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Application of Lie Symmetry for Mathematical Model of HIV Infection of CD_4^+T Cells

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Abstract: In this paper, we have studied the mathematical model of HIV infection of CD_4^+T cells. This model corresponds to initial value problem consisting a class of nonlinear ordinary differential equations. The main objective of this paper is to apply Lie symmetry approach to evaluate the uninfected CD_4^+T cells in the body. Lie symmetry method has been studied to introduce an approximated solutions of the systems of ordinary differential equations. The approximated solutions of the mathematical model of HIV infection has been studied using Lie symmetry.

Keywords: Lie symmetry method, Ordinary differential equation, Approximated solution, system, LSM.

INTRODUCTION

The most important mathematical models for physical phenomena is the differential equation. Motion of objects, Fluid and heat flow, bending and cracking of materials, vibrations, chemical reactions and nuclear reactions are all modeled by systems of differential equations (DEs). Moreover, Numerous mathematical models in science and engineering are expressed in terms of unknown quantities and their derivatives. Many applications of DEs, particularly ordinary differential equations (ODEs) of different orders, can be found in the mathematical modeling of real life problems (mechee2014direct).

Several studies, including mathematical modeling, have been devoted to understand the transmission of the infection. [2] studied the global properties of a class of human immunodeficiency virus (HIV) models. The basic model is a 5-dimensional nonlinear ODEs that describes the interaction of the HIV with two target cells, CD_4^+T cells and macrophages. [**?**] modeled HIV disease progression as a function of changes in viral load and CD_4 count over time among ART naïve persons. The disease progression Markov model was nested within a dynamic model of HIV transmission at population level. [11] introduced fractional order into an HIV model with considering the effect of viral diversity on the human immune system with frequency dependent rate of proliferation of cytotoxic T-lymphocytes (CTLs) and rate of elimination of infected cells by CTLs, based on a fractional-order differential equation model. HIV models can be classified into two categories: population-level models and within-host models ([8]). In this study, we consider the HIV infection model of CD_4^+T cells is examined ([12]). This model is given by the components of the basic three-component model are the concentration of susceptible CD_4^+T cells, they infected by the HIV viruses and free HIV virus particles in the blood. CD_4^+T cells are also called as leukocytes or T helper cells. These with order cells in human immunity systems fight against diseases. HIV use cells in order to propagate. In a healthy person, the number of CD_4^+T cells is $800-1200$ mm^3 ([12]).

Many nonlinear mathematical models have been developed to describe infection by the human immunodeficiency virus (HIV). A model for the infection of the human immune system by HIV was developed by [10]. This model of virus spread has three variables: the population sizes of uninfected cells, infected cells, and free virus particles.

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[10], [12], [3], and [7] extended HIV model and developed a new model by considering four variables:

1. Cells that are uninfected,
2. Cells that are latently infected,
3. Cells that are actively infected, and
4. Free virus particles. Their model is described by a system of four ordinary differential equations.

It was noted that the model can replicate many of the symptoms of AIDS observed clinically. [1] reduced the HIV model described to a system of three ordinary differential equations by assuming that all the infected cells are capable of producing the virus ([3]).

Dynamic of a model for HIV infection of CD_4^+T cells has been examined by ([5] while [**?**] studied global stability and periodic solution of a model for HIV infection of CD_4^+T cells and [4] solved a fractional order model of HIV infection of CD_4^+T cells. The components of the basic four-component model are the concentration of CD_4^+T cells, the concentration of infected CD_4^+T cells by the HIV viruses and free HIV viruses partialness are denoted respectively by $x(t)$, $y(t)$ and $z(t)$. In this paper, many nonlinear mathematical models have been developed to describe infection by the human immunodeficiency virus (HIV). We consider the HIV infection model of CD_4^+T cells is examined. This model is given by the components of the basic three-component model are the concentration of susceptible CD_4^+T cells, they infected by the HIV viruses and free HIV virus particles in the blood are denoted respectively by $x(t)$, $y(t)$ and $z(t)$. CD_4^+T cells are also called as leukocytes or T helper cells. In a healthy person, the number of CD_4^+T cells is $\frac{800}{1200}\ mm^3$. A Lie symmetry model of for HIV infection of CD_4^+T cells is reviewed ([3]).

PRELIMINARY

Mathematical Model for HIV System

In this work, we have studied the mathematical model of HIV infection of CD_4^+T cells in vivo. This model is characterized by a system of the nonlinear ordinary differential equations

$$\frac{dx}{dt} = s - ax + rx\left(1 - \frac{x+y}{T_{max}}\right) - kzx, \quad (1)$$

$$\frac{dy}{dt} = kzx - by; \qquad 0 \le t \le R < \infty, \quad (2)$$

$$\frac{dz}{dt} = nby - cz. \quad (3)$$

with the initial conditions:

$$x(0) = r_1,\ y(0) = r_2,\ z(0) = r_3,$$

Where R is any positive constant. Throughout this paper, we set;

$$s = 0.1,\ a = 0.02,\ b = 0.3,\ r = 3,\ c = 2.4,$$
$$k = 0.0027, T_{max} = 1500 \text{ and } n = 10.$$

The logistic growth of the healthy CD_4^+T cells is described by$(1 - \frac{x+y}{T_{max}})$, and proliferation of infected CD_4^+T cells is neglected. Natural turnover rates of uninfected T cells, infected T cells and virus particles, respectively, denoted by a, b and c respectively. For $k > 0$ is the infection rate. Each infected $CD4^+T$ cells is assumed to produce n virus particles during its lifetime,including any of its daughter cells. The body is believed to produce CD_4^+T cells from precursors in the bone marrow and thymus at a constant rate s. When stimulated by antigen or mitogen, CD_4^+T cells multiply through mitosis with a rate r. The maximum CD_4^+T cells concentration in the body denoted by T_{max} ([1]) and [9] solved the model of HIV infection of CD_4^+T cells. [9] introduced and applied the Laplace Adomian decomposition method for solving a model for HIV infection of CD_4^+T cells, this method yields very accurate approximate solutions by use only a few iterations. It was reported that the method was accurate. Nonlinear models can be studied through approximations such as linearisation and numerical methods since such models are used for describing a plethora of phenomena ranging from physics to systems chemistry, biology, ecology and economics, a construction of exact solutions for nonlinear systems is an important necessity. In the second half of the 19th century, the Norwegian mathematician Scopus Lie began to create a remarkable body of work that unified virtually all known methods of solving differential equation. He discovered that symmetries of differential equations can be found and exploited systematically. Over many years considerable research effort has been directed at understanding the elegant algebraic structure of symmetry groups.

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ANALYSIS OF LIE SYMMETRY METHOD

Consider the following system of ordinary differential equations:

$$\dot{x} = f(t,x,y,z) \quad (4)$$
$$\dot{y} = g(t,x,y,z) \quad (5)$$
$$\dot{z} = h(t,x,y,z) \quad (6)$$

where an over dot denoted $\frac{d}{dt}$. we consider invariance of this system under the infinitesimal transformations

$$\bar{t} = t + \lambda T(t,x,y,z) + O(\lambda^2) \quad (7)$$
$$\bar{x} = x + \lambda X(t,x,y,z) + O(\lambda^2) \quad (8)$$
$$\bar{y} = y + \lambda Y(t,x,y,z) + O(\lambda^2) \quad (9)$$
$$\bar{z} = z + \lambda Z(t,x,y,z) + O(\lambda^2) \quad (10)$$

the invariance of the equations (4) , (5) & (6) under the transformations (7), (8), (9) & (10) leads to Lie's invariance condition

$$\Gamma^{(1)}\Delta \mid_{\Delta=0} = 0 \quad (11)$$

where Δ refers to the system. The infinitesimal operator Γ is defined as:

$$\Gamma = T\frac{\partial}{\partial t} + X\frac{\partial}{\partial x} + Y\frac{\partial}{\partial y} + Z\frac{\partial}{\partial z} \quad (12)$$

with the first extension defined as

$$\Gamma^{(1)} = \Gamma + X_{[t]}\frac{\partial}{\partial \dot{x}} + Y_{[t]}\frac{\partial}{\partial \dot{y}} + Z_{[t]}\frac{\partial}{\partial \dot{z}} \quad (13)$$

In Equation (13), the extended infinitesimal transformations are

$$X_{[t]} = D_t(X) - \dot{x}D_t(T) \quad (14)$$
$$y_{[t]} = D_t(Y) - \dot{y}D_t(T) \quad (15)$$
$$Z_{[t]} = D_t(Z) - \dot{z}D_t(T) \quad (16)$$

where the total differential operator D_t is defined as

$$D_t = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} + \dot{z}\frac{\partial}{\partial z} + \ddot{x}\frac{\partial}{\partial \dot{x}} + \ddot{y}\frac{\partial}{\partial \dot{y}} + \ddot{z}\frac{\partial}{\partial \dot{z}} + ...$$

Once the infinitesimals T, X, Y and Z have been found, we must solve

$$Tr_t + Xr_x + Yr_y + Zr_z = 0 \quad (17)$$
$$Tu_t + Xu_x + Yu_y + Zu_z = 0 \quad (18)$$
$$Tv_t + Xv_x + Yv_y + Zv_z = 0 \quad (19)$$
$$Tw_t + Xw_x + Yw_y + Zw_z = 1 \quad (20)$$

This will then give a set of new variables that will transform the given system to one that is independent of w.

METHOD OF SOLUTION OF HIV MODEL

The HIV Model has been introduced in the system of ordinary differential equations(ODEs) (1)-(3). For solving this system of ODEs using Lie symmetry method, applying equations (12) & (13) to the equations of the system (1)-(3) yields

$$\Gamma\left(s - ax + rx\left(1 - \frac{x+y}{T_max}\right) - kzx\right) = \left(-a + r - \frac{2rx}{T_max} - \frac{ry}{T_max} - kz\right)X - \frac{rx}{T_max}Y - kxZ \quad (21)$$

$$\Gamma(kzx - by) = kzX - bY + kxZ \quad (22)$$

$$\Gamma(nby - cz) = nbY - cZ \quad (23)$$

Substituting the extended infinitesimal transformations in Equation (13)

$$X_t + \dot{x}X_X + \dot{y}X_y + \dot{z}X_z - \dot{x}(T_t + \dot{x}T_x + \dot{y}T_y + \dot{z}T_z) = \left(-a + r - \frac{2rx}{T_max} - \frac{ry}{T_max} - kz\right)X - \frac{rx}{T_max}Y - kxZ \quad (24)$$

$$Y_t + \dot{x}Y_x + \dot{y}Y_y + \dot{z}Y_z - \dot{y}(T_t + \dot{x}T_x + \dot{y}T_y + \dot{z}T_z) = kzX - bY + kxZ \quad (25)$$

$$Z_t + \dot{x}Z_x + \dot{y}Z_y + \dot{z}Z_z - \dot{z}(T_t + \dot{x}T_x + \dot{y}T_y + \dot{z}T_z) = nbY - cZ \quad (26)$$

Substituting $\dot{x}$, $\dot{y}$ and $\dot{z}$ into equations (24), (25) & (26) we get

$$X_t + \left(s - ax + rx\left(1 - \frac{x+y}{T_max}\right) - kzx\right)(X_x - T_t) + (kzx - by)X_y + (nby - cz)X_z - \left(s - ax + rx\left(1 - \frac{x+y}{T_max}\right) - kzx\right)^2 T_x - (kzx - by)\left(s - ax + rx\left(1 - \frac{x+y}{T_max}\right) - kzx\right)T_y - (nby - cz)\left(s - ax + rx\left(1 - \frac{x+y}{T_max}\right) - kzx\right)T_z = \left(-a + r - \frac{2rx}{T_max} - \frac{ry}{T_max} - kz\right)X - \frac{rx}{T_max}Y - kxZ \quad (27)$$

$$Y_t + \left(s - ax + rx\left(1 - \frac{x+y}{T_max}\right) - kzx\right)Y_x + (kzx - by)(Y_x - T_t)$$

$$+ (nby - cz)Y_z - (kzx - by) \times \left(s - ax + rx\left(1 - \frac{x+y}{T_{max}}\right) - kzx\right) T_x - (kzx - by)^2 T_Y - (kzx - by)(nby - cz)T_z$$

$$= kzX - bY + kxZ \tag{28}$$

$$Z_t + (s - ax + rx(1 - \frac{x+y}{T_{max}}) - kzx)Z_X + (kzx - by)Z_y + (nby - cz)(Z_z - T_t) - (nby - cz)\left(s - ax + rx\left(1 - \frac{x+y}{T_{max}}\right) - kzx\right)T_x - (nby - cz)(kzx - by)T_y - (nby - cz)^2 T_z = nbY - cZ \tag{29}$$

As the system is difficult to solve in general, we seek special solutions by assuming that

$$T = T(t),\ X = X(x),\ Y = Y(y) \tag{30}$$

We note that these are chosen only to reduce the complexity of equations (27), (28) & (29) and other choices could be made. Under the assumptions in (30), we get

$$\left(s - ax + rx\left(1 - \frac{x+y}{T_m ax}\right) - kzx\right)(X_x - T_t) = \left(-a + r - \frac{2rx}{T_m ax} - \frac{ry}{T_m ax} - kz\right)X - \frac{rx}{T_m ax}Y - kxZ \tag{31}$$

$$(kzx - by)(Y_x - T_t) = kzX - bY + kxZ \tag{32}$$

$$(nby - cz)(Z_z - T_t) = nbY - cZ \tag{33}$$

Which is obviously much simpler. Now, we need some non-trivial infinitesimals (those that are not identically zero). If we take the partial derivative of (33) from with respect to t, yield the equation partial differential equation $T_{tt} = 0$ implies to the equation $T_t = g$ and then,

$$T(t) = gt + h \tag{34}$$

where g and h are constants. Inserting (34) into (31), (32) & (33), we find

$$\left(s - ax + rx\left(1 - \frac{x+y}{T_m ax}\right) - kzx\right)(X_x - g) = \left(-a + r - \frac{2rx}{T_m ax} - \frac{ry}{T_m ax} - kz\right)X - \frac{rx}{T_m ax}(py + q) - kxZ \tag{35}$$

$$(kzx - by)(Y_x - g) = kzX - bY + kxZ \tag{36}$$

$$(nby - cz)(Z_z - g) = nbY - cZ \tag{37}$$

Partially differentiating Equation (36) with respect to y twice yields

$$Y"(y) = 0.$$

Thus,

$$Y(y) = py + q \tag{38}$$

where p and q are are constant. In the process of annihilating (35) by differentiating, it is possible that we introduced additional information into the solution. Therefore, we insert(38) into our starting point (35), yielding

$$\left(s - ax + rx\left(1 - \frac{x+y}{T_m ax}\right) - kzx\right)(X_x - g) = \left(-a + r - \frac{2rx}{T_m ax} - \frac{ry}{T_m ax} - kz\right)X - \frac{rx}{T_m ax}(py + q) - kxZ \tag{39}$$

As Equation (39) should be satisfied for all values of y.

We immediately see that

$$y : -\frac{rx}{T_{max}}(X_x - g) = \frac{-rx}{T_{max}}p$$

which leads to $X = (p+g)x + c_1$, then substituting X and Y in Equation (36) and satisfied for all values of x we get

$$x : -2kzg = kZ.$$

Then,

$$Z = -2zg$$

Therefore, the infinitesimals are given in following equations:

$$T = gt + h, X = (p + g)x + c_1, \quad Y = py + q, Z = -2zg \tag{40}$$

Of course, other infinitesimals could be found. In fact, there is an infinite set of infinitesimals. our next task is to find a change of variables, for convenience, We set

$$g = 1,$$

$$p = 1,$$

$$q = 0$$

and

$$h = 0$$

in Equation (40) to get the following

$$\begin{aligned} X &= 2x, \\ Y &= y, \\ T &= t, \\ Z &= -2z. \end{aligned}$$

Now, substituting X, Y, T and Z in these equations and then, we have to solve

$$tr_t + 2xr_x + yr_y - 2zr_z = 0 \quad (41)$$
$$tu_t + 2xu_x + yu_y - 2zu_z = 0 \quad (42)$$
$$tv_t + 2xv_x + yv_y - 2zv_z = 0 \quad (43)$$
$$tw_t + 2xw_x + yw_y - 2zw_z = 1 \quad (44)$$

Equations (41)-(44) have the following solutions:

$$\begin{aligned} r &= \frac{t}{f(xyz)} \\ u &= \frac{t}{f(xyz)} \\ v &= \frac{t}{f(xyz)} \\ w &= ln(t) + \frac{t}{f(xyz)} \end{aligned}$$

For special case we have the solutions:

$$\begin{aligned} r &= \frac{t}{xyz} \\ u &= \frac{t}{xyz} \\ v &= \frac{t}{xyz} \\ w &= ln(t) + \frac{t}{xyz} \end{aligned}$$

DISCUSSION AND CONCLUSION

In this paper, a mathematical model of HIV infection of CD_4^+T cells has been studied. Lie symmetry has been introduced for solving the HIV infection of CD_4^+T cells. Lie symmetry approach has been used to evaluate the uninfected CD_4^+T cells in the vivo. Lie symmetry method has been studied to derived an approximated solutions of the systems of ordinary differential equations. The approximated solutions of the mathematical model of HIV infection has been derived using Lie symmetry.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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