then d *is called a a* $C^* -$*algebra valued metric on* X *and* (X, A, d) *is called* $C^* -$*algebra valued metric space*

***Definition* 2.2 .** *Let* $(X, \mathcal{A}, d)$ *be a* C^**_algebra valued metric space and* $\{x_n\} \subset X$ *is a sequence in* X. *If* $x \in X$ *and* $\varepsilon > 0$ *there is* N *such that for all* $n > N$, $\|d(x_n, x)\| \leq \varepsilon$, *then* $\{x_n\}$ *is called a convergent sequence in* X *to* x *and denote it by* $\lim_{n\to\infty} x_n = x$.

Moreover, if for any $\varepsilon > 0$*there is* N *such that for all* $n, m > N$, $\|d(x_n, x_m)\| \leq \varepsilon$, *then* $\{x_n\}$ *is called a cauchy sequence in* X.

Definition 2.3. *The tripled* $(X, \mathcal{A}, d)$ *is a completed* C^**_algebras valued metric space if every cauchy sequence is convergent.*

Example 2.4 . *If*X *is a Banach space, then* $(X, \mathcal{A}, d)$ *is a completed* C^**_algebras valued metric space with the metric*

$d(x, y) = \|x - y\|$, $x, y \in X$.

Example 2.5 . *Let* $X = \mathbb{C}$ *and* $\mathcal{A} = M_{2\times2}(\mathbb{C})$ *,Then (X,* $\mathcal{A}$*, d) is a C*-algebra valued metric space, where*

$$d(x, y) = \begin{bmatrix} |x - y| & 0 \\ 0 & \alpha|x - y| \end{bmatrix}$$

and partial ordering on $\mathcal{A}$ *is given as*

$$\begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \succcurlyeq \begin{bmatrix} y_1 & y_2 \\ y_3 & y_4 \end{bmatrix} \Leftrightarrow x_i \geq y_i$$

for i = 1, 2, 3, 4, Let $\alpha \epsilon \mathbb{C}$, $\alpha \geq 0$.

MAIN RESULTS

Common coupled fixed point for real valued –metric space was introduced and studied in [1] and [5], by using different ways to choose the contraction , here we generalized these results in case of C*-algebras valued metric spaces.

Definition 3.1 . Let $(\mathrm{X}, \mathcal{A}, \mathrm{d})$ be a C^*_algebras valued metric space. The mapping $\mathrm{T}: \mathrm{X} \to \mathrm{X}$ is called contractive mapping on X if there is an $a \in \mathcal{A}$such that $\|a\| < 1$ and satisfy

$$d\big(T(x), T(y)\big) \leq a^* d(x, y)a, \;\; x, y \in X$$

Theorem 3.2: Let (X, A, d) be a complete C*-valued metric Space, and let $\mathrm{F}, \mathrm{G}: \mathrm{X} \to X$ be a mappings such that $d\big(F(x, y), G(u, v)\big) \leq a w(x, y, u, v)a^*$ for all $x, y, u, v \in X$, where $w(x, y, u, v) \in [d(x, u), d(y, v), \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u)), \frac{1}{2}(d(F(x, y), u) + d(G(u, v), x))$, then F, G have a unique common coupled fixed point.

Proof: let x_0, y_0 be two arbitrary elements in X, choose $x_1, y_1 \in X$ such that $x_1 = F(x_1, y_1)$ and $y_1 = F(x_0, y_0)$, again choose $x_2, y_2 \in X$ such that $x_2 = G(x_1, y_1)$ and $y_2 = G(x_1, y_1)$, Containing this places we can construct two sequences $(x_n)$and (y_n) in X such that $x_{2n+1} = F(x_{2n}, y_{2n})$, $y_{2n+1} = F(y_{2n}, x_{2n})$, $x_{2n+2} = G(x_{2n+1}, y_{2n+1})$, $y_{2n+2} = G(y_{2n+1}, x_{2n+1})$ for $n = 0,1,2, \ldots$ then we have the following cases:

Case1: $w(x, y, u, v) = d(x, u)$.

From

$$d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}) \leq a\, d(x_{2n}, x_{2n+1})a^* , \tag{1}$$

and

$$d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1}) \leq a\, d(y_{2n}, y_{2n+1})a^* , \tag{2}$$

Then from (1) and (2) we have

$$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq a\, (d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}))a^* \tag{3}$$

Case2: $w(x, y, u, v) = d(y, v)$.

By similar arguments to case 1 we get

$$d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}) \leq a\, d(x_{2n}, x_{2n+1})a^* , \tag{4}$$

and

$$d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1}) \leq a\, d(y_{2n}, y_{2n+1})a^* , \tag{5}$$

Then from (1) and (2) we have

$$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq a\, (d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}))a^* \tag{6}$$

Case 3:$w(x, y, u, v) = \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u)$, we get

$$d(x_{2n+1}, x_{2n+2}) = d(F(x_{2n}, y_{2n}), G(x_{2n+1}, y_{2n+1}) \leq \frac{a}{2}(\, d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}))a^* \tag{7}$$

Similarly we have

$$d(y_{2n+1}, y_{2n+2}) = d(F(y_{2n}, x_{2n}), G(y_{2n+1}, x_{2n+1}) \leq \frac{a}{2}(d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))a^* \tag{8}$$

From (7), (8)

$$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})) \leq \frac{a}{2}(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))a^* \tag{9}$$

Case 4:$w(x, y, u, v) = \frac{1}{2}(d(F(x, y), u) + d(G(u, v), x)$, Similarly we obtain

$$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2})) \leq \frac{a}{2}(d(x_{2n}, x_{2n+1}) + d(x_{2n+1}, x_{2n+2}) + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2}))a^* \tag{9}$$

Recall all above argument for each case we get

Case1:

$d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) \leq a\ (d(x_{2n}, x_{2n+1}) + d(y_{2n}, y_{2n+1}))a^*$

$\leq a^2\ (d(x_{2n-1}, x_{2n}) + d(y_{2n-1}, y_{2n}))(a^*)^2 \leq$

$\dots . \leq\ a^{2n+1}\ (d(x_0, x_1) + d(y_0, y_1))(a^*)^{2n+1}$ (10)

Put $d(x_{2n+1}, x_{2n+2}) + d(y_{2n+1}, y_{2n+2}) = d_{2n+1,2n+2}$

And $d(x_0, x_1) + d(y_0, y_1) = d_{0,1}$

We can rewrite eqn (9) in the following form

$d_{2n+1,2n+2} \leq a^{2n+1}\ (d_{0,1})(a^*)^{2n+1}$and then for each $n \in N_1$we obtain

$d_{n\ ,n+1} \leq a^n\ (d_{0,1})(a^*)^n$ from lemma 1.4, we have

$\|d_{n\ ,n+1}\| \leq \|a^n\|\ \|d_{0,1}\| \|(a^*)^n\|$

Since A is multiplicative, we get $\|d_{n\ ,n+1}\| \leq \|a\|^n\ \|d_{0,1}\| \|a^*\|^n$ also $\|a\| = \|a^*\|$ so we get $\|d_{n\ ,n+1}\| \leq \|a\|^{2n}\ \|d_{0,1}\|$ choose $\|a\|^2 = h < 1$ se we have

$\|d_{n\ ,n+1}\| \leq h^n\ \|d_{0,1}\|$, for $m > n$ we get

$\|d_{n\ ,m}\| \leq h^n(1 + h + h^2 + \cdots + h^{n-m})\|d_{0,1}\| = \frac{h^n}{1-h}\ \|d_{0,1}\| \to 0\ as\ n, m\ \to \infty$

So $\|d_{n\ ,m}\| \leq 0, as\ n, m\ \to \infty$, therefore $d_{n\ ,m} = 0, \quad as\ n, m\ \to \infty$

So (x_n), (y_m), are Couchy sequences in X. Since X is complete we get xand y such that $x_m \to x$, $y_m \to y$, as $n, m \to \infty$ now, we prove that $F(x, y) = G(x, y) = x$ and $F(y, x) = G(y, x) = y$, for that $d(F(x, y), x) \leq d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x) = d\big(F(x, y), G(x_{2n+1}, y_{2n+1})\big) + d(x_{2n+2}, x) \leq$

$a\ \ d(x, x_{2n+1})a^* + d(x_{2n+2}, x)$ since (x_n) is Couchy sequence $d(F(x, y), x) \leq\ a\ \ d(x, x_{2n+1})a^*$ by using lemma 2.1

$\|d(F(x, y), x)\| \leq\ \|a\|\ \ \|d(x, x_{2n+1})\| \|a^*\| = \|a\|^2\ \ \|d(x, x_{2n+1})\| \to 0$ as $n \to \infty$

Therefore $d(F(x, y), x) = 0$ and then $F(x, y) = x$.

Using similar arguments, we get

$d(G(x, y), x) \leq d(G(x, y), x_{2n+1}) + d(x_{2n+1}, x) = d\big(G(x, y), F(x_{2n}, y_{2n})\big) + d(x_{2n+1}, x) \leq a\ \ d(x, x_{2n})a^* + d(x_{2n+1}, x)$ since (x_n) is Couchy sequence $d(G(x, y), x) \leq\ a\ \ d(x, x_{2n})a^*$ by using lemma 2.1

$\|d(G(x, y), x)\| \leq\ \|a\|\ \ \|d(x, x_{2n})\| \|a^*\| = \|a\|^2\ \ \|d(x, x_{2n})\| \to 0$ as $n \to \infty$

Therefore $d(G(x, y), x) = 0$ and then $G(x, y) = x$.

Similarly we can show that $F(y, x) = G(y, x) = y$ thus $(x, y)$is a common coupled fixed point of a mappings $F\ and\ G$

To see (x, y) is unique let (x_0, y_0) be other common coupled fixed point of a mappings $F\ and\ G$

Let $d(\mathrm{x}, x_0) = d(F(\mathrm{x}, x_0), G(\mathrm{x}, x_0) \leq a\ \ d(\mathrm{x}, x_0)a^*$.

$\|d(\mathrm{x}, x_0)\| \leq \|a\|^2 \; \|d(x, x_0)\| \to 0$ as $n \to \infty$

So $d(\mathrm{x}, x_0) = 0$ therefore $\mathrm{x} = x_0$ and similarly $\mathrm{y} = y_{0.}$

Case 2: If $w(x, y, u, v) = d(y, v)$, we follow the same arguments as in case 1, and get the Uniqueness and existence of the common coupled fixed points of the mappings $F \; and \; G$.

Case 3: $w(x, y, u, v) = \frac{1}{2}(d(F(x, y), x) + d(G(u, v), u)$ from equation (9)

$d_{2n+1,2n+2} \leq \frac{a}{2}(d_{2n,2n+1} + d_{2n+1,2n+2})a^*$ from lemma 1.4

$\|d_{2n+1,2n+2}\| \leq \frac{1}{2}\|a\|(\| d_{2n,2n+1}\| + \|d_{2n+1,2n+2}\|)\|a^*\| \leq \frac{1}{2}\|a\|^2(\| d_{2n,2n+1}\| + \|d_{2n+1,2n+2}\|)$

$\|d_{2n+1,2n+2}\|(1 - \frac{1}{2}\|a\|^2) \leq \frac{1}{2}\|a\|^2(\| d_{2n,2n+1}\|)$ put $\frac{1}{2}\|a\|^2 = k$

$\|d_{2n+1,2n+2}\| \leq \frac{k}{1-k}(\| d_{2n,2n+1}\|) \leq h(\| d_{2n,2n+1}\|)$ where $\frac{k}{1-k} = h < 1$

Apply similar analogues, we obtain

$\|d_{2n+1,2n+2}\| \leq h^{2n}(\| d_{0,1}\|)$ so for each n we have

$\|d_{n,n+1}\| \leq h^n(\| d_{0,1}\|) \to 0 \; as \; n \to \infty$

For $n > m$ one can find $\|d_{n,m}\| \to 0 \; as \; n, m \to \infty$

So $(x_n), (y_m)$, are Couchy sequences in X. *Since* X is complete we get $x \; and \; y$ such that $x_n \to x$, $y_m \to y$, $as \; n, m \to \infty$.

$d(F(x, y), x) \leq d(F(x, y), x_{2n+2}) + d(x_{2n+2}, x) \leq d(F(x, y), x_{2n+2}) = d\big(F(x, y), G(x_{2n+1}, y_{2n+1})\big) \leq$
$\frac{a}{2}(d(F(x, y), x) + d(G(x_{2n+1}, y_{2n+1}), x))a^*$ (11)

$\leq \frac{a}{2}(d(F(x, y), x) + d(x_{2n+2}, x))a^*$

$d(F(x, y), x) \leq \frac{a}{2}d(F(x, y), x)a^*$

$\|d(F(x, y), x)\| \leq \frac{\|a\|^2}{2}\|d(F(x, y), x)\|$

$\|d(F(x, y), x)\|(1 - \frac{1}{2}\|a\|^2) \leq 0$

$\|d(F(x, y), x)\| = 0$ this implies that $d(F(x, y), x) = 0$ and this gives $F(x, y) = x$ by a similar way $G(x, y) = x$, $F(y, x) = y \; and \; G(y, x) = y$.

Thus $(x, y)$is a common coupled fixed point of a mappings $F \; and \; G$

To see (x, y) is unique let (x_0, y_0) be other common coupled fixed point of a mappings F and G

Let $d(\mathrm{x}, x_0) = d(F(\mathrm{x}, x_0), G(\mathrm{x}, x_0) \leq a \; d(\mathrm{x}, x_0)a^*$.

$d(\mathrm{x}, x_0) = d(F(\mathrm{x}, x_0), G(\mathrm{x}, x') \leq \frac{a^*}{2}[\; d(F(\mathrm{x}, x_0), x) + d(G(\mathrm{x}, x_0), x) \; a$

$d(\mathrm{x}, x_0) \leq \frac{a^*}{2}[\; d(F(\mathrm{x}, x_0), x) + d(G(\mathrm{x}, x_0), x)] \; a$

$\|d(\mathrm{x}, x_0)\| \leq \frac{\|a\|^2}{2}\|d(\mathrm{x}, x_0)\| \to 0$

$\|d(\mathrm{x}, x_0)\| \to 0$ then $x = x_0$ similarly $y = y_0$

Thus $(x, y)$is a common coupled fixed point is a unique

Case 4: If $w(x, y, u, v) = \frac{1}{2}(d(F(x, y), u) + d(G(u, v), x))$, we follow the same arguments as in case 3, and get the Uniqueness and existence of the common coupled fixed points of the mappings $F \; and \; G$.

CONCLUSIONS

We proved a main theorems in common coupled fixed point theorem in C*-algebras valued metric space using suitable contraction condition that generalize the results obtained in case of real valued metric space.

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